Casson–Whitney unknotting numbers

Unknotting 2-spheres in S^4 with Finger & Whitney moves

with Jason Joseph, Michael Klug & Hannah Schwartz

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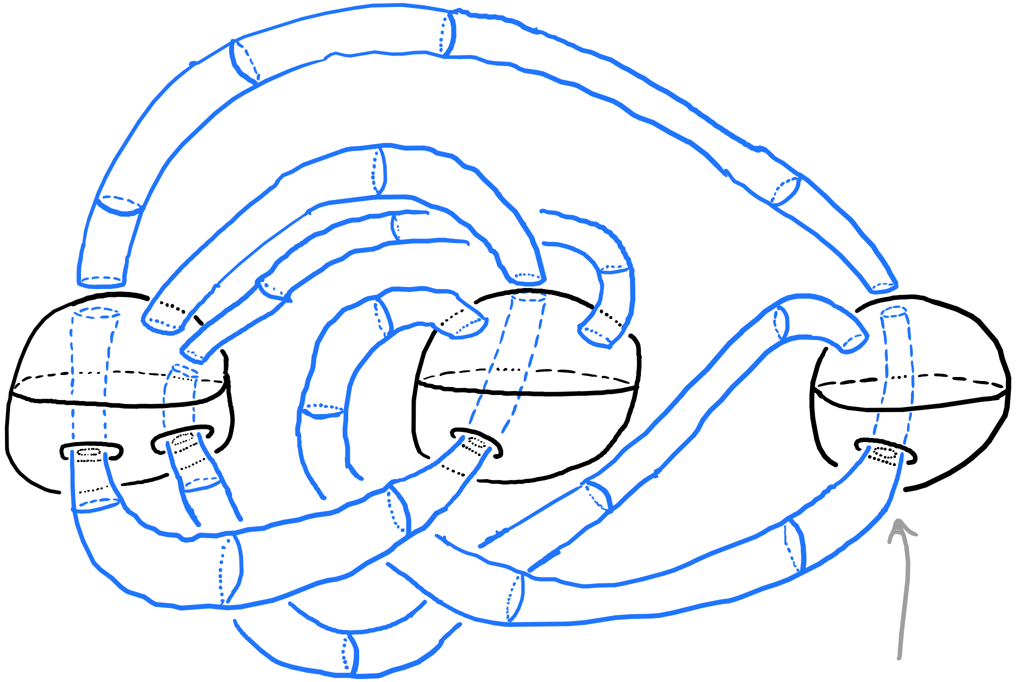
NCNGT 2021 Lightning talk (5 min.)

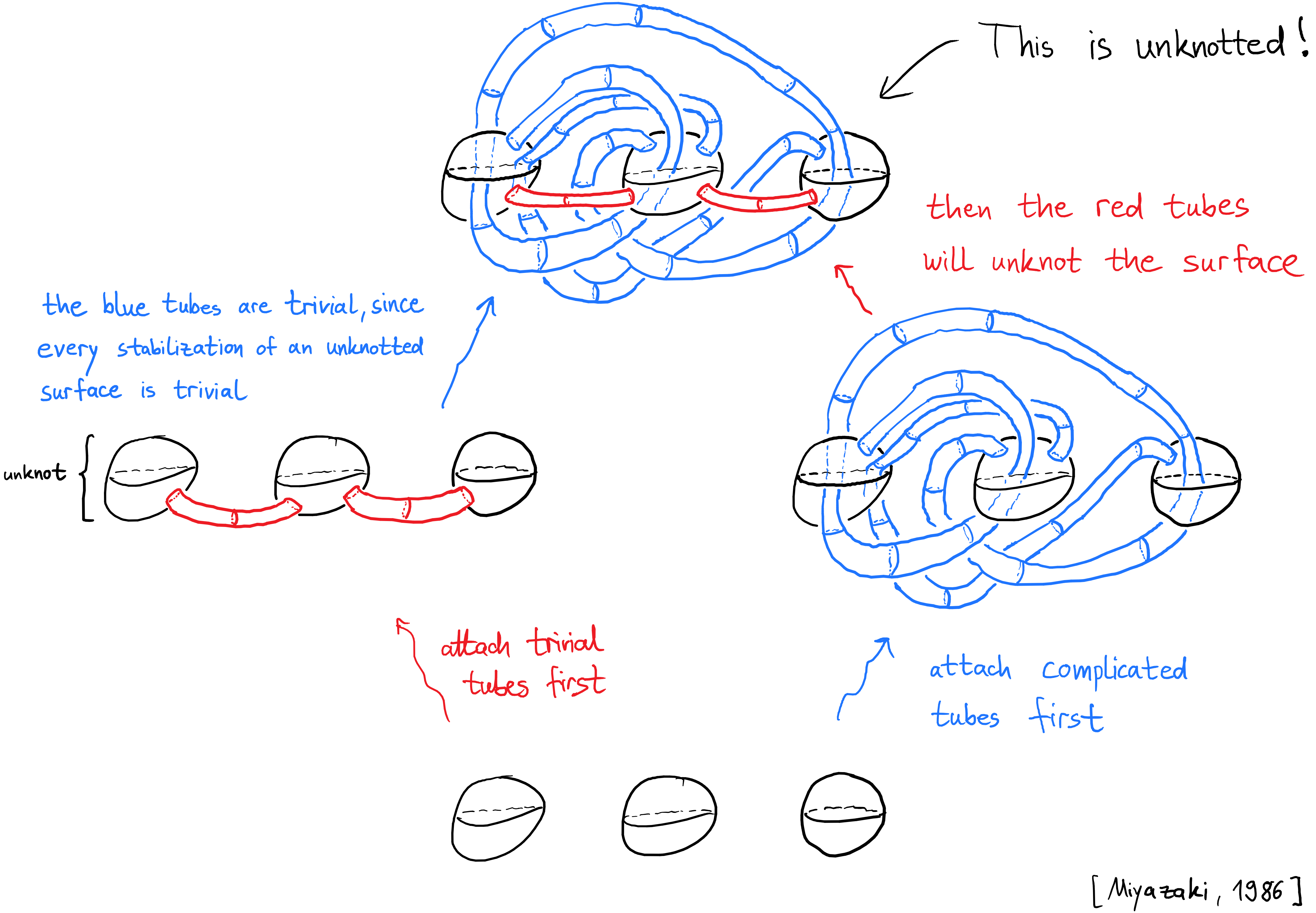
Ribbon 2-knots $S^2 \hookrightarrow S^4$

Start with an unlink of 2-spheres in S^4

Attach fusion tubes



blue tubes can <u>link</u> with the black spheres


This is unknotted!
then the red tubes
will unknot the surface
the blue tubes are trivial, since
every stabilization of an unknotted
surface is trivial
unknot
attach trivial
tubes first
attach complicated
tubes first
[Miyazaki, 1986]

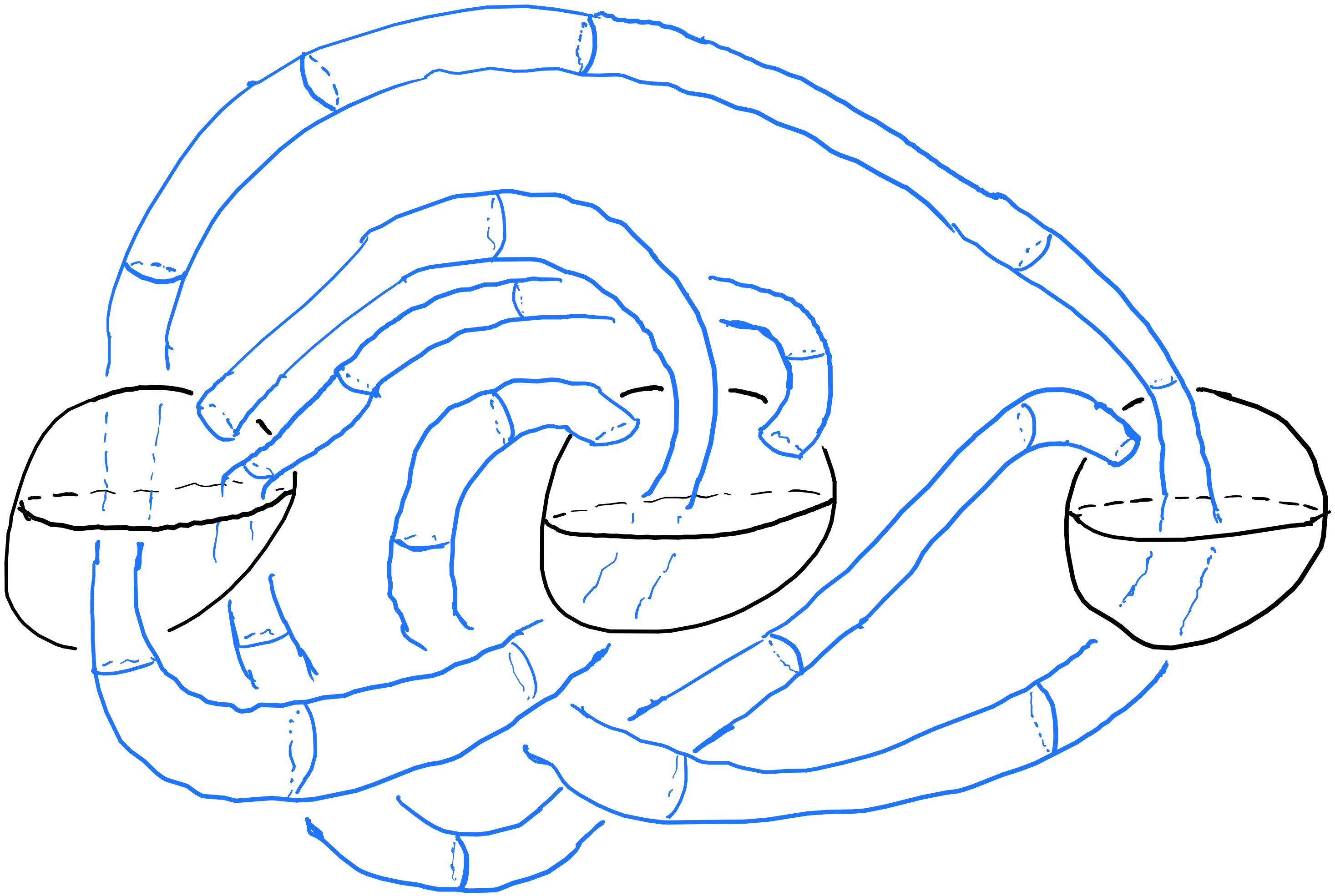
Thm. [Miyazaki, 1985]: For a ribbon 2-knot $K: S^2 \hookrightarrow S^4$

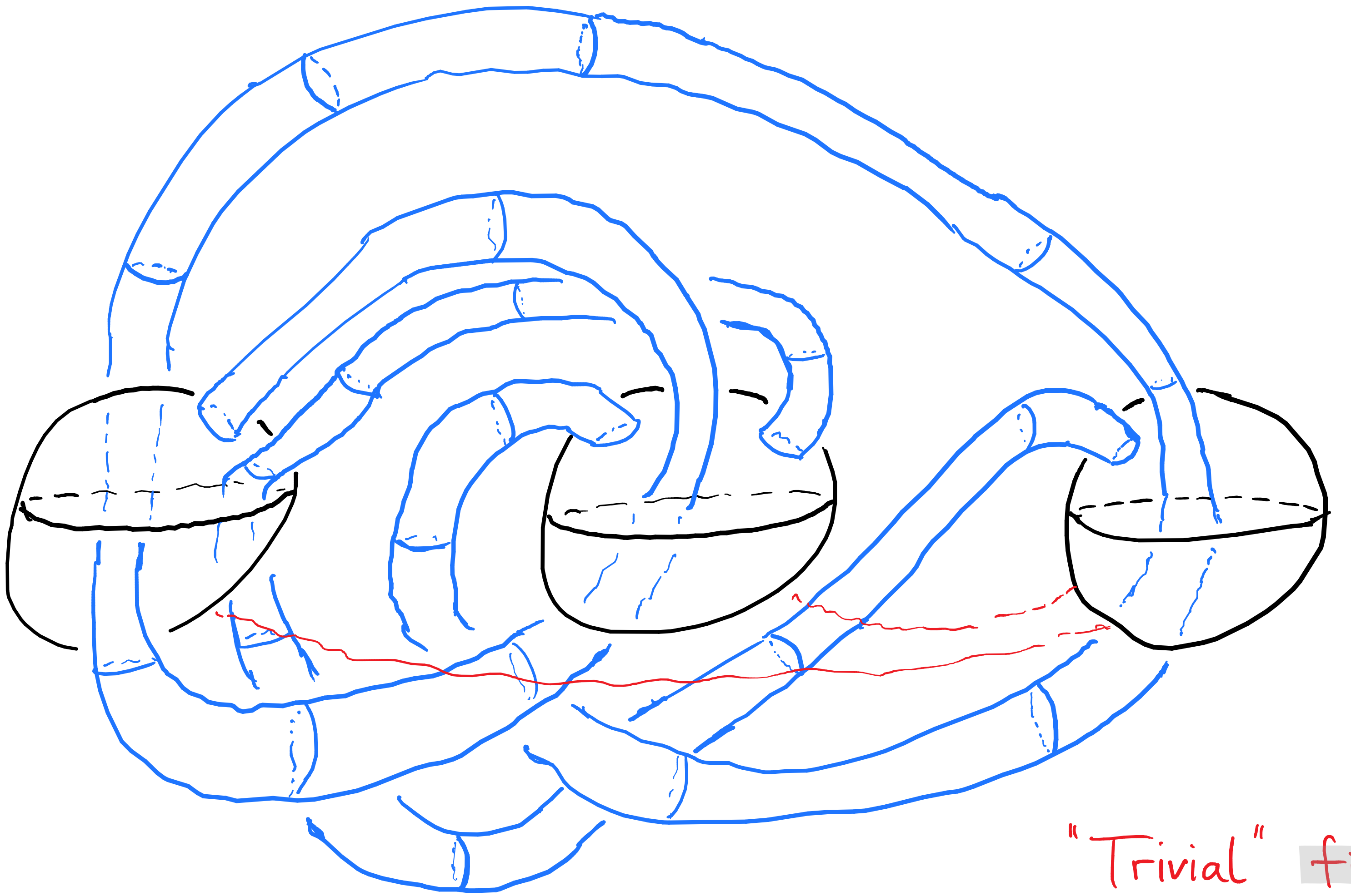
$$u_{1\text{-handle}}(K) \leq \text{fusion-}\#(K)$$

minimal # of fusion tubes in a ribbon presentation of K

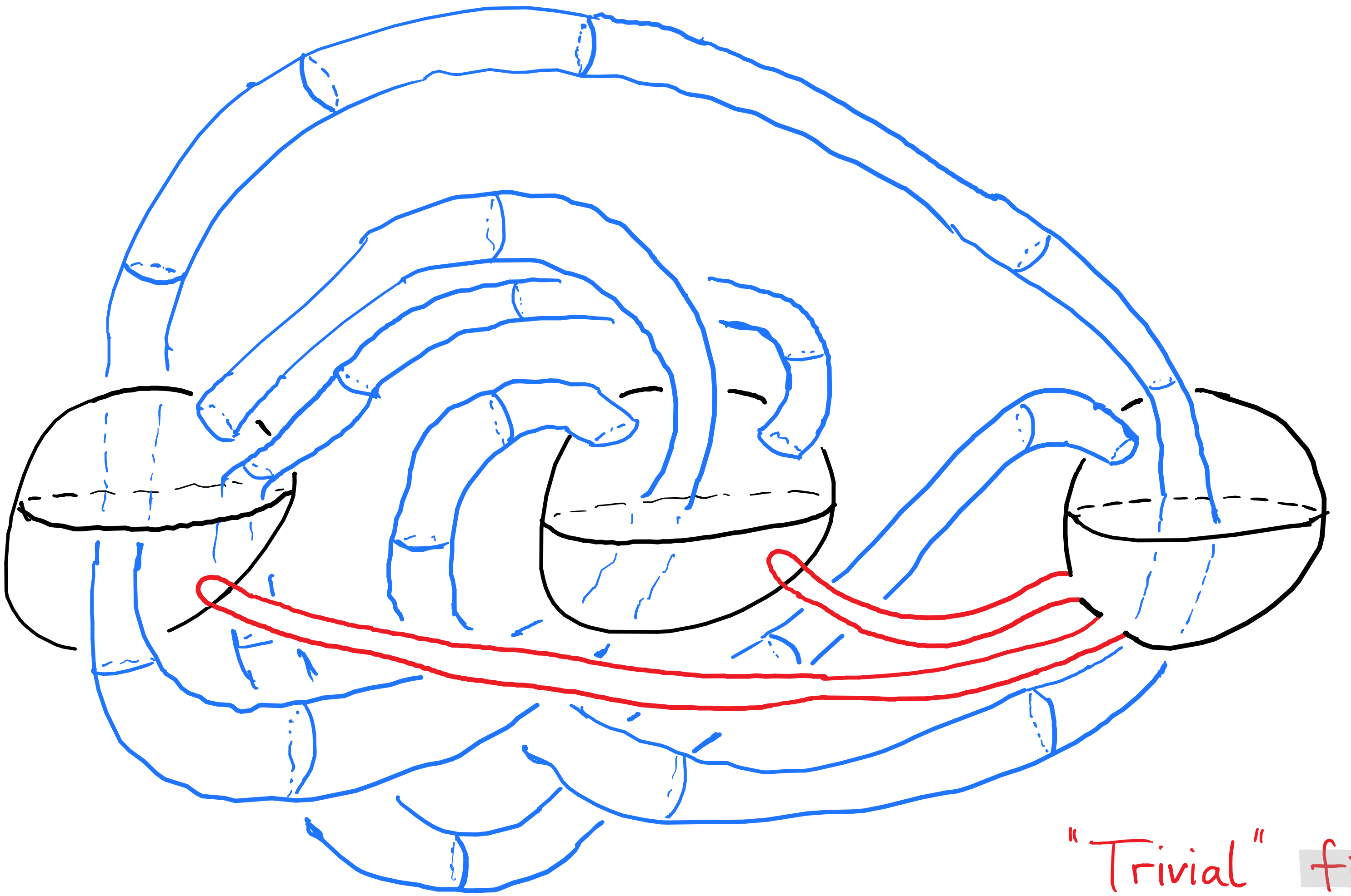
Here, we would like to present a similar lower bound for another 2-knot "unknotting number" defined in terms of the "length" of a regular homotopy to the unknot.

The regular homotopy for ribbon 2-knots:





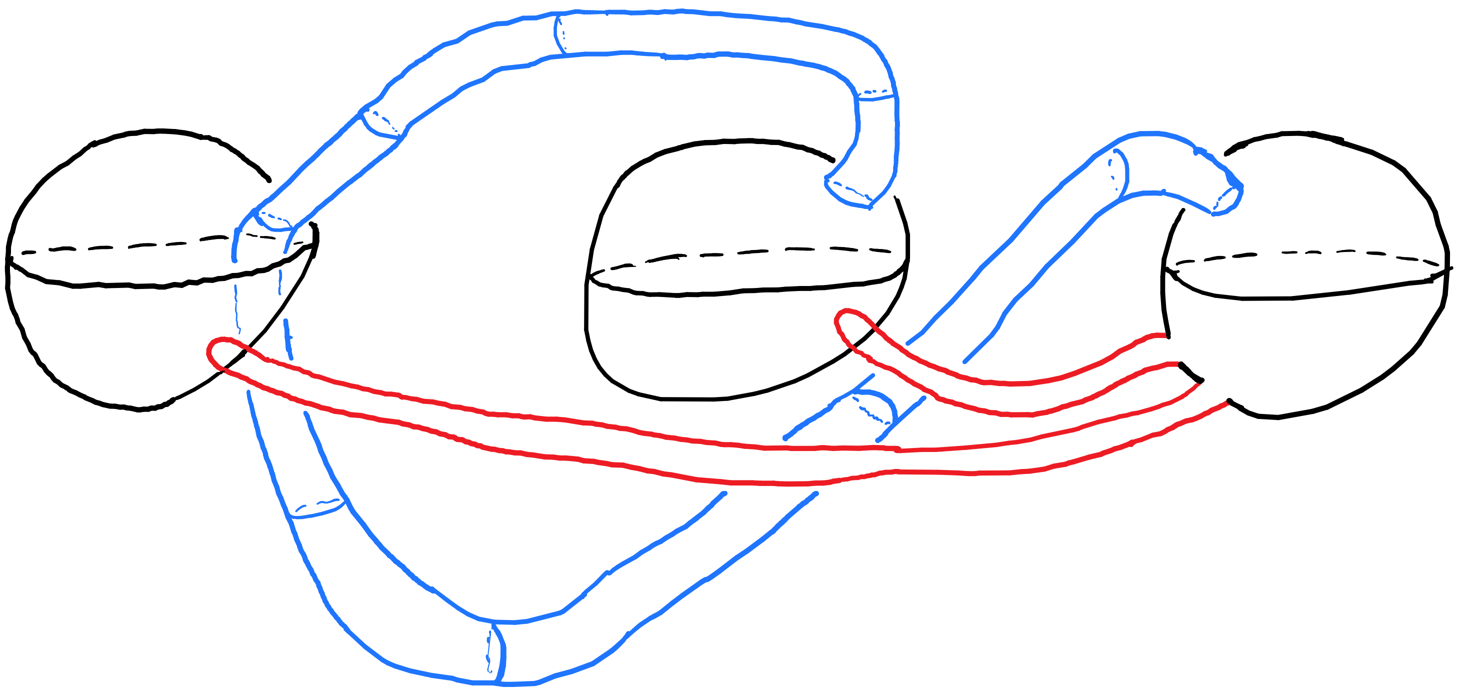
"Trivial" finger moves from the last minimum to all the others.

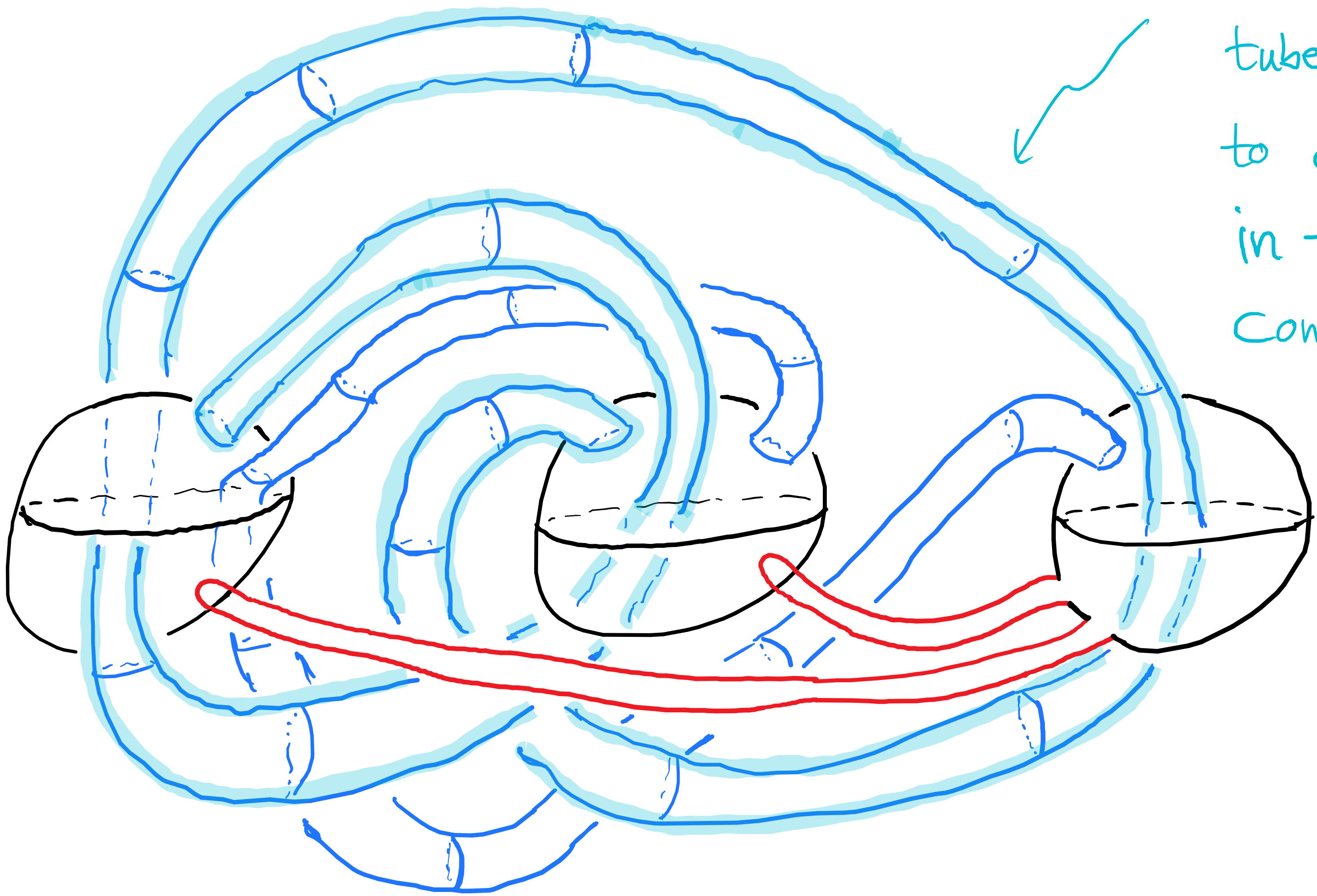


"Trivial" finger moves
from the last minimum
to all the others.

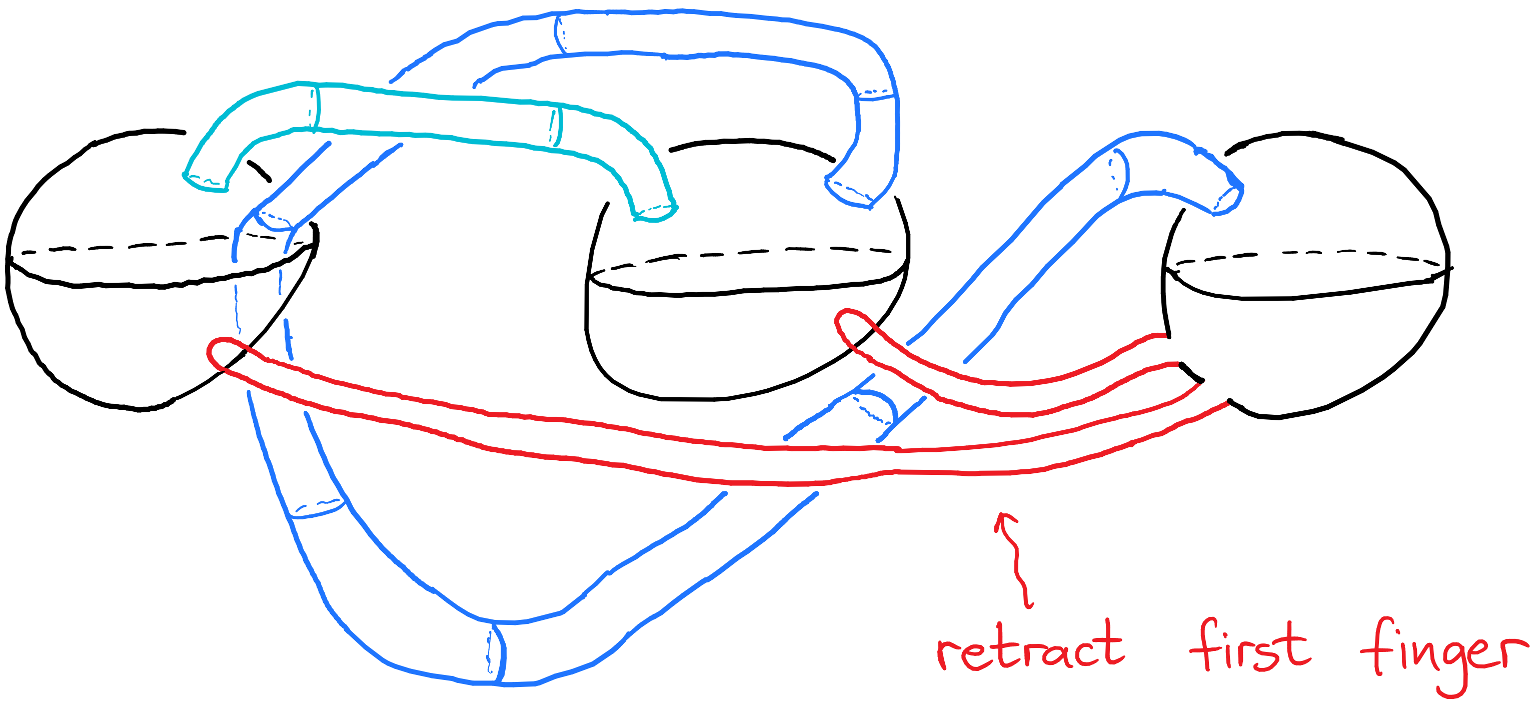
Forget the first fusion tube

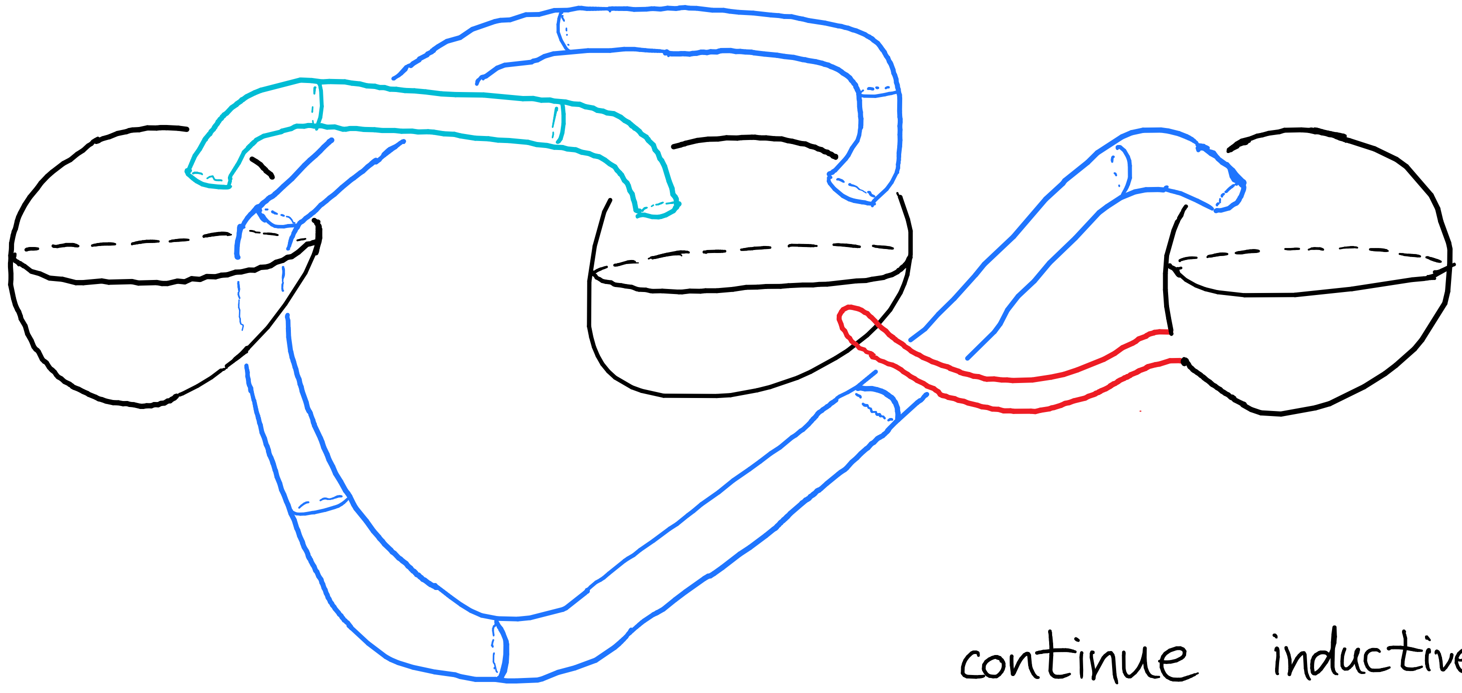
$\rightsquigarrow$ The group $\pi_1(S^4 - L, *)$ of this 2-component immersed Link is $\mathbb{Z} \oplus \mathbb{Z}$





So this complicated tube is isotopic to a trivial fusion in the immersion complement


retract first finger



continue inductively ...

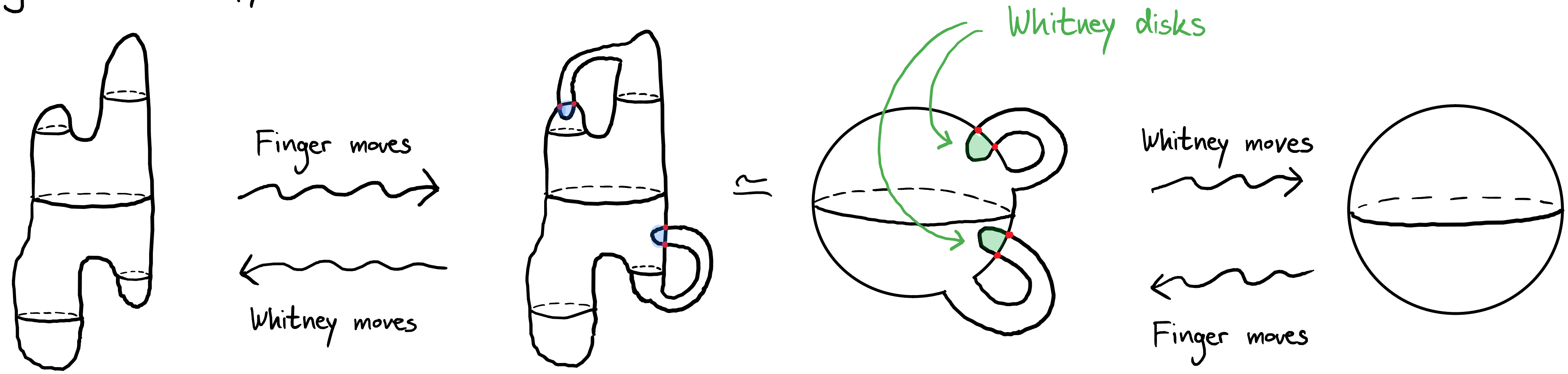
We [Joseph - Klug - R. - Schwartz] define the <u>Casson–Whitney number</u>

$$u_{CW}(K) \quad \text{of} \quad K : S^2 \hookrightarrow S^4$$

as the minimal number of Finger moves in a regular homotopy $K \rightsquigarrow$ unknot

[Unknotting numbers of 2-knots in the 4-sphere, arXiv:2007.13244]

Schematic of a regular homotopy:



knotted 2-sphere K

immersed middle level

unknot

<u>Thm.:</u> For a ribbon 2-knot $S^2 \overset{K}{\hookrightarrow} S^4$: $u_{CW}(K) \leq \text{fusion-}\#(K)$