Casson-Whitney unknotting numbers & fundamental groups of knotted surface complements

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2021-07-07, 16:40 – 18:00, 15 min talk
MPIM visit Fachbeirat, Geometry & Topology room

Collaboration with

Jason Joseph, Michael Klug & Hannah Schwartz

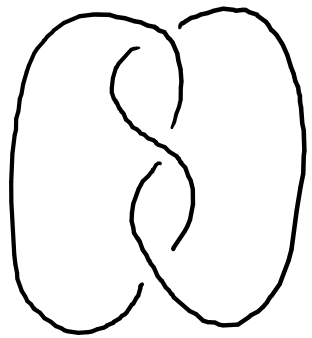
(Rice) (Berkeley & MPIM) (Princeton)

started at MPIM in Spring 2020

[Unknotting numbers of 2-knots in the 4-sphere, arXiv:2007.13244]

Knotted circles in 3-space

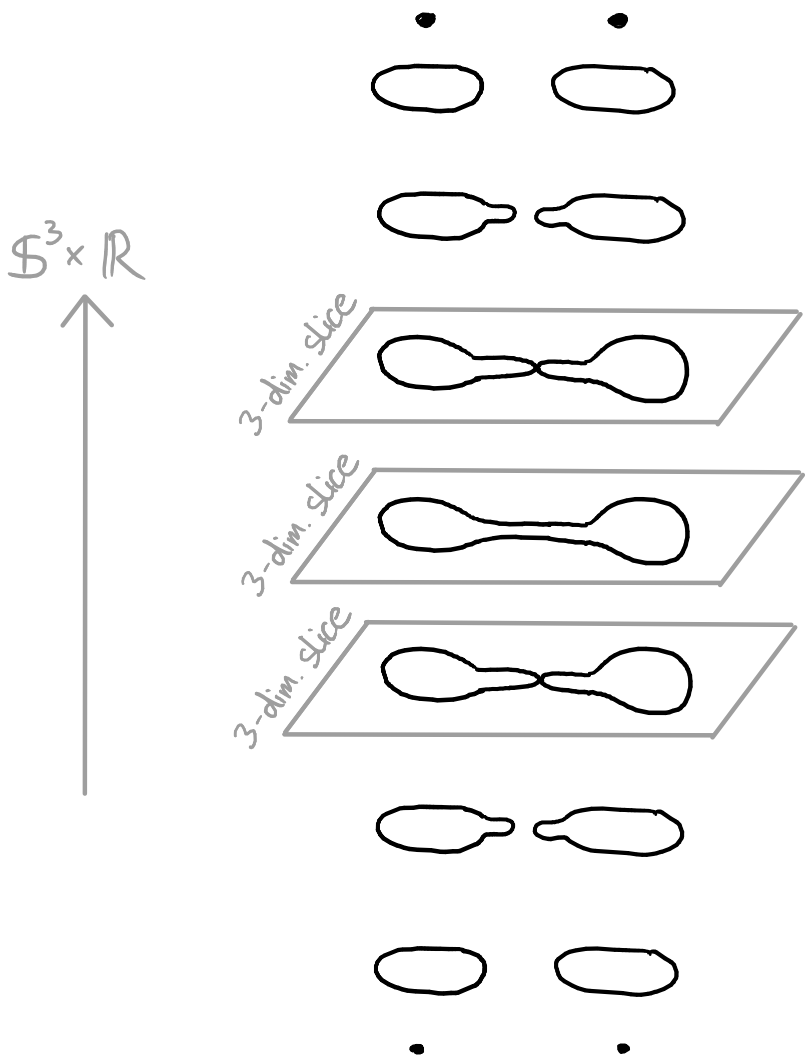
classical **knot** $k : S^1 \hookrightarrow S^3$ smooth embedding



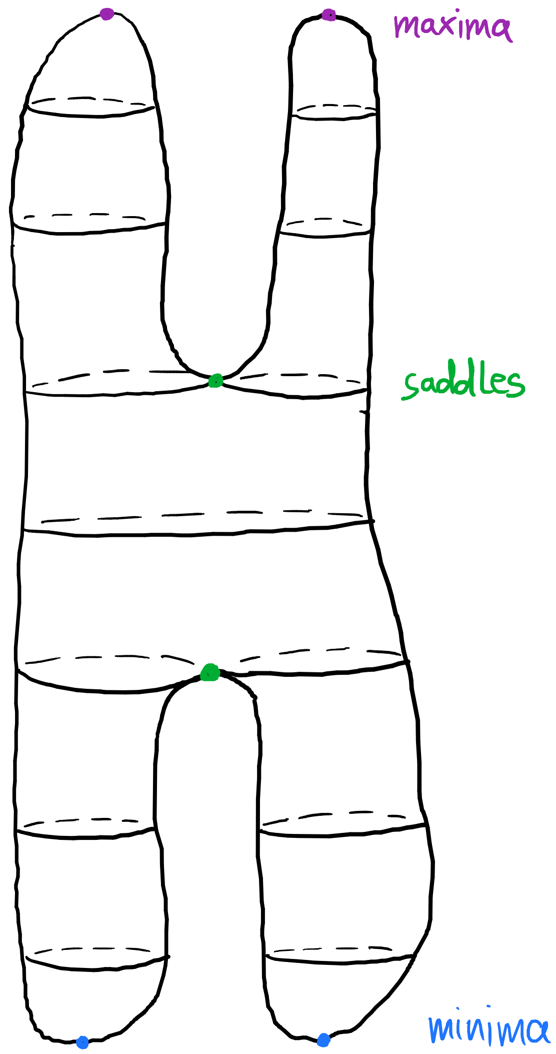
usually considered up to isotopy

Knotted 2-spheres in 4-space: Movies of Links

$S^2 = \ominus \hookrightarrow S^4$ up to smooth isotopy



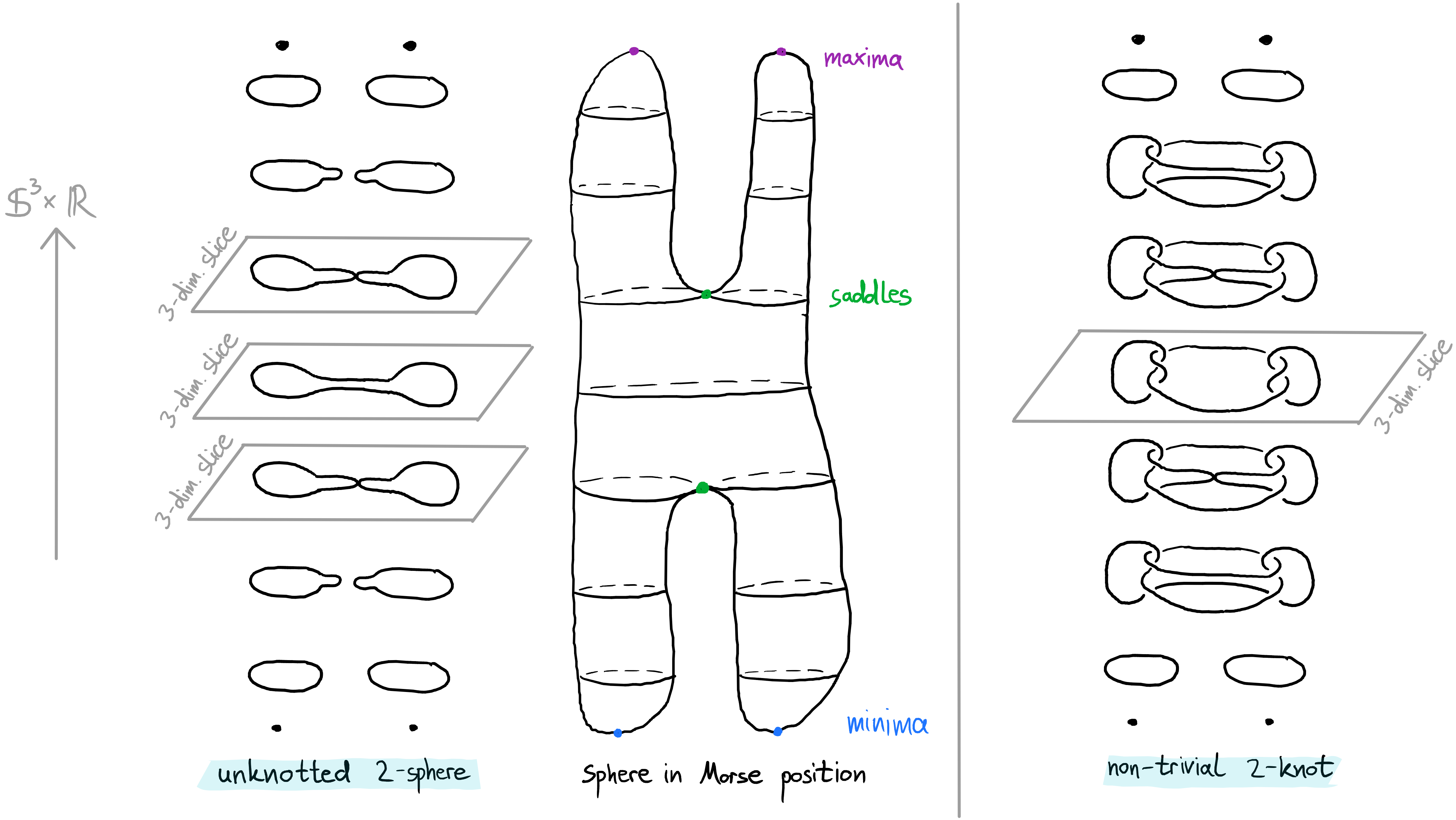
unknotted 2-sphere



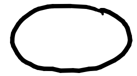
Sphere in Morse position

Knotted 2-spheres in 4-space: Movies of Links

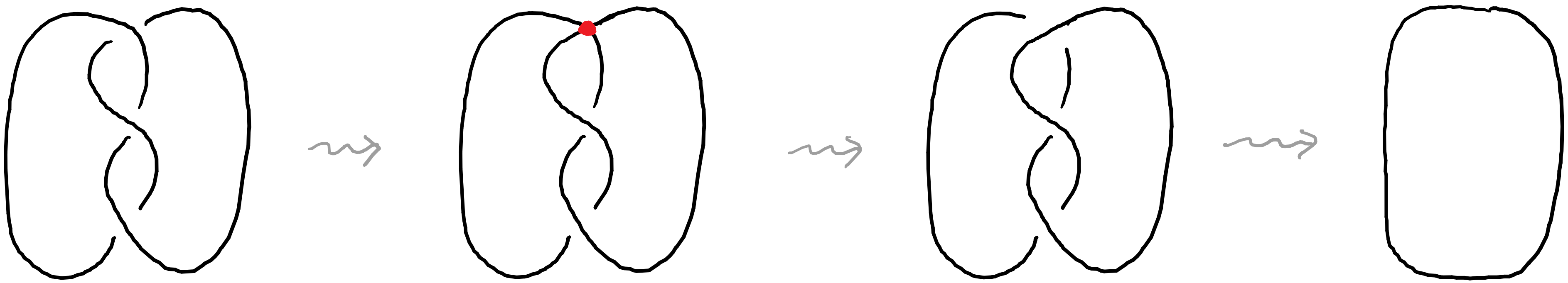
$S^2 = \ominus \hookrightarrow S^4$ up to smooth isotopy



The unknotting number of a classical knot

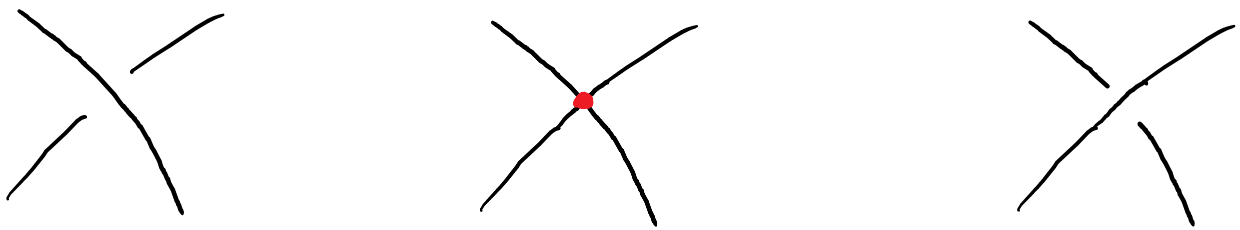
Every classical knot $k: S^1 \hookrightarrow S^3$ is homotopic to the unknot 

(of course if k non-trivial, not isotopic to unknot) $\quad H: S^1 \times [0,1] \longrightarrow S^3$



$t = 0$ $\qquad$ $t = 1$

Generically, the homotopy is a sequence of isotopies and crossing changes:



Question: How could we define an unknotting number of knotted 2-spheres?

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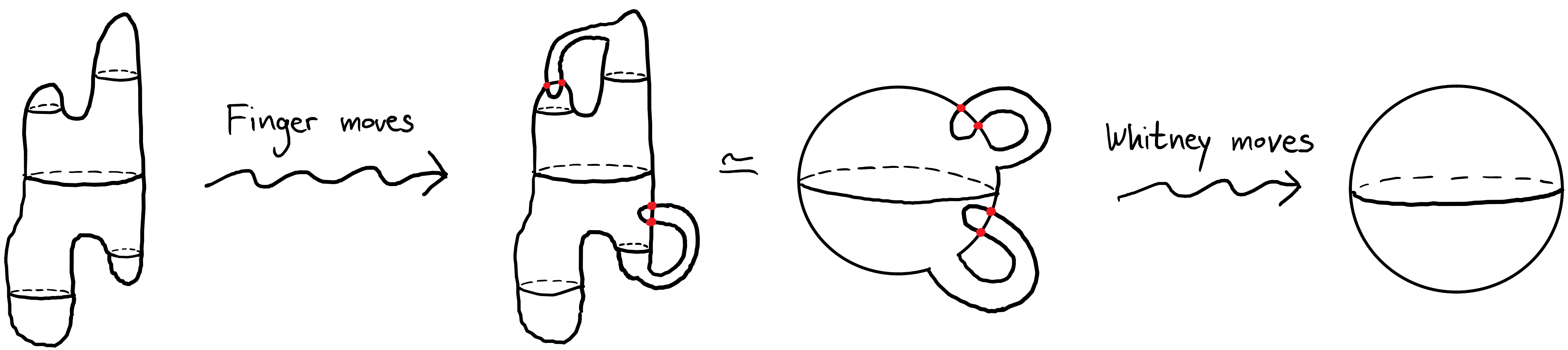
[Smale]: For every smoothly knotted 2-sphere $K : S^2 \hookrightarrow S^4$ there is a regular homotopy starting at the embedding K and ending at the unknot.

$$\text{regular homotopy} \;=\; \text{homotopy } H : S^2 \times [0,1] \longrightarrow S^4 \text{ through immersions}$$

Schematic of a regular homotopy:



knotted 2-sphere K

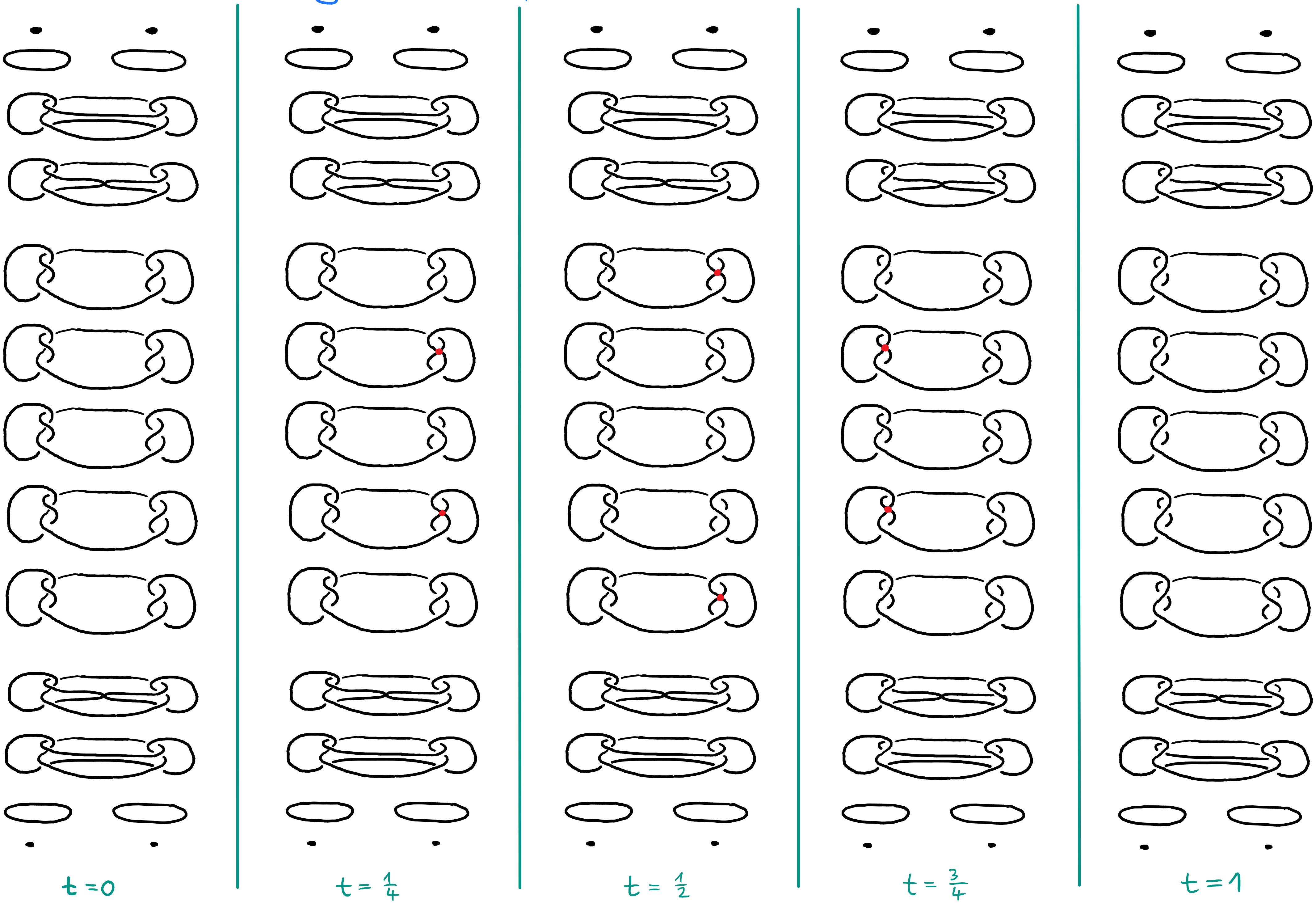
$t = 0$

immersed middle level

unknot

$t = 1$

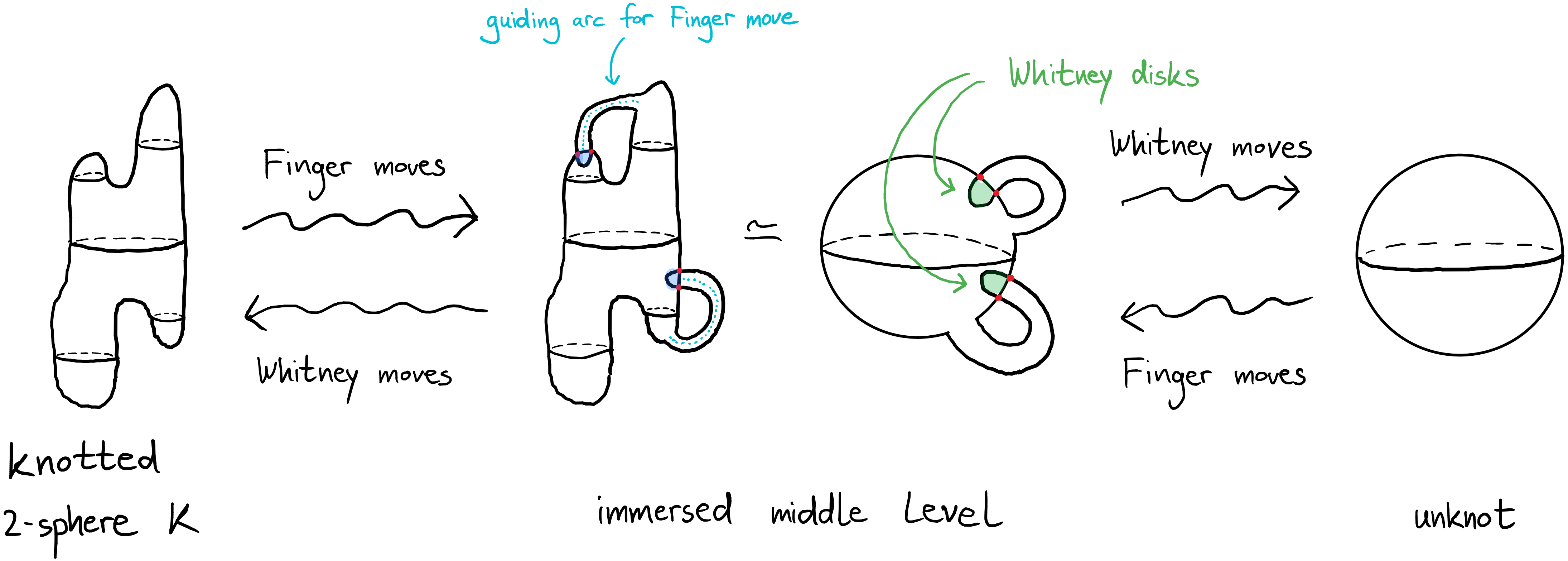
Regular homotopies of 2-knots: Movies of movies



We [Joseph - Klug - R. - Schwartz] define the Casson-Whitney number

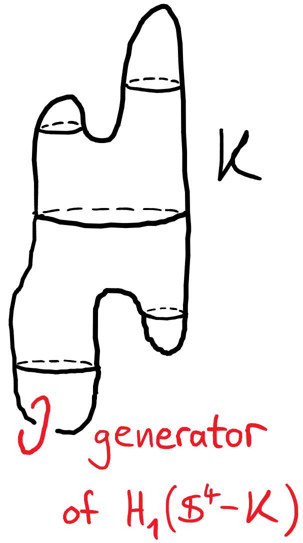
$$u_{CW}(K) \quad \text{of} \quad K : S^2 \hookrightarrow S^4$$

as the minimal number of Finger moves in a regular homotopy $K \rightsquigarrow$ unknot



Lower bounds on u_{cw} from the Alexander Module

$$\pi_1(S^4 - K) \xrightarrow{\text{abelianize}} H_1(S^4 - K) \cong \mathbb{Z} \cong \langle t \rangle$$



$\rightsquigarrow$ Study the homology of the associated infinite cyclic cover

$$\langle t \rangle \circlearrowleft \widetilde{S^4 - K}_{\mathbb{Z}} \longrightarrow S^4 - K$$

$$H_1(\widetilde{S^4 - K}_{\mathbb{Z}}) \cong [\pi_1(S^4-K), \pi_1(S^4-K)] \big/ [[\pi_1(S^4-K), \pi_1(S^4-K)], [\pi_1(S^4-K), \pi_1(S^4-K)]]$$

$\circlearrowleft$

$\mathbb{Z}[t, t^{-1}]$

$$\cong \pi_1(S^4-K)^{(1)} \big/ \pi_1(S^4-K)^{(2)}$$

Algebraic effect of Finger move:

$$\pi_1(S^4 - K') \cong \pi_1(S^4 - K) \big/ \langle\langle [w^{-1} a w, b] \rangle\rangle$$

Immersion after Finger move on K

Slogan: Finger moves can make a pair of meridians commute.

$u_{cw}(K) \geq$ minimal # of Finger move relations

$[w_i^{-1} a_i w_i, a_i]$, a_i = meridian

to make $\pi_1(S^4 - K)$ abelian

$\vee\!|$

minimal # of relations of the form

$w_i^{-1} a_i w_i = a_i$, a_i = meridian

to make $\pi_1(S^4 - K)$ abelian

$\vee\!|$

minimal size of generating set of Alexander module of K

(Nakanishi index)

u_{cw} can be arbitrarily big

Proposition: There are 2-knots K_n with $u_{cw}(K_n) \geq n$.

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Beyond the Alexander module

K = σ-twist spin of $\big(T(2,p) \# T(2,q)\big)$ is a 2-knot

$q = p+2$ or $q = p+4$
or ($q = p+6$ and $\gcd(p, p+6) = 1$)

- with cyclic Alexander module $\pi_1(S^4 - K)^{(1)} / \pi_1(S^4 - K)^{(2)}$
- but nevertheless we can show $u_{cw}(K) = 2$

Thm.: For K_1, K_2 2-knots with determinants $\Delta(K_i)\big|_{-1} \neq 1$

have $u_{cw}(K_1 \# K_2) \geq 2$

determinant = positive generator of the evaluation of the Alexander ideal at $t = -1$

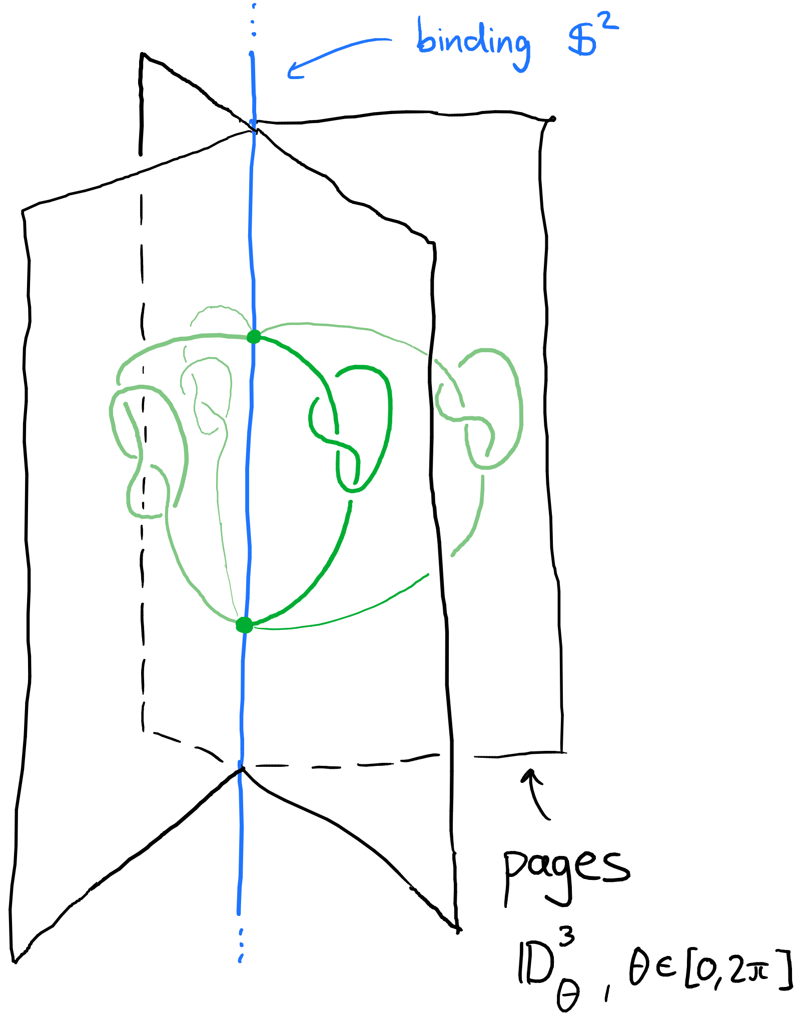
Idea: $\pi_1\big(S^4 - (K_1 \# K_2)\big) \twoheadrightarrow Dih_{2p} *_{\langle \text{meridian} \rangle} Dih_{2 \cdot q}$ ⟵ use a "Freiheitssatz" to show that the image of a single Finger move relation cannot abelianize this group.

Teaser: Further results

Prop.: $u_{CW}\left(\begin{matrix}\text{(twist) spin of}\\ k: S^1 \hookrightarrow S^3\end{matrix}\right) \leq u(k)$

classical unknotting number of the 1-knot $k: S^1 \hookrightarrow S^3$

open book decomposition of S^4

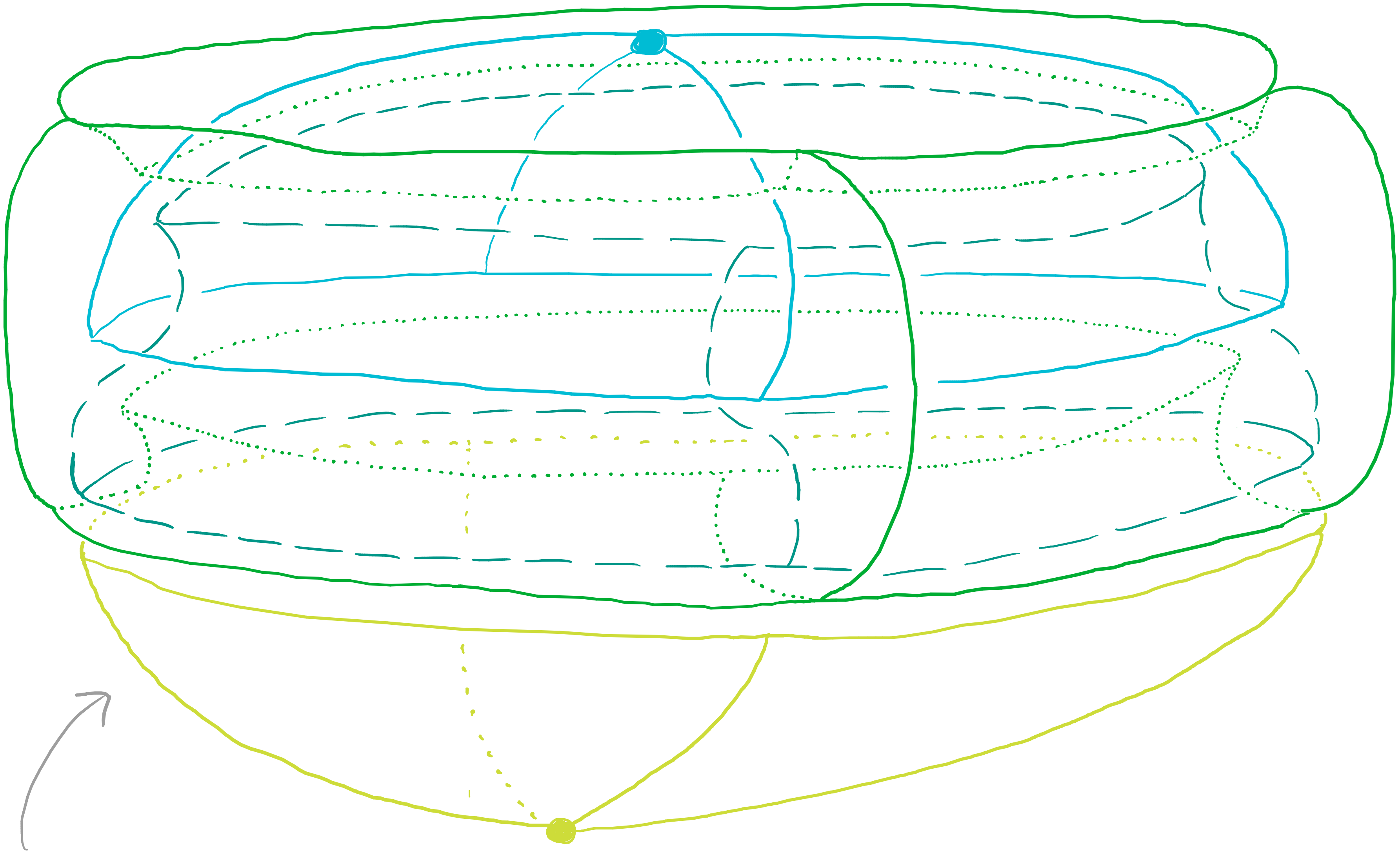


Corollary: The (algebraic) Casson-Whitney number of every (twist)-spin of k is a lower bound for the classical unknotting number of the original knot k.

Corollary: [Special case of Scharlemann (1985)]

k_1, k_2 classical knots with nontrivial determinant, then the classical unknotting number is $u(k_1 \# k_2) \geq 2$.

Thanks!



broken surface diagram of a spun trefoil knot