Ben Ruppik, Max-Planck Institute for Mathematics, Bonn   2021-07-15, 18:30–19:00 CEST

Markov's theorem: [1950s]

There cannot be an algorithm to distinguish all pairs of smooth, closed, compact 4-manifolds.

Adian–Rabin theorem: Let $\mathcal{J}$ be a Markov property of finitely presented groups.

•) $\mathcal{J}$ preserved under group iso. •) $\exists$ f.p. A_+ with property $\mathcal{J}$ •) $\exists$ f.p. A_- s.th. $\forall$ f.p. T with property $\mathcal{J}$: $A_- \not\hookrightarrow T$

Then there is no algorithm which can decide whether a group given by a finite presentation $\pi = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$ has property $\mathcal{J}$.

Proof idea: Reduce to the unsolvability of the word problem [Novikov-Boone theorem]

Given a word w in the generators $\{g_1, g_1^{-1}, \dots, g_n, g_n^{-1}\}$ of π

$\rightsquigarrow$ construct a finitely presented group π_w such that $\begin{cases} \text{if } w =_\pi 1 & \text{then } \pi_w \cong A_+ \\ \text{if } w \neq_\pi 1 & \text{then } A_- \hookrightarrow \pi_w \end{cases}$

$\Rightarrow$ π_w has property $\mathcal{J}$ iff. $w =_\pi 1$ $\leftarrow$ $\exists$ groups where the word problem is undecidable □

Important special case: [Novikov, Boone, based on ideas of Gödel]

There is no algorithm to decide whether a given finitely presented group $\pi = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$ is trivial.

$$\mathrm{Man}^4 := \left\{ \begin{matrix} \text{smooth, compact,} \\ \text{closed } 4\text{-manifolds} \end{matrix} \right\}$$

(arbitrary fundamental group)

Def.: A manifold $X^4 \in \mathrm{Man}^4$ is called recognizable if there exists an algorithm which given as input some manifold $Y \in \mathrm{Man}^4$ decides whether $Y \cong_{\text{diffeo.}} X$.

Goal:

Thm.: $\exists k > 0$ so that $\#^k S^2 \times S^2$ is not recognizable in Man^4.

Corollary: There cannot be an algorithm which distinguishes compact, smooth, closed 4-manifolds.

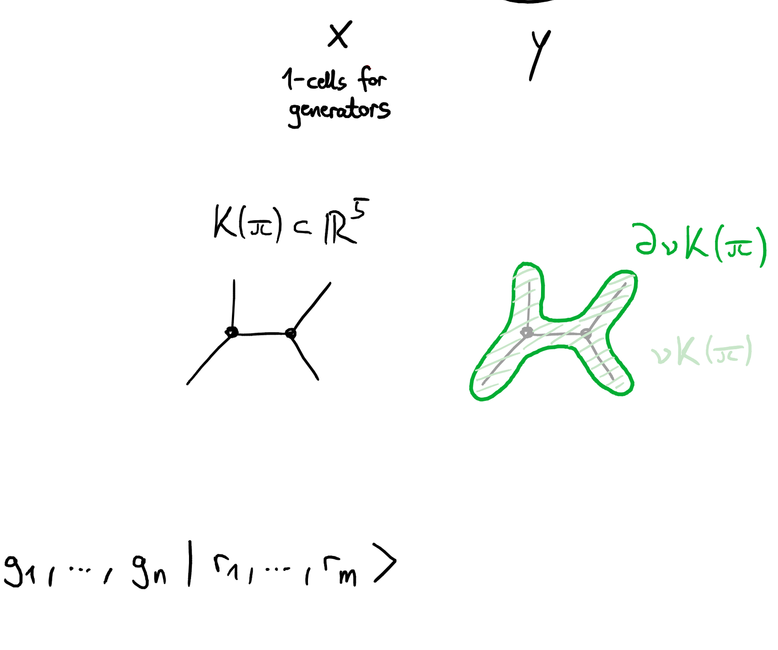
Fun fact:

Any finitely presented group appears as $\pi_1 \left(\begin{matrix} \text{closed, smooth, oriented} \\ 4\text{-manifold} \end{matrix} \right)$

•) Given presentation

$\pi = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

Ex.: $\pi = \langle x, y \mid x^n y^{-2}, xyxy^{-1}, y^2 \rangle$



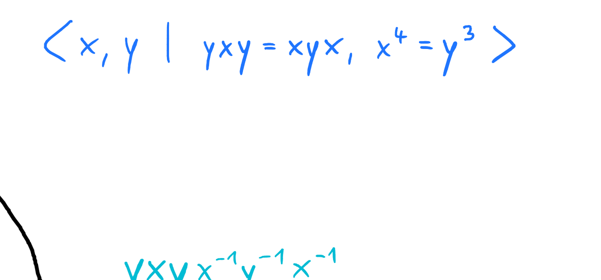
•) build 2-complex

$$K(\pi) = \Big(\bigvee_{\substack{\text{generators} \\ g_i}}^{n} S^1 \Big) \cup_{\partial = \text{relations}} \bigcup^{m} D^2$$

•) $K(\pi) \hookrightarrow \mathbb{R}^5$

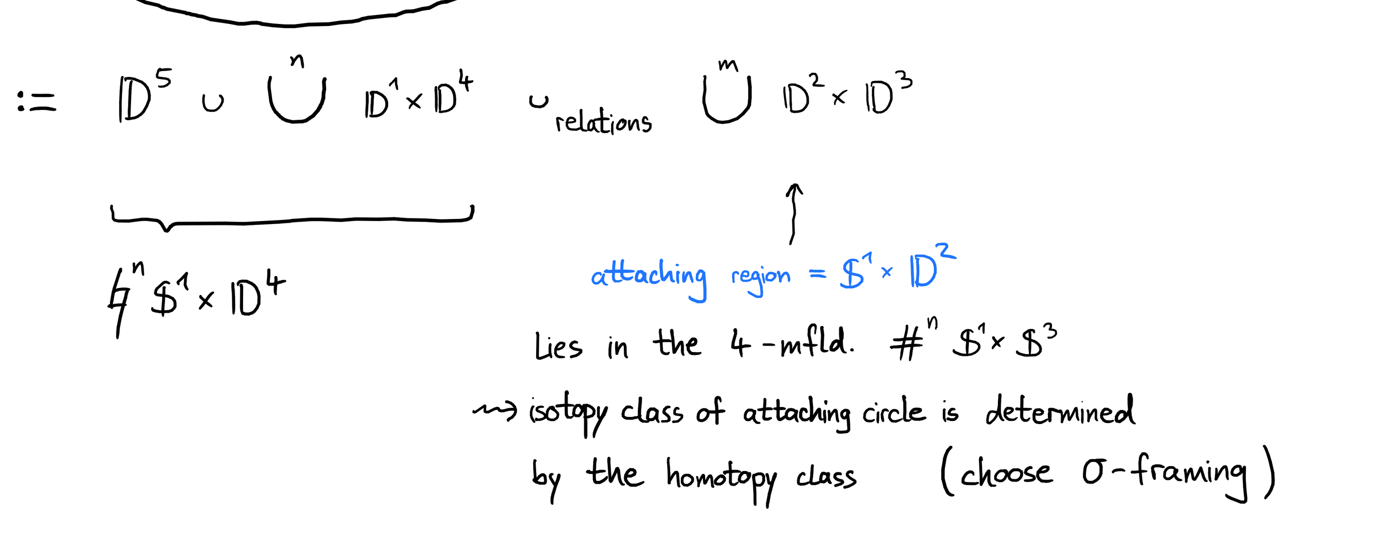
•) take a closed tubular neighborhood $\nu K(\pi) = 5$-mfld.

$\rightsquigarrow$ boundary $\partial \nu K(\pi) =$ closed 4-mfld. with fundamental group π



Alternative description of this construction: $\pi_{\mathcal{P}} = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

Ex.: $\langle x, y \mid yxy = xyx, x^{k+1} = y^k \rangle$, say for $k = 3$: $\langle x, y \mid yxy = xyx, x^4 = y^3 \rangle$



$$Y^5_\pi := \underbrace{D^5 \cup \bigcup^{n} D^1 \times D^4}_{\natural^n S^1 \times D^4} \cup_{\text{relations}} \bigcup^{m} D^2 \times D^3$$

attaching region $= S^1 \times D^2$

Lies in the 4-mfld. $\#^n S^1 \times S^3$

$\rightsquigarrow$ isotopy class of attaching circle is determined by the homotopy class (choose 0-framing)

Different presentations $\mathcal{P}, \mathcal{P}'$ of the group π differ by Tietze moves

correspond to •) birth or death of 1-2 handle pair
•) handle slide
} do not change the 5-mfld. $Y^5_{\pi_{\mathcal{P}}}$

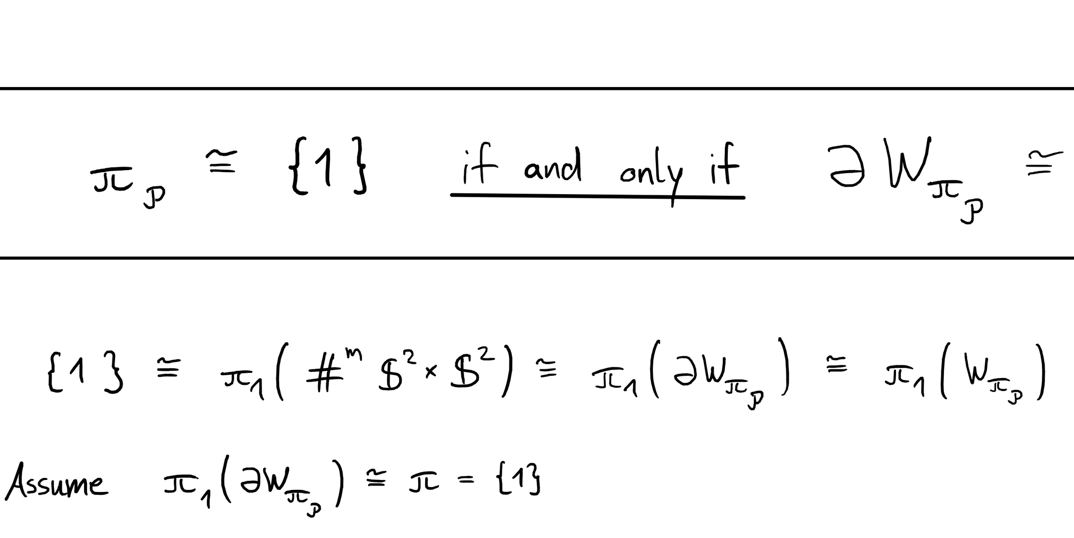
$\pi_{\mathcal{P}} = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

general position: Loops in π_1 and homotopies can be pushed away from the spine of Y_π to lie in ∂Y_π

$Y^5_{\pi_{\mathcal{P}}} = 0\text{-h.} \cup \bigcup^{n} 1\text{-h.} \cup \bigcup^{m} 2\text{-h.}$

smooth 5-manifold with $\pi_1(\partial Y_\pi) \xrightarrow[\text{incl}_*]{\cong} \pi_1(Y_\pi) \cong \pi$

$$\boxed{W_{\pi_{\mathcal{P}}} := Y_{\pi_{\mathcal{P}}} \cup \bigcup^{n} \text{trivial 2-handles} = Y_{\pi_{\mathcal{P}}} \natural^n S^2 \times D^3}$$

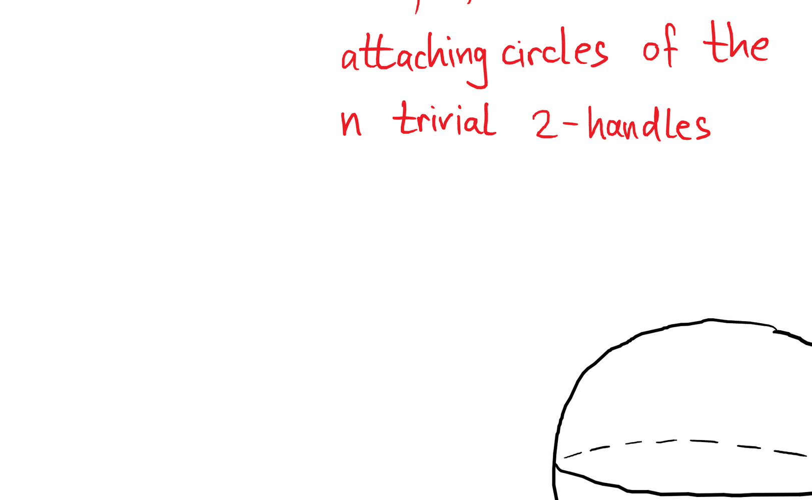


number of relations

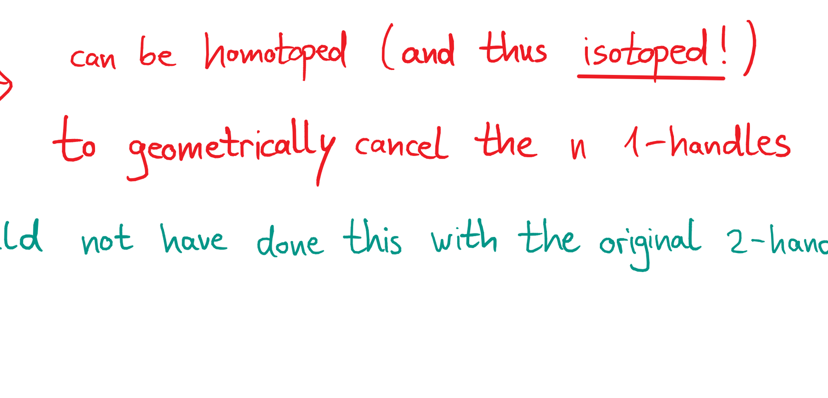
$$\boxed{[\text{Markov}] \quad \pi_{\mathcal{P}} \cong \{1\} \quad \underline{\text{if and only if}} \quad \partial W_{\pi_{\mathcal{P}}} \cong \#^m S^2 \times S^2}$$

"$\Leftarrow$": $\{1\} \cong \pi_1(\#^m S^2 \times S^2) \cong \pi_1(\partial W_{\pi_{\mathcal{P}}}) \cong \pi_1(W_{\pi_{\mathcal{P}}}) \cong \pi_1(Y_{\pi_{\mathcal{P}}}) \cong \pi$

"$\Rightarrow$": Assume $\pi_1(\partial W_{\pi_{\mathcal{P}}}) \cong \pi = \{1\}$



attaching circles of the n trivial 2-handles



$\rightsquigarrow$ can be homotoped (and thus isotoped!) to geometrically cancel the n 1-handles

Could not have done this with the original 2-handles

Cancel the 1-handles $\rightsquigarrow$ what remains is $D^5 \cup \bigcup^{m}$ trivial 2-handles (with zero-framing) $\cong \natural^m S^2 \times D^3$

with $\partial(\natural^m S^2 \times D^3) = \#^m S^2 \times S^2$

□

Thm.: $\exists k > 0$ so that $\#^k S^2 \times S^2$ is not recognizable in Man^4.

Because if it were, we would have an algorithm to decide whether a group given by a presentation $\pi_{\mathcal{P}}$ is trivial, which is impossible.

Open question: Is the 4-sphere recognizable in Man^4 ?

INPUT: integral homology 4-sphere H

OUTPUT: decide in finite time whether $H \cong_{\text{diffeo.}} \mathbb{S}^4$

For $n \geq 5$, the n-sphere is not recognizable.

Idea: [Kervaire] Every finitely presented group π with $\left.\begin{matrix} H^1(\pi) = 0 \\ H^2(\pi) = 0 \end{matrix}\right\}$ "superperfect group"

is the fundamental group of a $\mathbb{Z}H_* \mathbb{S}^n$; $n \geq 5$.

Poincaré conjecture: H integral homology sphere, so

$$H \cong_{\text{homeo.}} \mathbb{S}^n \quad \text{iff.} \quad \pi_1(H) = \{1\}$$

But (consequence of a variation of the Adian-Rabin thm.):

there is no algorith to recognize the trivial group in the class of superperfect groups. $\square$

[Gordon: On the homeomorphism problem for 4-manifolds, arXiv: 2106.06006]

$\#^{12}\, S^2 \times S^2$ cannot be recognized in Man^4.

Previously known: $\#^{14}\, S^2 \times S^2$ cannot be recognized.

$\exists$ finite 12-relator presentation of a group with unsolvable word problem [Borisov]

[Gordon] m-relator presentation of a group with unsolvable word problem

$\rightsquigarrow$ $(m+2)$-relator Adian-Rabin set

recursively enumerable set $\mathcal{S}$ of finite $(m+2)$-relator presentations

s.th. there is no algorithm to decide

whether the group presented by $P \in \mathcal{S}$ is trivial

New idea in this preprint: Clever tricks to reduce the number of summands

in Markov's argument by 2 (from $\#^k\, S^2 \times S^2$ to $\#^{k-2}\, S^2 \times S^2$)

Further reading

•) [Kirby: Markov's theorem on the nonrecognizability of 4-manifolds: an exposition]

appears in "Celebratio mathematica" for Martin Scharlemann

•) [Gompf, Stipsicz: 4-manifolds and Kirby calculus, Exercise on page 149]

•) [Gordon: Some embedding theorems and undecidability questions for groups (1980, 1994)]

•) [Miller: Decision problems for groups – survey and reflections (1992)]