80 min talk in the Topological manifolds seminar, Bonn

2021-06-23, 10:00

Explicit immersion of $\mathbb{T}^n - \mathbb{D}^n$ into $\mathbb{R}^n$

[Kirby-Siebenmann, Essay I, Appendix B: Submerging a punctured torus (by J. Milnor, 1969)]

[Barden] as presented in [Rushing: Topological Embeddings (1973)]

[Ferry]

<u>Plan for Today</u>: We construct immersions $\mathbb{T}^n - \mathbb{D}^n \looparrowright \mathbb{R}^n$

but won't actually see applications

<u>Daniel</u> & <u>Weizhe</u>:

[Hatcher: The Kirby Torus Trick for Surfaces (2013)]

Existence and uniqueness of smooth structures on topological surfaces

(so in particular PL structures)

"every homeomorphism between smooth surfaces is isotopic to a diffeomorphism"

[Hamilton: The triangulation of 3-manifolds (1976)]

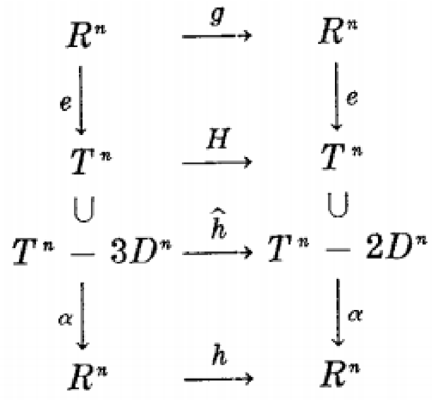
Existence and uniqueness of PL structures on 3-manifolds

originally proved by [Moise, 1952]

Where did we use immersions $\mathbb{T}^n - \mathbb{D}^n \looparrowright \mathbb{R}^n$ last semester?

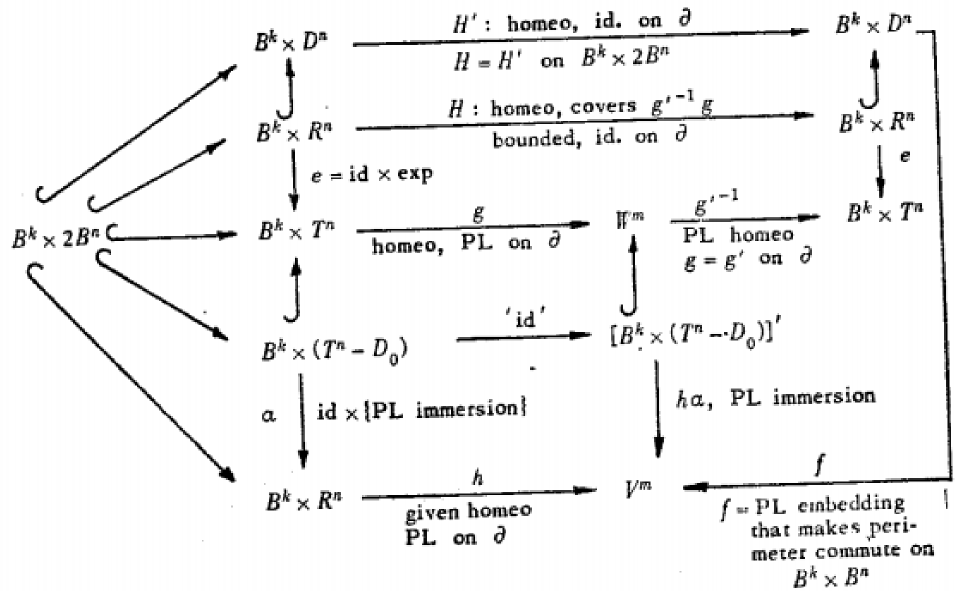
The original wrapping up/unwrapping diagrams

- From Kirby's 1969 Annals paper



- From the Kirby-Siebenmann 1969 AMS Bulletin paper

MAIN DIAGRAM



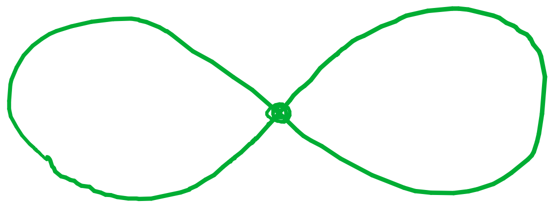
smooth immersion (= locally a smooth embedding)

smooth function whose derivative is everywhere injective

$f: M \longrightarrow N$ $D_p f: T_p M \hookrightarrow T_{f(p)} N$

M, N: smooth manifolds

f itself does not need to be injective



topological immersion:

continuous map which is locally an embedding

$f: M \longrightarrow N$

M, N: topological manifolds

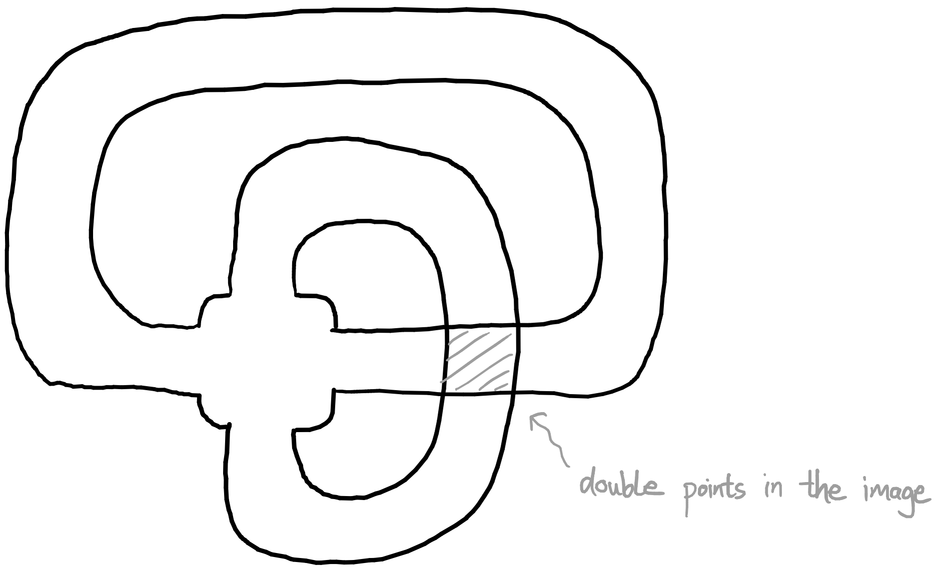
for all $x \in M$ there is a neighborhood $x \in U \subset M$

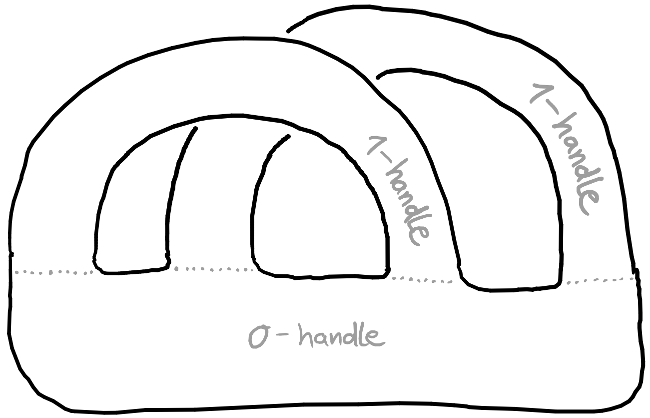
s.th. $f|_U : U \hookrightarrow N$ is an embedding

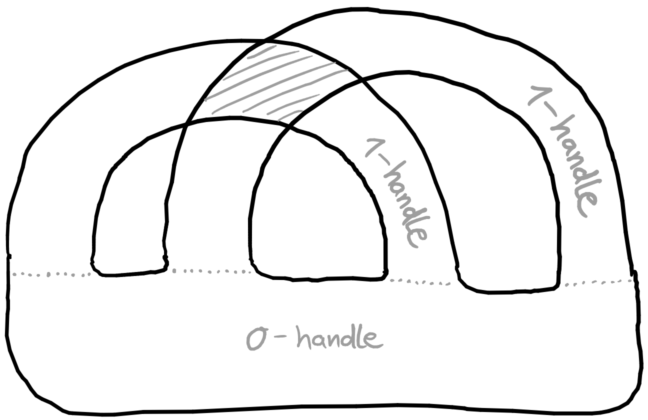
$\boxed{\text{\underline{immersion}: continuous map which is locally an embedding}}$

$$\mathbb{T}^2 = \underbrace{\text{0-h.} \cup \bigcup^2 \text{1-h.}}_{\mathbb{T}^2 - \mathring{\mathbb{D}}^2} \cup \text{2-h.}$$

Immersion $\mathbb{T}^2 - \mathring{\mathbb{D}}^2 \looparrowright \mathbb{R}^2$







[Smale], [Hirsch] studied spaces of immersions

(they described the homotopy type of those spaces in terms of mapping spaces of vector bundles)

Thm.: Every smooth parallelizable n-manifold M^n admits a smooth immersion into $\mathbb{R}^{n+1}$.

trivial tangent bundle
$TM \cong M \times \mathbb{R}^n$

Thm.: [Hirsch]

Every smooth open parallelizable n-manifold admits a smooth immersion into $\mathbb{R}^n$.

each component is non-compact
and with empty boundary

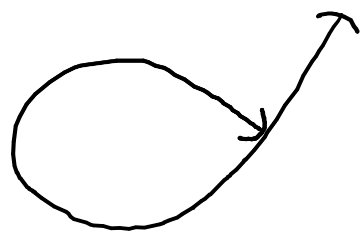
(such manifolds can be deformed into a neighborhood of their $(n-1)$-skeleton)

Fun facts:

- $M \xhookrightarrow{f} N$ injective immersion $\Rightarrow$ f is an embedding

  compact

- not true if M is not compact



- **codimension 0:** immersion is the same as a submersion
  (= differential is surjective)

  Example: A codimension 0 immersion of a closed manifold is a covering map.

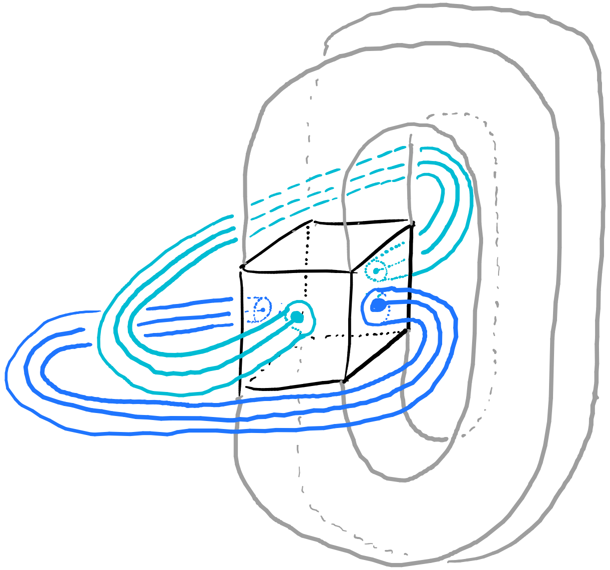
- No immersion $\mathbb{T}^2 \to \mathbb{S}^2$ or $\mathbb{T}^3 \to \mathbb{S}^3$ or ...

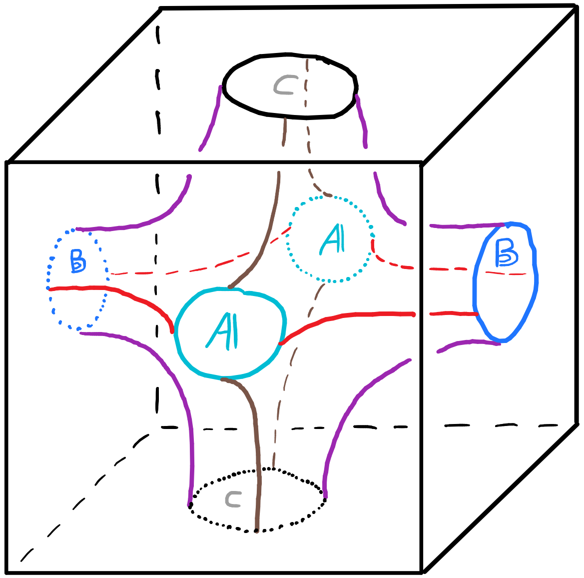
Trying to visualize the 3-dimensional case

$$\mathbb{T}^3 - \mathring{\mathbb{D}}^3 \hookrightarrow \mathbb{R}^3$$

Can embedd 1-skeleton into $\mathbb{R}^3$

$\rightsquigarrow$ Now have to add the three 2-handles

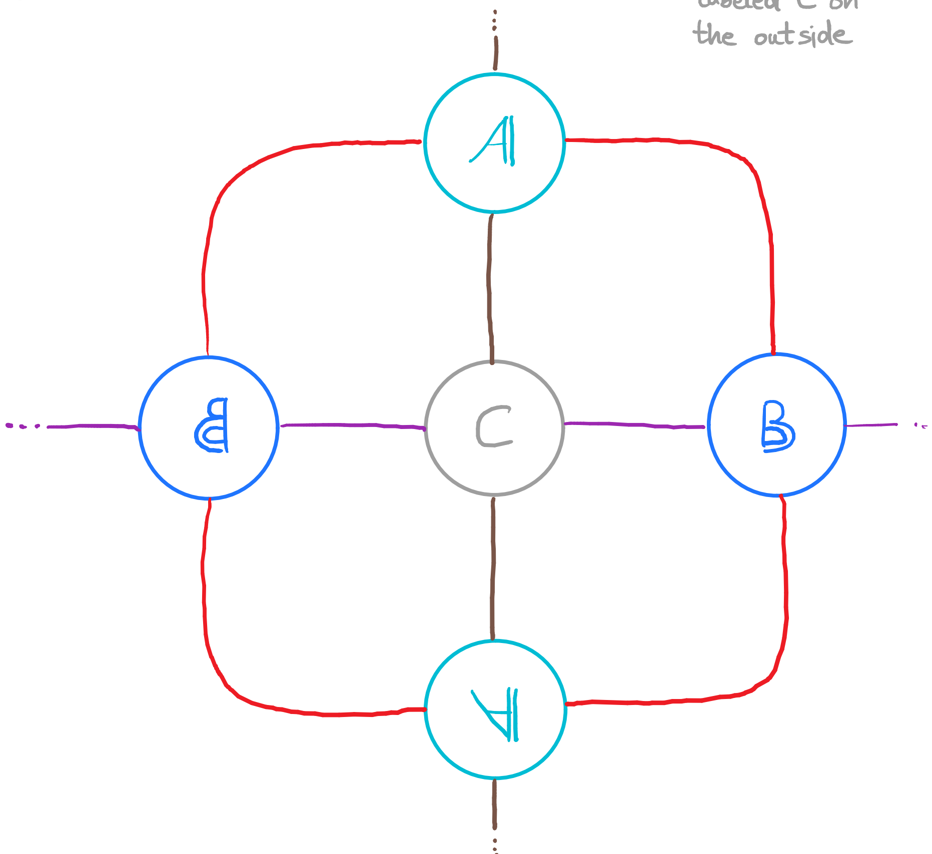




$$\underbrace{\mathbb{T}^3 = 0\text{-h.} \cup \bigcup^3 1\text{-h.} \cup \bigcup^3 2\text{-h.}}_{\mathbb{T}^3 - \mathring{\mathbb{D}}^3} \cup 3\text{-h.}$$

Heegaard diagram of $\mathbb{T}^3$:

another circle labeled C on the outside



1 Answer

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The general case is very much like the 2-dimensional case, it just takes time to process the picture, to see how you could do the same constructions in the higher-dimensional case.

A punctured $S^1 \times S^1$ looks like a wedge of two circles, but fattened up a little bit. Precisely, around each circle you have an annulus neighbourhood. To immerse the punctured torus into $\mathbb{R}^2$ what you do is you embed the first annulus, and then embed the 2nd annulus so that it overlaps the first in the same way that the two annuli live in the torus itself. In doing this you create an extra overlap, but that's fine as we're only looking for an immersion.

A punctured $S^1 \times S^1 \times S^1$ has the same kind of decomposition. It looks like the union of three "annuli" , precisely, $[0,1] \times S^1 \times S^1$, $S^1 \times [0,1] \times S^1$ and $S^1 \times S^1 \times [0,1]$, where here $[0,1]$ is shorthand for a small interval in S^1. Each of these spaces you can embed in $\mathbb{R}^3$ as tubular neigbhourhoods of embedded tori. You just have to make the embeddings overlap in the same way they overlap in $S^1 \times S^1 \times S^1$ -- and that is to make $[0,1] \times S^1 \times S^1 \cap S^1 \times [0,1] \times S^1 = [0,1]^2 \times S^1$, i.e they intersect along one of the coordinate circles. So the idea is to draw a picture of an embedded torus, then along each of the two coordinate (longitude/latitude) axis, draw the boundary of a tubular neighbourhood of that axis. Suitably interpreted, you can think of this picture as the image of the coordinate tori under your immersion.

The general picture goes like that.

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answered Mar 21 '13 at 20:55
Ryan Budney
21.7k ● 3 ■ 63 ▲ 103

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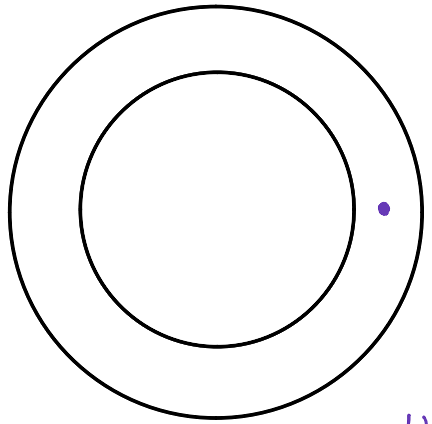
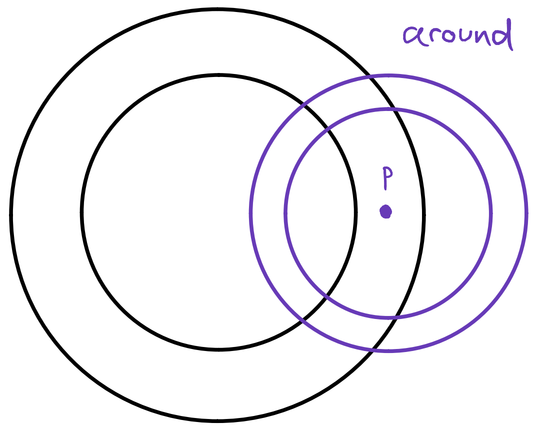
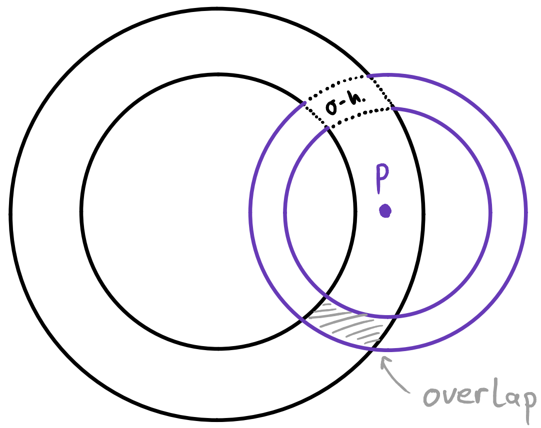
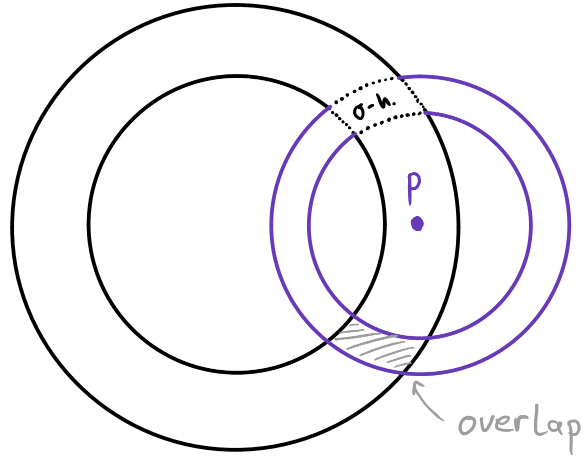


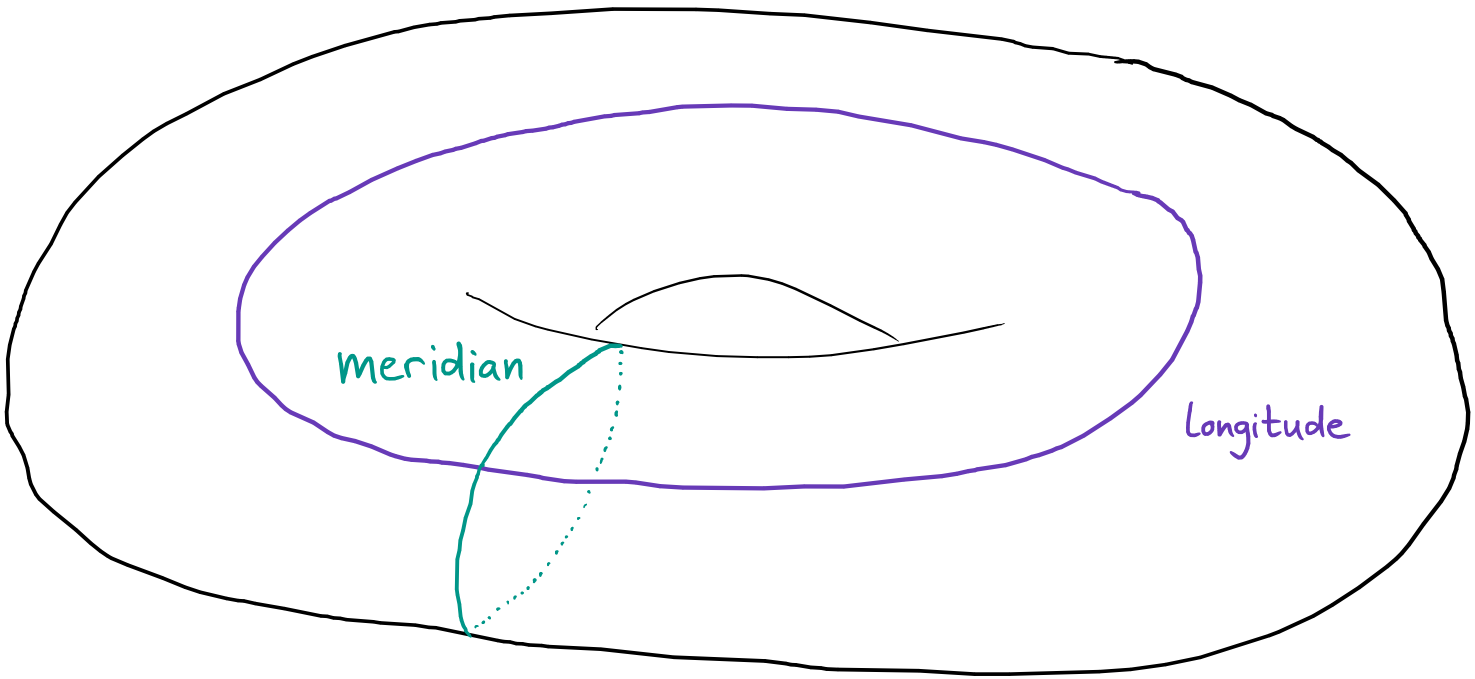


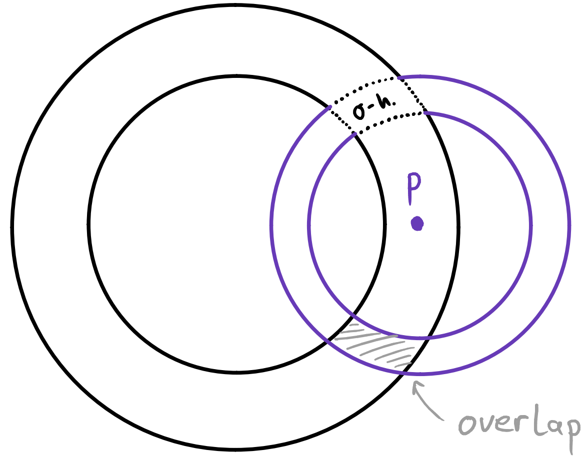
$$T^2 - \mathring{D}^2 \hookrightarrow \mathbb{R}^2 \rightsquigarrow$$



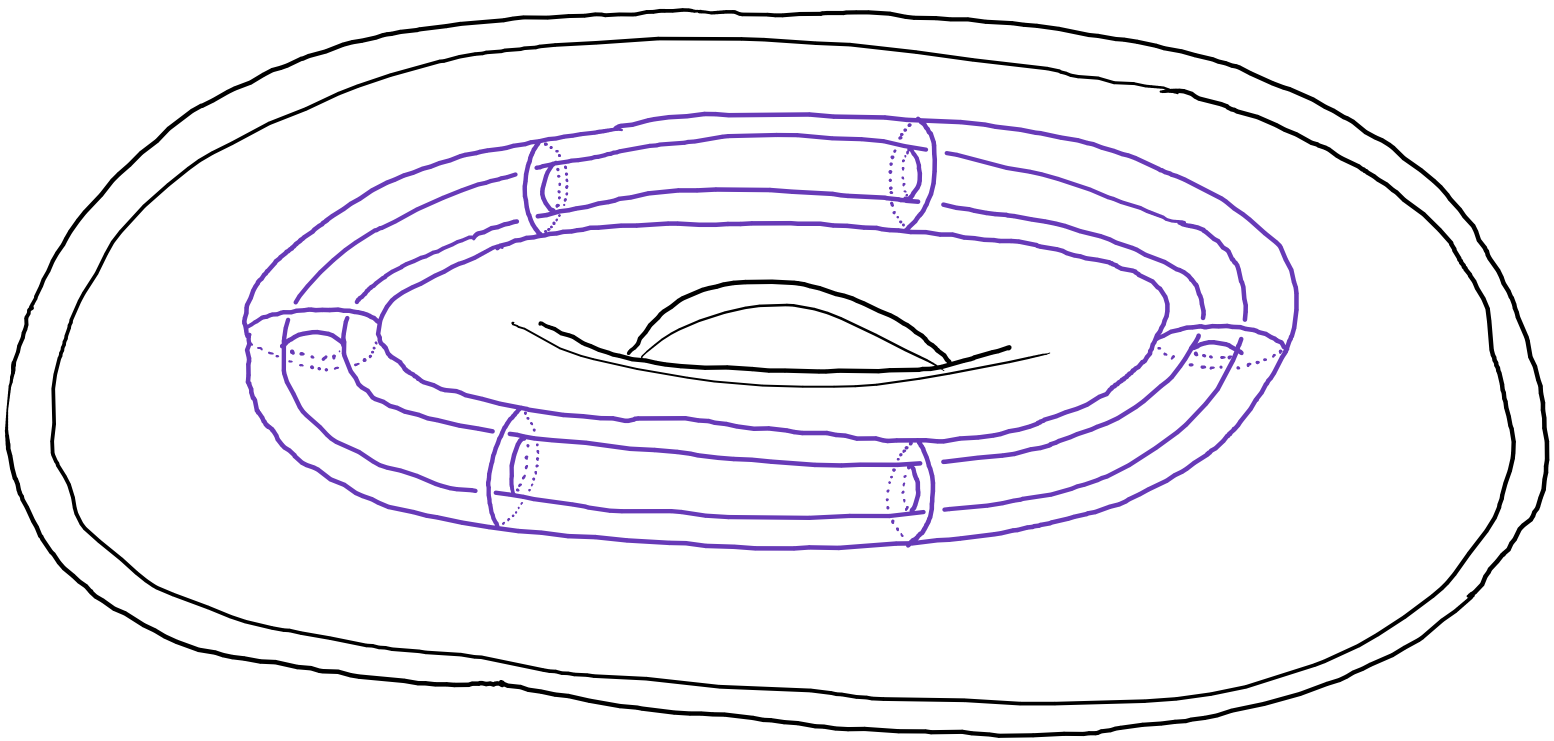


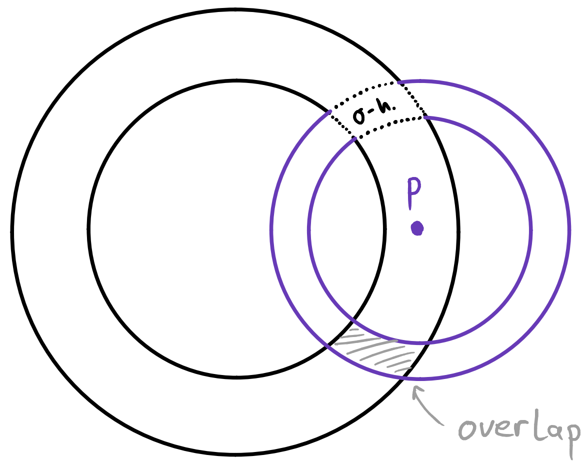
$$T^2 \hookrightarrow \mathbb{R}^3$$





$$\mathbb{T}^2 \times [-\varepsilon, \varepsilon] \hookrightarrow \mathbb{R}^3$$




0-h.
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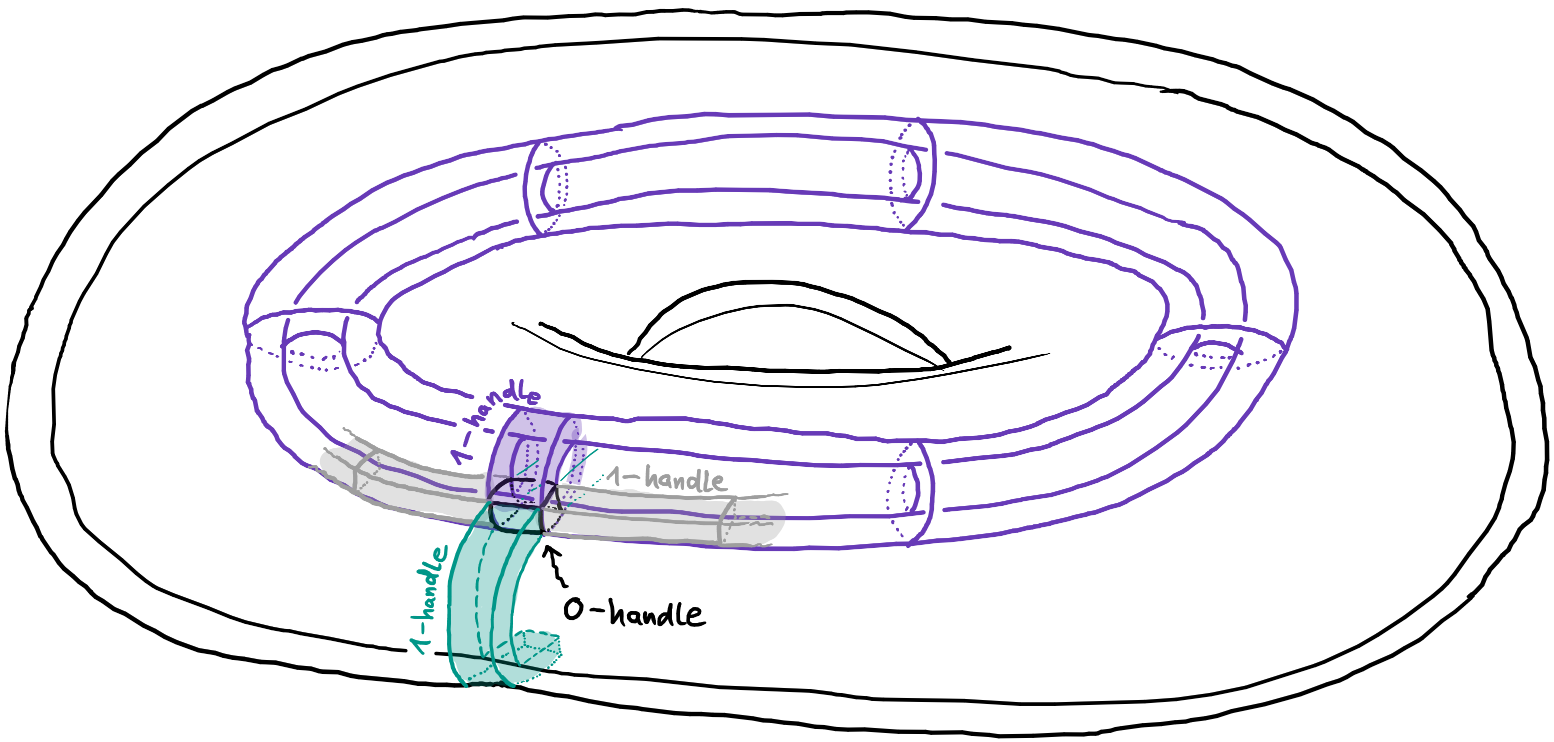
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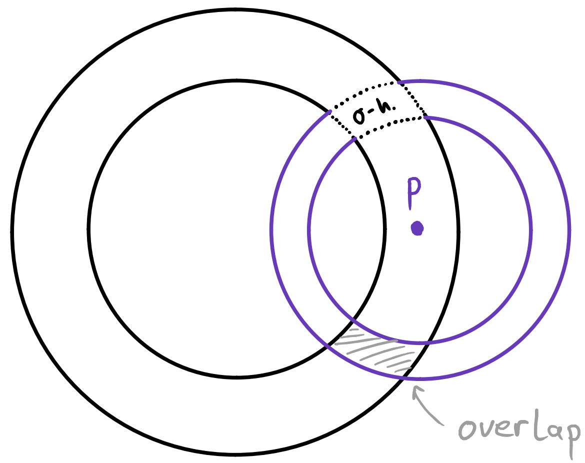
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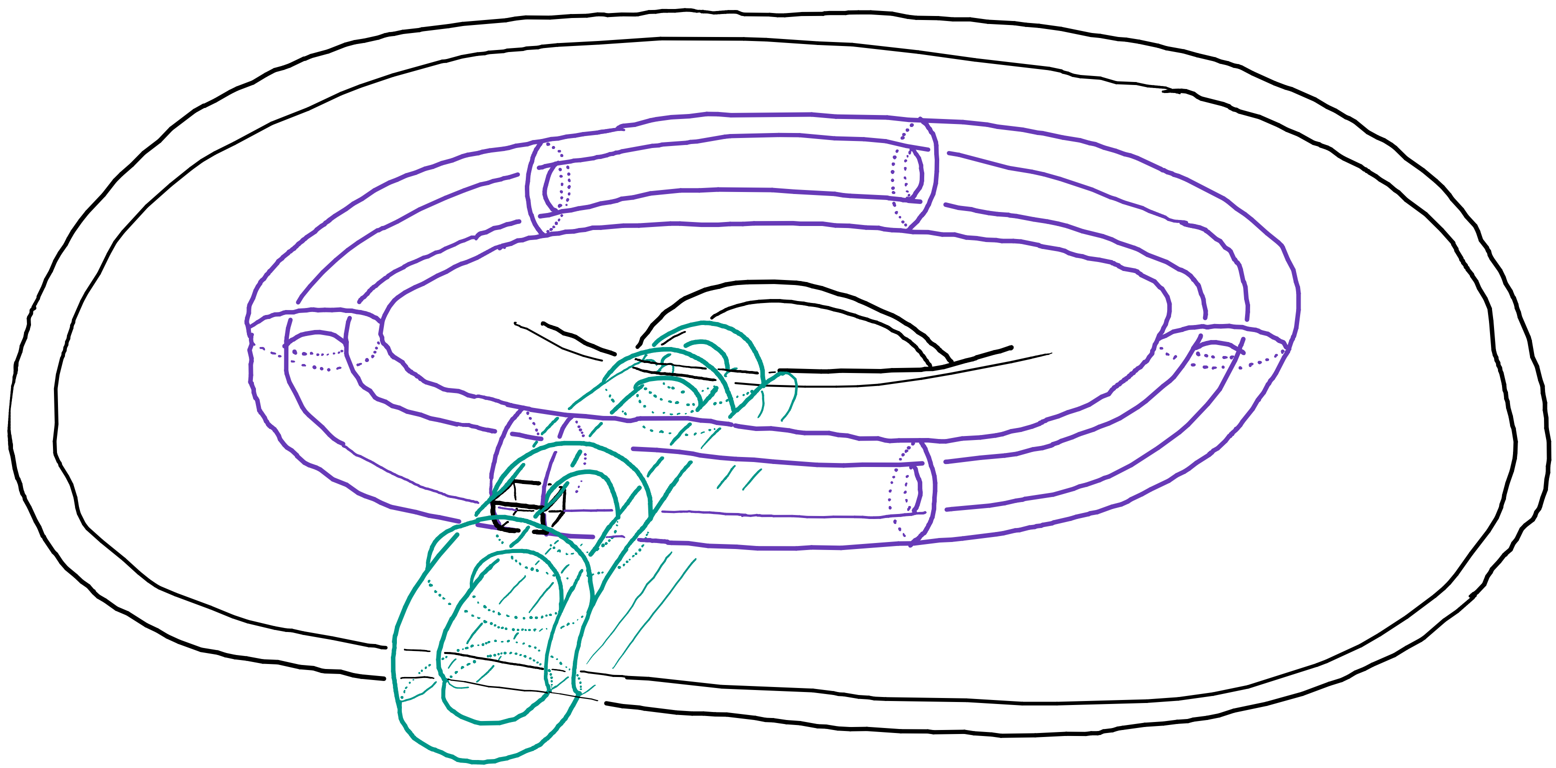


Plan for today:

(a) Milnor's clever inductive argument:

Start with $M_1 = S^1$ where $M_1 - D \hookrightarrow \mathbb{R}^1$

show that $M_k \times S^1$ also has $(M_k \times S^1) - D \hookrightarrow \mathbb{R}^{k+1}$

(b) Ferry's explicit version:

Standard embedding $\mathbb{T}^n \times (-1,1) \hookrightarrow \mathbb{R}^{n+1}$ via explicit coordinates

Perturb image of $\mathbb{T}^n \times \{0\}$ in its normal bundle

so that projection to $\mathbb{R}^n$ is an immersion in a neighborhood of $S = (n-1)$-skeleton of $\mathbb{T}^n$

neighborhood of S in $\mathbb{T}^n$ is a punctured torus

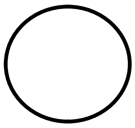
(c) Barden: builds immersions $\mathbb{T}^n \times \mathbb{I} \xrightarrow{f} \mathbb{R}^n \times \mathbb{I}$ inductively which are

$\cup$

a product map on $\mathbb{T}^n - (n\text{-cell}) \times \mathbb{I}$

[Milnor]

$M = S^1$ satisfies the hypothesis codim = 1 embedding in $\mathbb{R}^2$



Appendix B : SUBMERGING A PUNCTURED TORUS

Submersion = differential everywhere surjective

This contains verbatim a letter from J. Milnor of October, 1969 , which gives an elementary construction of a submersion of the punctured torus T^n–point into euclidean space R^n . It is used in §3 . A different elementary construction was found by D. Barden [Bar] [Ru] earlier in 1969 , and another by S. Ferry , [Fe] 1973†. Milnor produces a smooth C^∞ (= DIFF) submersion . A secant approximation to it in the sense of J. H. C. Whitehead [Mu_1 , §9] provides a piecewise-linear (= PL) submersion .

"Let M be a smooth compact manifold .

HYPOTHESIS . M has a codimension 1 embedding in euclidean space so that , for some smooth disk $D \subset M$ and some hyperplane P in euclidean space , the orthogonal projection from $M - D$ to P is a submersion .

THEOREM . *If M satisfies this hypothesis , so does $M \times S^1$.*

It follows inductively that every torus satisfies the hypothesis .

PROOF . Suppose that $M = M^{k-1}$ embeds in R^k so that $M - D$ projects submersively to the hyperplane $x_1 = 0$. We will assume that the subset $M \subset R^k$ lies in the half-space $x_k > 0$. Hence , rotating R^k about R^{k-1} in R^{k+1}, we obtain an embedding $(x, \theta) \leftrightarrow (x_1, \ldots, x_{k-1}, x_k \cos\theta , x_k \sin\theta)$ of $M \times S^1$ in R^{k+1} . This embedding needs only a mild deformation in order to satisfy the required property .

Let $e_1, \ldots, e_{k+1}$ be the standard basis for R^{k+1} . Let r_θ be the rotation

$$e_i \leftrightarrow e_i \text{ for } i < k , \quad \begin{aligned} e_k &\leftrightarrow e_k \cos\theta + e_{k+1} \sin\theta , \\ e_{k+1} &\leftrightarrow -e_k \sin\theta + e_{k+1} \cos\theta . \end{aligned}$$

Let $n(x) = n_1(x)e_1 + \cdots + n_k(x)e_k$ be the unit normal vector to M in R^k.

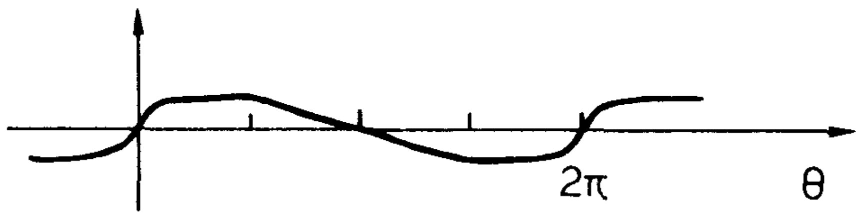
† And still another by A. Gramain [Gra] 1973 .

For $x \in M-D$ we can assume that n_1 is bounded away from zero. Say $n_1 \geqslant 2\alpha > 0$.

Suppose that M lies in the open slab $0 < x_k < \beta$ of R^k. Choose $\epsilon > 0$ so that the correspondence $(x, t) \leftrightarrow x + tn(x)$ embeds $M \times (-\epsilon, \epsilon)$ diffeomorphically in this slab.

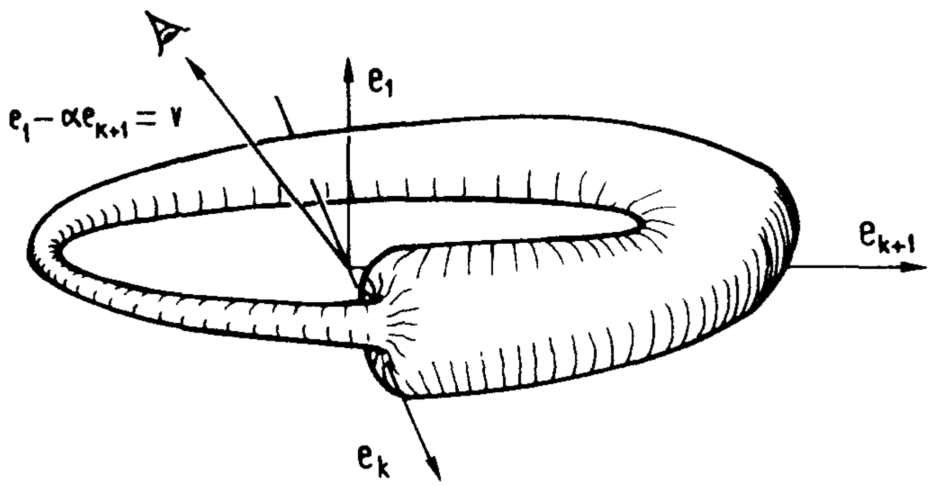
Choose a smooth map $t : S^1 \to (-\epsilon, \epsilon)$ so that

$$\frac{dt}{d\theta} \geqslant 2\beta/\alpha \text{ when } \theta = 0\,; \qquad \cos\theta\,\frac{dt}{d\theta} \geqslant 0 \text{ always}.$$



The required embedding $M \times S^1 \to R^{k+1}$ is now given by

$$(x, \theta) \leftrightarrow r_\theta(x + t(\theta)n(x))\,.$$



Computation shows that the normal vector to this embedding is $p/\|p\|$ where

$$p(x, \theta) = (x_k + tn_k)r_\theta(n) - \frac{dt}{d\theta}$$

Let $v = e_1 - \alpha e_{k+1}$. Then $p \cdot v = A + B$ where

$$A = (x_k + tn_k)(n_1 - \alpha \sin\theta\, n_k) \quad\text{and}\quad B = \alpha\cos\theta\,\frac{dt}{d\theta} \geqslant 0\,.$$

Thus if $x \in M - D$ we have

$$A \geqslant (x_k + tn_k)(2\alpha - \alpha) > 0$$

hence $p \cdot v > 0$. On the other hand , for any $x \in M$, if $\theta = 0$, we have

$$A \geqslant -\beta , \qquad B \geqslant \alpha(2\beta/\alpha) .$$

Hence $p \cdot v > 0$ for $\theta = 0$, and therefore $p \cdot v > 0$ for all sufficiently small θ ; say for $|\theta| \leqslant \eta$.

It now follows that the complement $(M \times S^1) - (D \times [\eta, 2\pi - \eta])$ projects submersively to the hyperplane $v^{\perp}$. This completes the proof .
∎

A smooth, compact manifold M^{k-1} satisfies Property I @ [Milnor]

if it has a codim = 1 embedding into Euclidean space

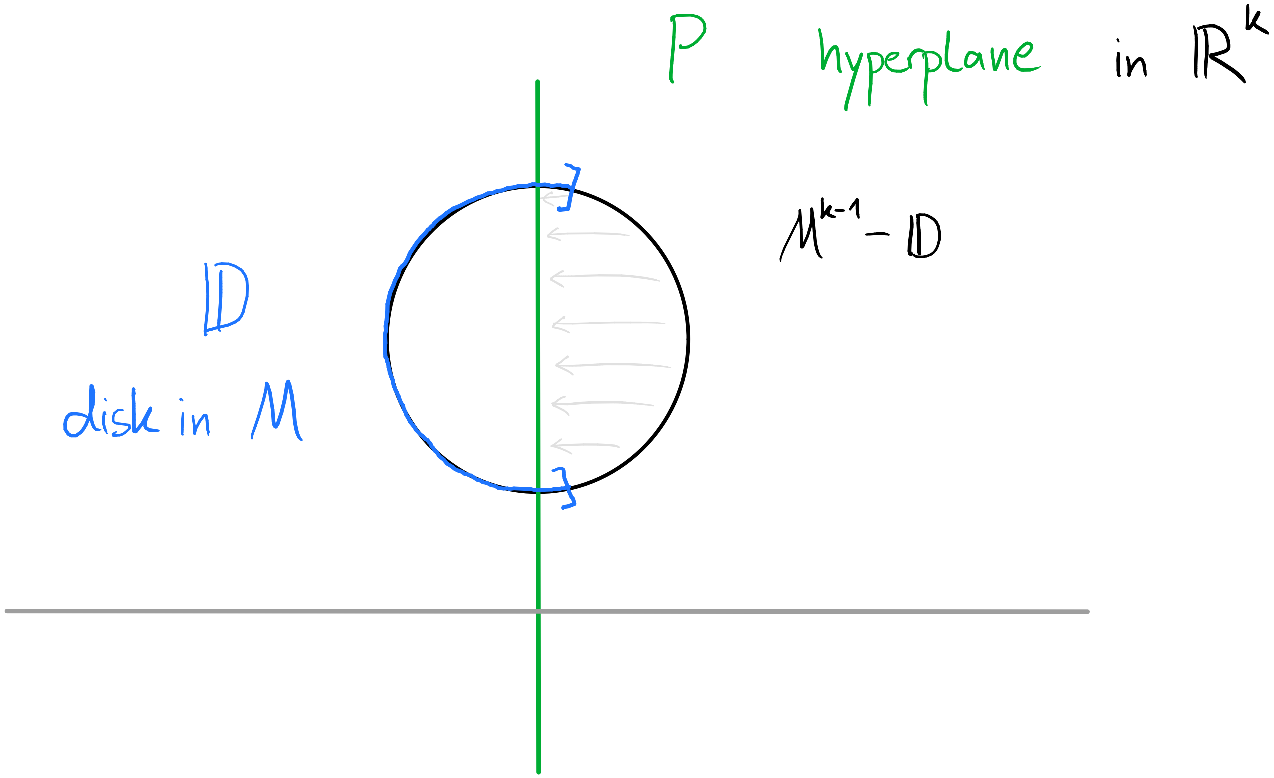
$$M \hookrightarrow \mathbb{R}^k$$

s.th. for some smooth closed disk $\mathbb{D} \subset M$

there exists a $(k-1)$ dimensional hyperplane $P \subset \mathbb{R}^k$ so that orthogonal projection

$$M - \mathbb{D} \xrightarrow{pr_P} P \cong \mathbb{R}^{k-1}$$ is an immersion.

Ex.: $M = S^1$



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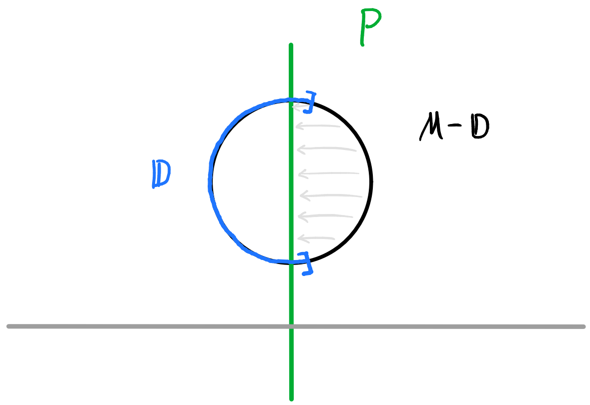


Thm.: If M satisfies Property I, then so does $M \times S^1$

Inductively, $\mathbb{T}^n = S^1 \times \dots \times S^1$ satisfies Property I

$$\rightsquigarrow \quad \mathbb{T}^n - \mathbb{D} \looparrowright \mathbb{R}^n$$

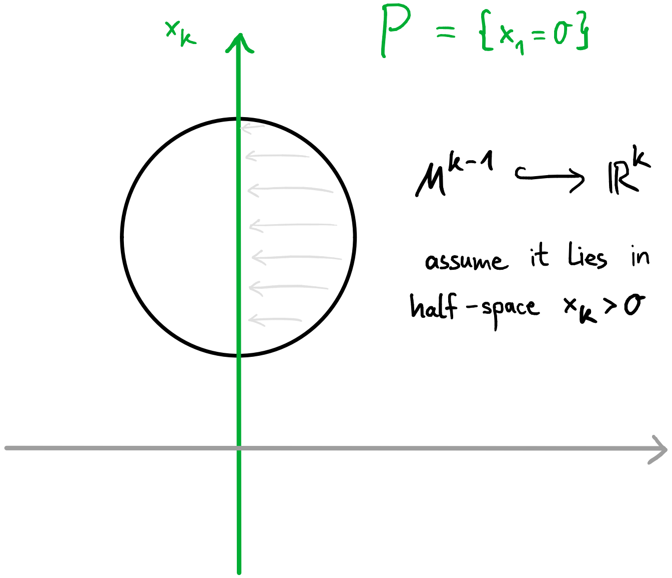
Thm.: If M satisfies Property $\mathcal{I}$, then so does $M \times S^1$

still needs slight deformation to satisfy Property $\mathcal{I}$

$$M \times S^1 \hookrightarrow \mathbb{R}^{k+1}$$

Pf. of the inductive step:

hyperplane $P = \{x_1 = 0\}$

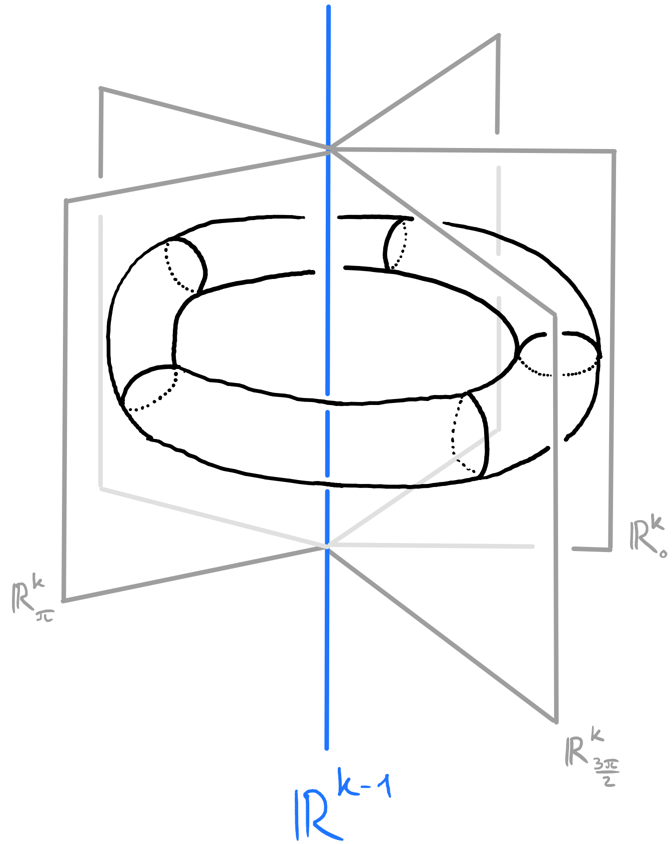


$M^{k-1} \hookrightarrow \mathbb{R}^k$

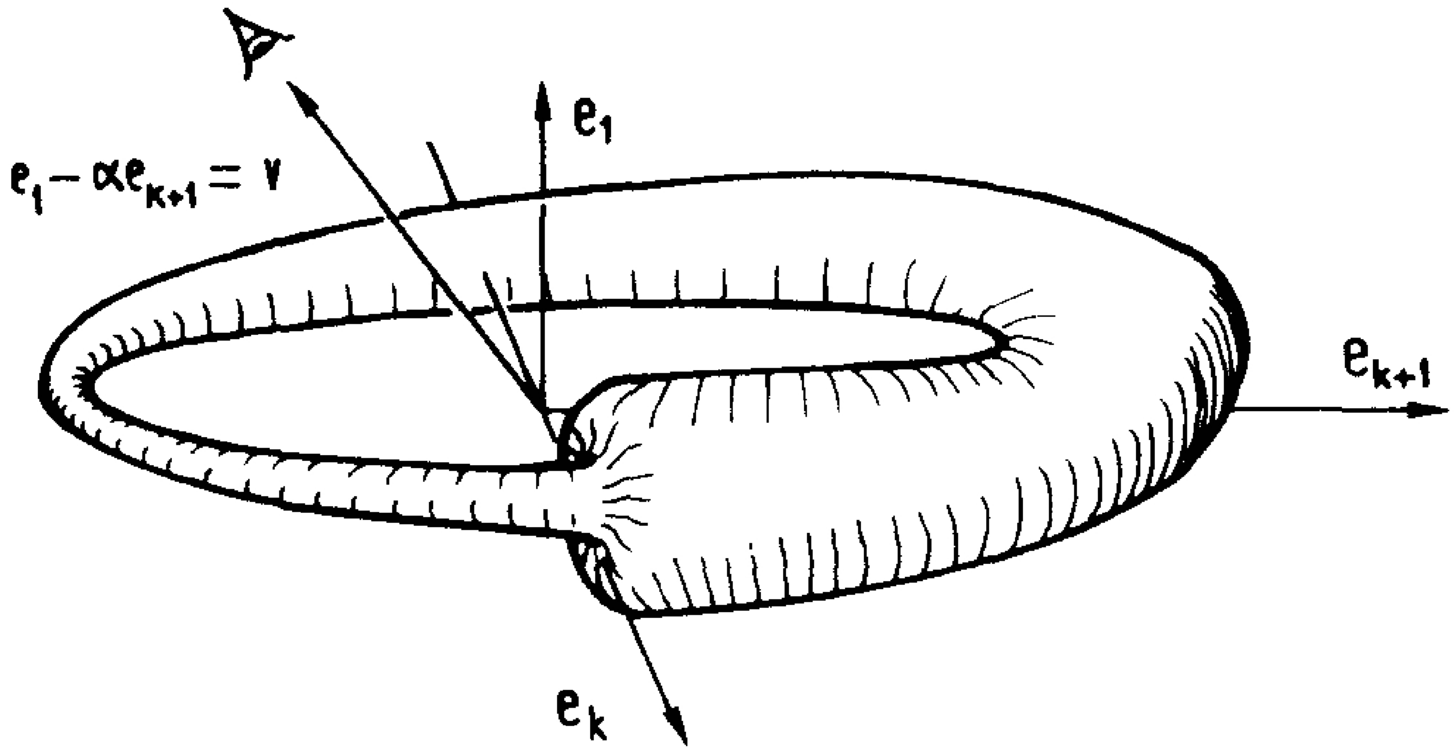
assume it lies in half-space $x_k > 0$

Spinning $\rightsquigarrow$

$$(x, \theta) \mapsto (x_1, \dots, x_{k-1}, x_k \cdot \cos\theta, x_k \cdot \sin\theta)$$



$$\begin{array}{ccc} \overset{\mathbb{R}^k}{\overset{\cup}{M^{k-1}}} \times S^1 & \hookrightarrow & \mathbb{R}^{k+1} \quad (\text{with standard basis } e_1, \dots, e_{k+1}) \\ (x, \theta) & \mapsto & \mathrm{rot}_\theta\big(x + t(\theta) \cdot n(x) \big) \end{array}$$

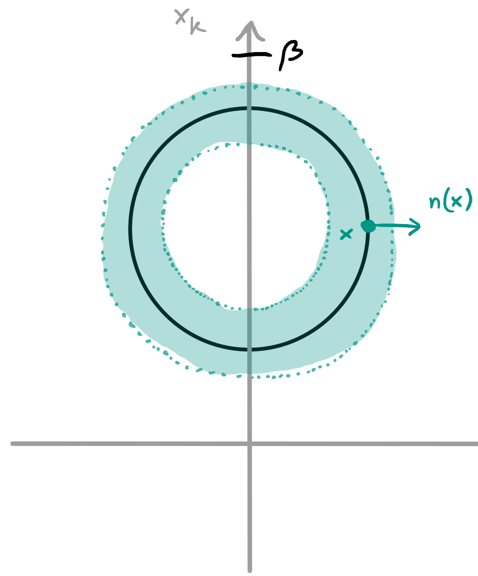


$$M \times S^1 \hookrightarrow \mathbb{R}^{k+1} \quad (\text{with standard basis } e_1, \dots, e_{k+1})$$

$$(x, \theta) \mapsto \mathrm{rot}_\theta\left(x + t(\theta) \cdot n(x) \right)$$

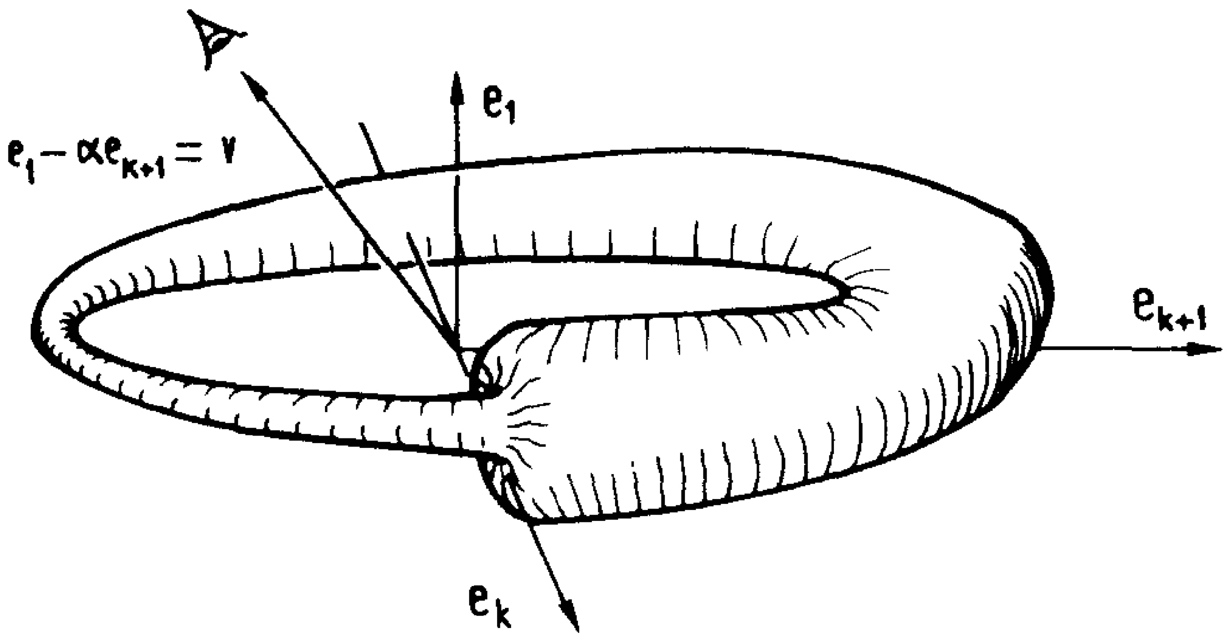
$$n(x) = \begin{pmatrix} n_1(x) \\ \vdots \\ n_k(x) \end{pmatrix}$$

unit normal vector to $x \in M$ in $\mathbb{R}^k$



$M \hookrightarrow \mathbb{R}^k$

assume it lies in the slab $\beta > x_k > 0$



Choose $\varepsilon > 0$ so that

$$M \times (-\varepsilon, \varepsilon) \hookrightarrow \mathbb{R}^k$$

$$(x, t) \mapsto x + t \cdot n(x)$$

is an embedding into $\{0 < x_k < \beta\}$

$$\begin{array}{ccc} M \times S^1 & \hookrightarrow & \mathbb{R}^{k+1} \\ (x,\theta) & \mapsto & \mathrm{rot}_\theta\big(x + t(\theta)\cdot n(x)\big) \end{array}$$

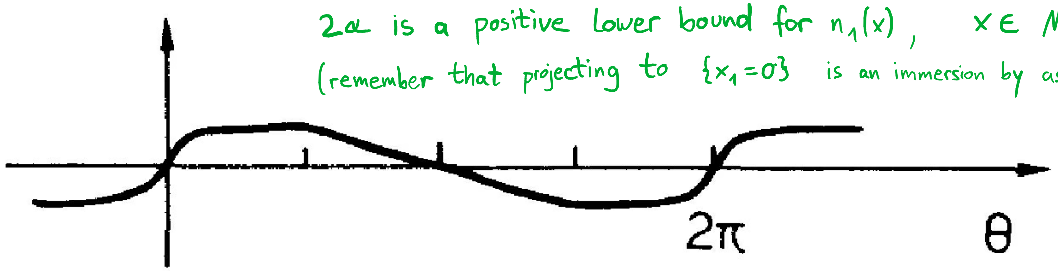
(with standard basis $e_1, \dots, e_{k+1}$)

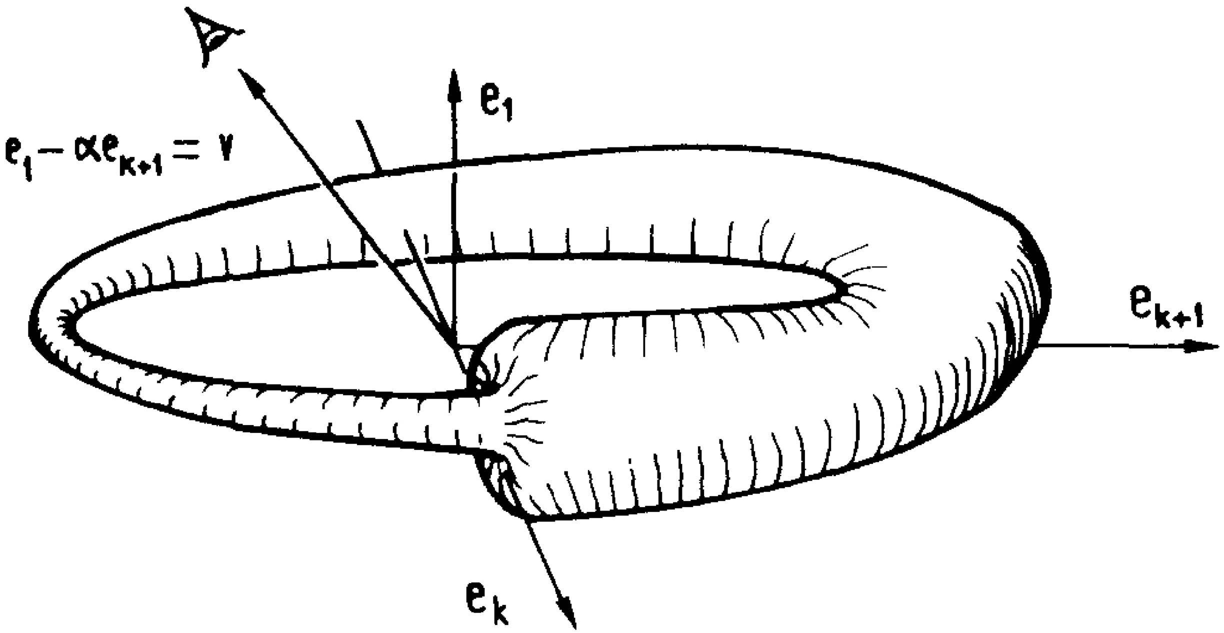
Choose a smooth map $t : S^1 \to (-\epsilon, \epsilon)$ so that

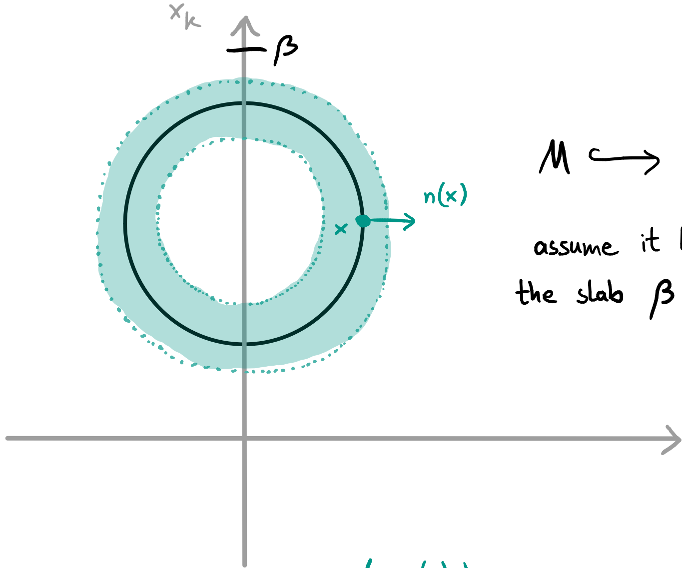
$$\frac{dt}{d\theta} \geqslant 2\beta/\alpha \text{ when } \theta = 0\,; \qquad \cos\theta\,\frac{dt}{d\theta} \geqslant 0 \text{ always}.$$

2α is a positive lower bound for $n_1(x)$, $x \in M - D$

(remember that projecting to $\{x_1 = 0\}$ is an immersion by assumption)







$M \hookrightarrow \mathbb{R}^k$

assume it lies in the slab $\beta > x_k > 0$

$$n(x) = \begin{pmatrix} n_1(x) \\ \vdots \\ n_k(x) \end{pmatrix}$$

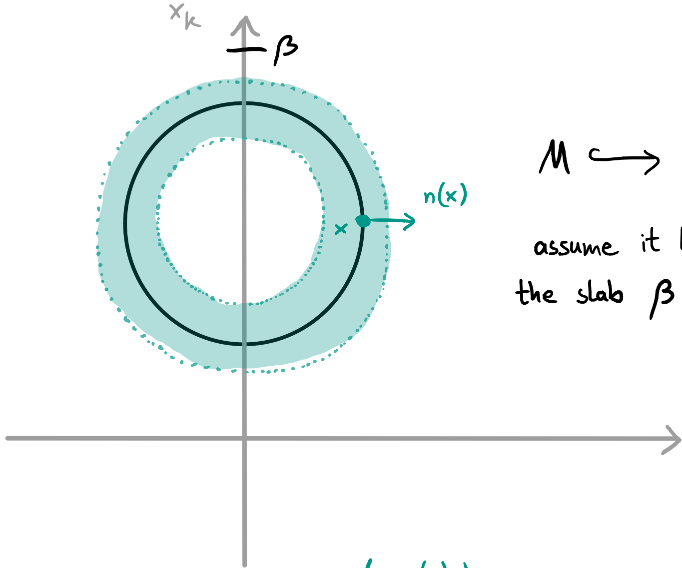
unit normal vector to $x \in M$ in $\mathbb{R}^k$

$$M \times S^1 \hookrightarrow \mathbb{R}^{k+1} \quad (\text{with standard basis } e_1, \dots, e_{k+1})$$

$$(x, \theta) \mapsto \mathrm{rot}_\theta\big(\, x + t(\theta) \cdot n(x) \,\big)$$

rotation $\quad \mathrm{rot}_\theta : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{k+1}$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos\theta & -\sin\theta \\ & & & \sin\theta & \cos\theta \end{pmatrix}$$

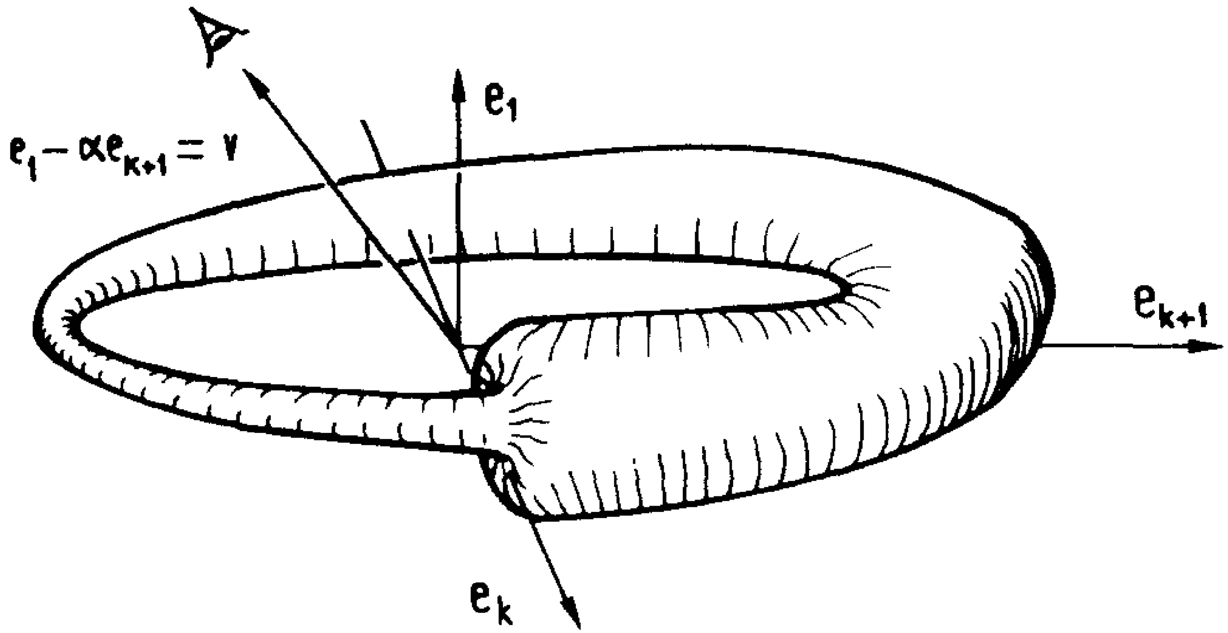


$M \hookrightarrow \mathbb{R}^k$

assume it lies in the slab $\beta > x_k > 0$

$$n(x) = \begin{pmatrix} n_1(x) \\ \vdots \\ n_k(x) \end{pmatrix}$$

unit normal vector to $x \in M$ in $\mathbb{R}^k$

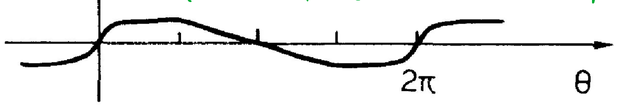


Choose a smooth map $t : S^1 \to (-\epsilon, \epsilon)$ so that

$$\frac{dt}{d\theta} \geqslant 2\beta/\alpha \text{ when } \theta = 0 ; \qquad \cos\theta \frac{dt}{d\theta} \geqslant 0 \text{ always}.$$

2α is a positive lower bound for $n_1(x)$, $x \in M - \mathbb{D}$

(remember that projecting to $\{x_1 = 0\}$ is an immersion by assumption)



$$M \times S^1 \hookrightarrow \mathbb{R}^{k+1} \qquad (\text{with standard basis } e_1, \dots, e_{k+1})$$

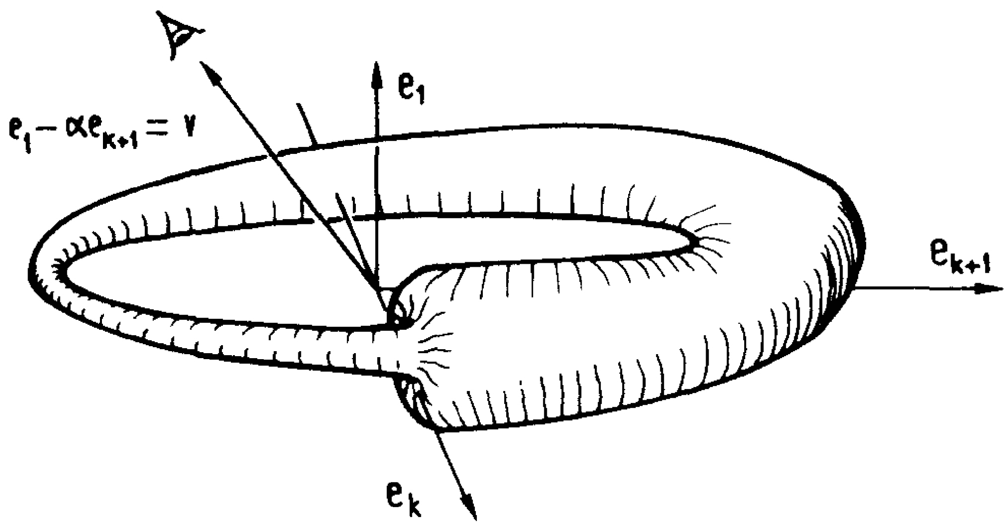
$$(x, \theta) \mapsto \mathrm{rot}_\theta\big(x + t(\theta) \cdot n(x)\big)$$

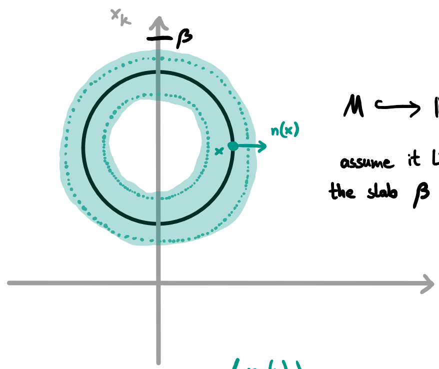
<u>Normal vector to this embedding</u> $= \dfrac{p(x,\theta)}{\|p(x,\theta)\|}$ where

$$p\big(\underbrace{(x_1, \dots, x_k)}_{= x}, \theta\big) = \big(x_k + t(\theta) \cdot n_k(x)\big) \cdot \mathrm{rot}_\theta\big(n(x)\big) - \frac{d\,t(\theta)}{d\theta}$$

rotation $\mathrm{rot}_\theta : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{k+1}$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos\theta & -\sin\theta \\ & & & \sin\theta & \cos\theta \end{pmatrix}$$





$M \hookrightarrow \mathbb{R}^k$

assume it lies in the slab $\beta > x_k > 0$

$$n(x) = \begin{pmatrix} n_1(x) \\ \vdots \\ n_k(x) \end{pmatrix}$$ unit normal vector to $x \in M$ in $\mathbb{R}^k$

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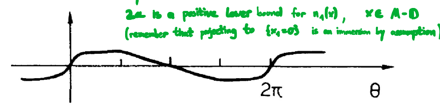
2α is a positive lower bound for $n_1(x)$, $x \in M - D$
(remember that projecting to $\{x_1 = 0\}$ is an immersion by assumption)



$$\boxed{\begin{array}{ccc} M \times S^1 & \hookrightarrow & \mathbb{R}^{k+1} \quad (\text{with standard basis } e_1, \dots, e_{k+1}) \\ (x, \theta) & \mapsto & \mathrm{rot}_\theta\big(x + t(\theta)\cdot n(x)\big) \end{array}}$$

normal vector $p\big((x_1, \dots, x_k), \theta\big) = \big(x_k + t(\theta)\cdot n_k(x)\big)\cdot \mathrm{rot}_\theta\big(n(x)\big) - \frac{d\,t(\theta)}{d\theta}$

$v := e_1 - \alpha \cdot e_{k+1}$

Reminder: 2α is a positive lower bound for $n_1(x)$ on $M - D$

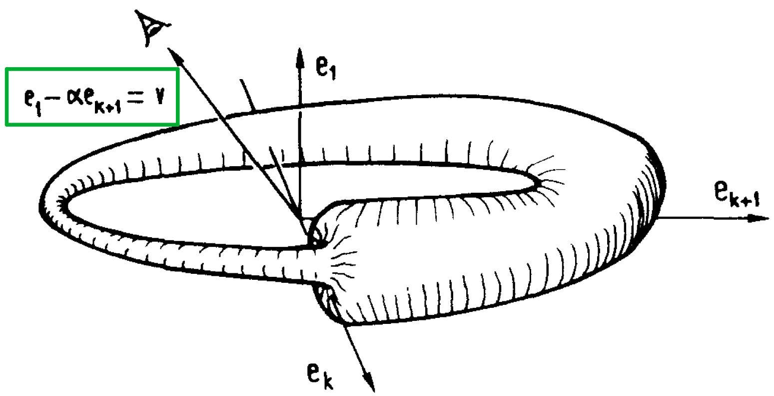
We will project to the hyperplane $v^\perp$

$$p(x,\theta)\cdot v = \underbrace{\big(x_k + t(\theta)\cdot n_k(x)\big)\cdot\big(n_1(x) - \alpha\cdot\sin\theta\cdot n_k(x)\big)}_{} + \underbrace{\alpha\cdot\cos\theta\cdot\frac{dt(\theta)}{d\theta}}_{\geq 0}$$

(a) for $x \in M - D$: $\geq \big(x_k + t(\theta)\cdot n_k(x)\big)(2\alpha - \alpha) > 0$

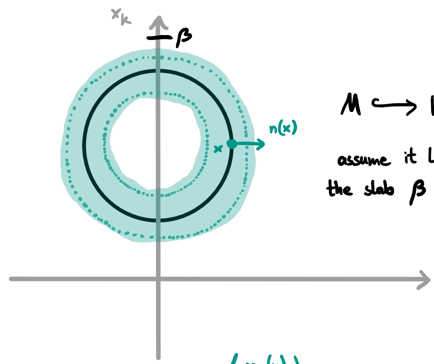
(b) for all $x \in M$, but $\theta = 0$: $\geq -\beta$ $\qquad \geq \alpha \cdot \frac{2\beta}{\alpha}$

So $p \cdot v \geq 0$ for small $|\theta| \leq \eta$



rotation $\mathrm{rot}_\theta : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{k+1}$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos\theta & -\sin\theta \\ & & & \sin\theta & \cos\theta \end{pmatrix}$$



$M \hookrightarrow \mathbb{R}^k$

assume it lies in the slab $\beta > x_k > 0$

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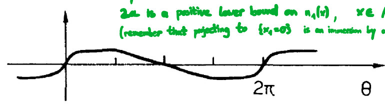

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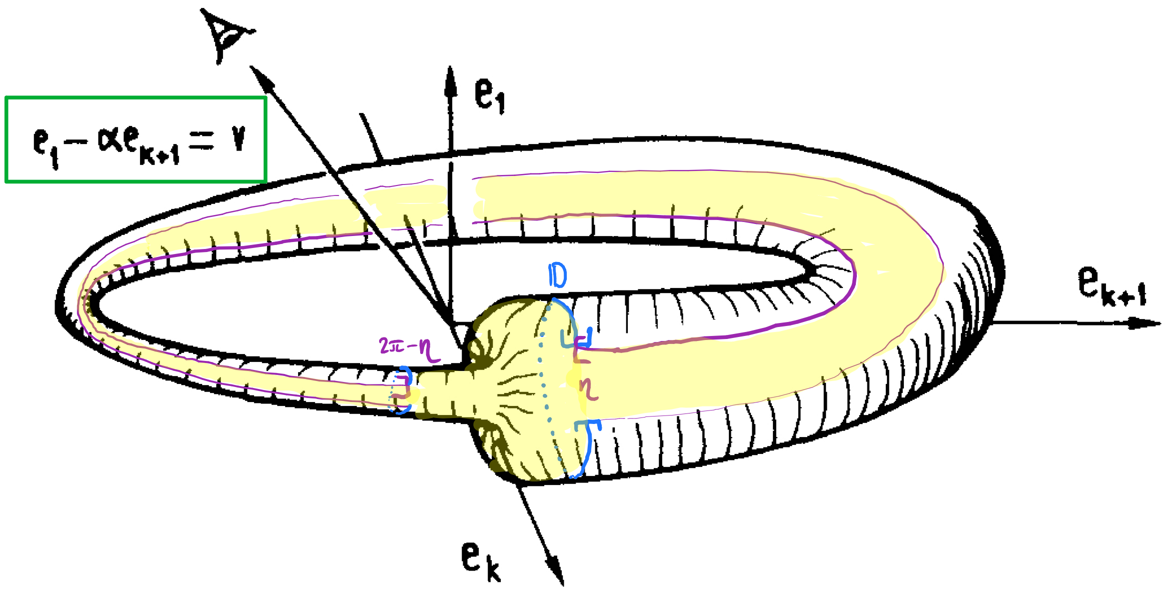
normal vector $p\big((x_1, \dots, x_k), \theta\big) = \big(x_k + t(\theta)\cdot n_k(x)\big)\cdot \mathrm{rot}_\theta\big(n(x)\big) - \frac{d\,t(\theta)}{d\theta}$

$$\boxed{\begin{array}{l} v := e_1 - \alpha \cdot e_{k+1} \\ \text{We will project to the hyperplane } v^\perp \end{array}}$$

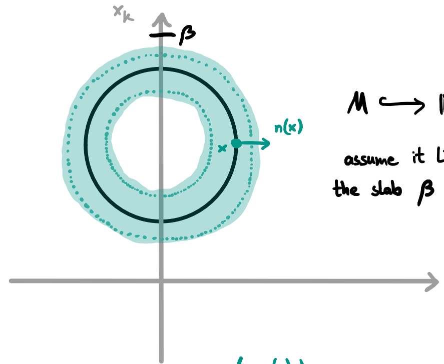
$p(x,\theta) \bullet v$ is >0 for $x \in M - D$ and $|\theta| \le \eta$ small

$\Rightarrow \big(M \times S^1\big) - \big(D \times [\eta, 2\pi - \eta]\big) \hookrightarrow v^\perp$ hyperplane



rotation $\mathrm{rot}_\theta : \mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{k+1}$

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos\theta & -\sin\theta \\ & & & \sin\theta & \cos\theta \end{pmatrix}$$



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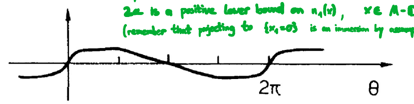

AN IMMERSION OF $T^n - D^n$ INTO R^n

by Steven Ferry [1973]

Let $T^n = S^1 \times \ldots \times S^1$ be the n-torus and let $D^n \subset T^n$ be an embedded disc. Kirby, Siebenmann, and Edwards use immersions of $T^n - D^n$ into R^n repeatedly in their work on stable homeomorphisms and triangulations. Although the existence of such immersions follows trivially from the work of Smale and Hirsch, it is appealing to have a more elementary construction. We will provide an explicit formula. Less explicit constructions have been given by D. Barden and J. Milnor.

Elements of T^n will be written as $(\theta_1, \ldots, \theta_n)$ with $\theta_1 \in S^1$. θ_1 will also be thought of as a real number $0 \leqslant \theta < 2\pi$. Occasionally $(\theta_1, \ldots, \theta_n)$ will be denoted by θ.

Lemma 1. Let $(\theta, t) \to \big(f_1(\theta, t), \ldots, f_{n+1}(\theta, t)\big)$ be an embedding of $T^n \times (-1, 1)$ into R^{n+1} such that $f_{n+1}(\theta, t) > 0$. The map of $T^{n+1} \times (-1, 1)$ into R^{n+2} defined by

$$(\theta, t) \to \big(f_1(\theta, t), \ldots, f_n(\theta, t), f_{n+1}(\theta, t) \cos \theta_{n+1}, f_{n+1}(\theta, t) \sin \theta_{n+1}\big)$$

is an embedding.

We define the *standard embedding* of $T^n \times (-1, 1)$ into R^{n+1} to be the embedding obtained by starting with $(\theta_1, t) \to \big((1+t) \cos \theta_1, (1+t) \sin \theta_1 + 2\big)$ and iterating the process described in lemma 1. At each stage we must add 2^n to the last term so that the condition $f_{n+1}(\theta, t) > 0$ will be satisfied. For example, in the standard embedding of $T^3 \times (1, 1) \to R^4$ we have

$$f_3(\theta, t) = \big(((1+t) \sin \theta_1 + 2) \sin \theta_2 + 4\big) \cos \theta_3$$

and

$$f_4(\theta, t) = \big(((1+t) \sin \theta_1 + 2) \sin \theta_2 + 4\big) \sin \theta_3 + 8\,.$$

θ_i

Let $S = \{\theta \in T^n \mid \theta_{\cancel{1}} = 0 \text{ for some } i, 1 \leqslant i \leqslant n\}$.
Let $\varphi : T^n \to R^1$ be defined by

$$\varphi(\theta) = \frac{\sin \theta_1 \ldots \sin \theta_n}{2^n} + \frac{\sin \theta_2 \ldots \sin \theta_n}{2^{n-1}} + \ldots + \frac{\sin \theta_n}{2}.$$

Theorem 1. Let $(\theta, t) \to (f_1(\theta, t), \ldots, f_{n+1}(\theta, t))$ be the standard embedding of $T^n \times (-1, 1)$ into R^{n+1}. For some $\varepsilon > 0$ the map $\theta \to$

$\theta \mapsto (f_1(\theta, \varepsilon\varphi(\theta)), \ldots, f_n(\theta, \varepsilon\varphi(\theta)))$ has nonsingular Jacobian on S. It therefore immerses a regular neighborhood of S (i.e. $T^n - D^n$) into R^n.

Proof. Using elementary properties of determinants, we compute:

$\mathbb{T}^n \times (-1,1) \hookrightarrow \mathbb{R}^{n+1}$ standard embedding

$(\theta, t) \mapsto (f_1(\theta,t), \ldots, f_n(\theta,t), f_{n+1}(\theta,t))$

$$\det\left(\frac{\partial f_i}{\partial \theta_j} + \varepsilon \frac{\partial f_i}{\partial t}\frac{\partial \varphi}{\partial \theta_j}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} + \varepsilon \dfrac{\partial f_i}{\partial t}\dfrac{\partial \varphi}{\partial \theta_j} & 0 \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 1 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} =$$

$$= \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 1 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 0 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} + \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon \dfrac{\partial f_i}{\partial t} \\ \hline 0 & 1 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}}$$

By construction, f_i involves only $\theta_1, \ldots, \theta_i$ and $\dfrac{\partial f_i}{\partial \theta_i}$ has a factor of $\sin \theta_i$. Thus, on S the upper left hand corner of the second matrix is triangular with at least one zero on the diagonal. We have

$$\det\left(\frac{\partial f_i}{\partial \theta_j} + \frac{\partial f_i}{\partial t}\frac{\partial \varphi}{\partial \theta_j}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = -\varepsilon \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 0 \end{array}\right)\Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}}$$

Notice that $f_{n+1}(\theta, 0) = 2^n \varphi(\theta)$ and that $\dfrac{\partial f_{n+1}}{\partial t}$ is identically zero on S. Thus, if the above determinant is evaluated at $\theta \in S$, $t = 0$ it is $\left(\dfrac{-\varepsilon}{2^n}\right)$ times the determinant of the Jacobian of the standard embedding. It is therefore nonsingular when evaluated at $\theta \in S$, $t = \varepsilon\varphi$ for sufficiently small ε. This completes the proof.

In essence, we have perturbed the image of $T^n \times 0$ in R^{n+1} along its normal bundle so that projection into R^n is an immersion on S. More precise calculations show that ε may be taken to be 1.

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Lexington, KY. 40506

(Reçu le 25 septembre 1973)

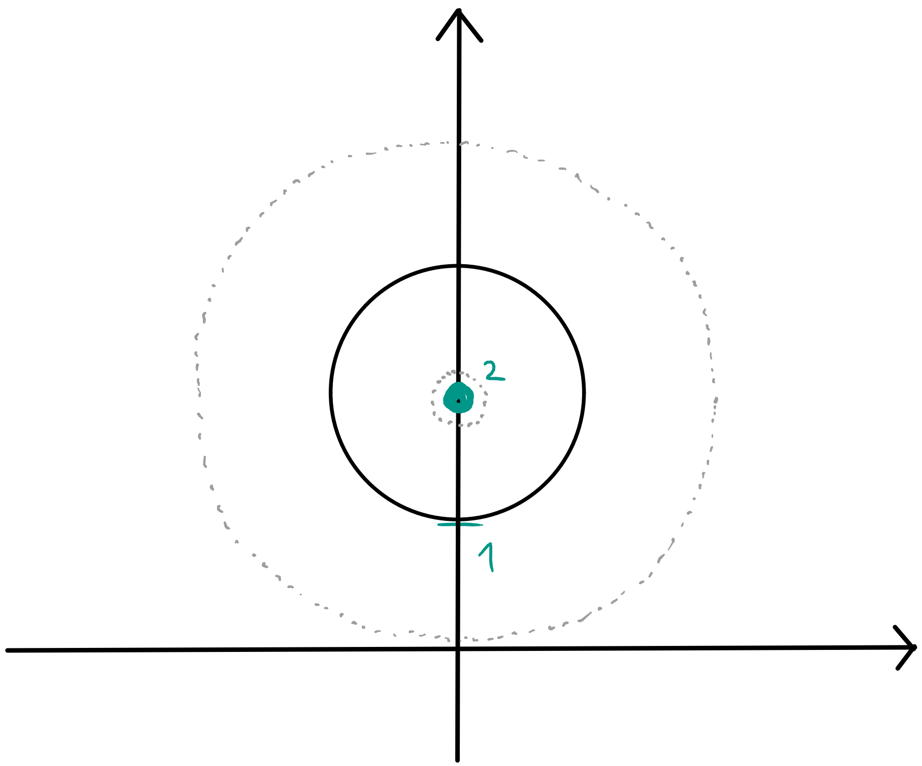
(b) [Ferry]

Standard embedding of $\mathbb{T}^n \times (-1,1)$ into $\mathbb{R}^{n+1}$ via iterated spinning:

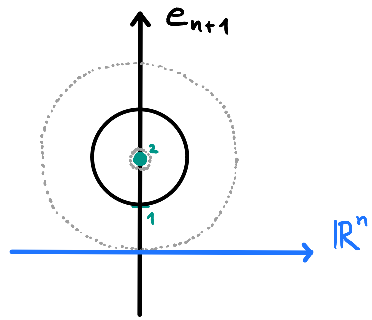
Start with

$$\begin{gathered} {}^{[0,2\pi]}/_{0\sim 2\pi} \\ \shortparallel \\ S^1 \times (-1,1) \hookrightarrow \mathbb{R}^2 \end{gathered}$$

$$(\theta_1, t) \longmapsto \left((1+t)\cdot \cos\theta_1 \,,\, (1+t)\cdot \sin\theta_1 + 2 \right)$$



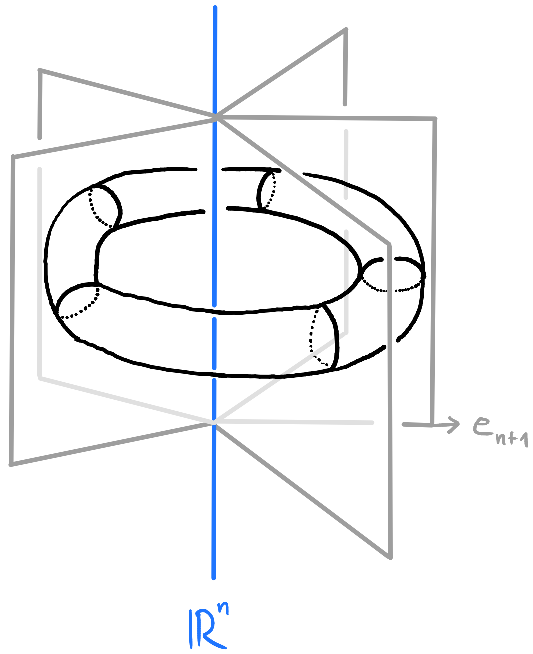
Standard embedding of $\mathbb{T}^n \times (-1,1)$ into $\mathbb{R}^{n+1}$



Start with

$$\overset{[0,2\pi]/_{0\sim 2\pi}}{\overset{\parallel}{S^1}} \times (-1,1) \hookrightarrow \mathbb{R}^2$$

$$(\theta_1, t) \longmapsto \left((1+t)\cdot \cos\theta_1 \,,\ (1+t)\cdot \sin\theta_1 + 2 \right)$$



Spinning step: Given an embedding

$$\mathbb{T}^n \times (-1,1) \hookrightarrow \mathbb{R}^{n+1}$$

$$(\vec{\theta}, t) \mapsto \Big(f_1(\vec{\theta},t), \dots, f_n(\vec{\theta},t), \underbrace{f_{n+1}(\vec{\theta},t)}_{\text{such that } >0} \Big)$$

$$\rightsquigarrow \quad \mathbb{T}^{n+1} \times (-1,1) \hookrightarrow \mathbb{R}^{n+2}$$

$$(\vec{\theta}, t) \mapsto \Big(f_1(\vec{\theta},t), \dots, f_n(\vec{\theta},t), f_{n+1}(\vec{\theta},t)\cdot\cos\theta_{n+1}, f_{n+1}(\vec{\theta},t)\cdot\sin\theta_{n+1} \Big)$$

After each spinning stage, add $+2^n$ to last coordinate to force $f_{n+1}(\vec{\theta},t) > 0$

$$\mathbb{T}^1 \times (-1,1) \hookrightarrow \mathbb{R}^2$$

$$(\theta_1, t) \longmapsto \left((1+t)\cdot \cos\theta_1 \,,\, (1+t)\cdot \sin\theta_1 + 2 \right)$$

$$\mathbb{T}^2 \times (-1,1) \hookrightarrow \mathbb{R}^3$$

$$\underset{\substack{\parallel \\ (\theta_1,\theta_2)}}{(\vec{\theta}}, t) \mapsto \left((1+t)\cdot \cos\theta_1 \,,\, \left((1+t)\cdot \sin\theta_1 + 2\right)\cdot \cos\theta_2 \,,\, \left((1+t)\cdot \sin\theta_1 + 2\right)\cdot \sin\theta_2 + 4 \right)$$

$$\mathbb{T}^3 \times (-1,1) \hookrightarrow \mathbb{R}^4$$

$$\underset{\substack{\parallel \\ (\theta_1,\theta_2,\theta_3)}}{(\vec{\theta}}, t) \mapsto \left((1+t)\cdot \cos\theta_1 \,,\, \left((1+t)\cdot \sin\theta_1 + 2\right)\cdot \cos\theta_2 \,,\, \left(\left((1+t)\cdot \sin\theta_1 + 2\right)\cdot \sin\theta_2 + 4\right)\cdot \cos\theta_3 \,,\, \left(\left((1+t)\cdot \sin\theta_1 + 2\right)\cdot \sin\theta_2 + 4\right)\cdot \sin\theta_3 + 8 \right)$$

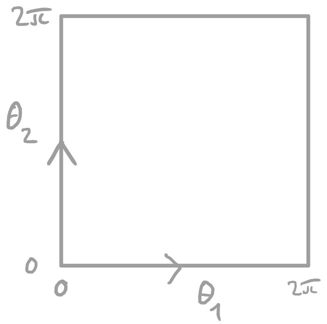
$\vdots$

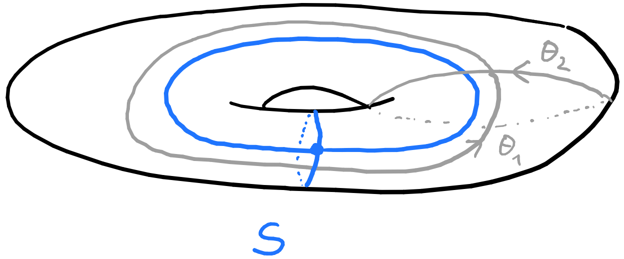
$$S = \{ (\underset{[0,2\pi]/_{0\sim 2\pi}}{\underset{\cap}{\theta_1}}, \ldots, \underset{[0,2\pi]/_{0\sim 2\pi}}{\underset{\cap}{\theta_n}}) \in \mathbb{T}^n \mid \theta_i = 0 \text{ for some } i \in \{1,\ldots,n\} \}$$

"$(n-1)$-skeleton of $\mathbb{T}^n$"

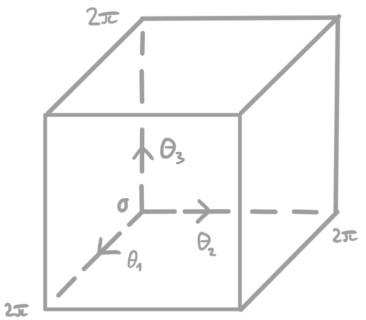
$\boxed{\text{neighborhood of } S \text{ in } \mathbb{T}^n \text{ is everything except the } n\text{-cell in } \mathbb{T}^n, \text{ i.e. it is } \mathbb{T}^n - \text{disk}}$

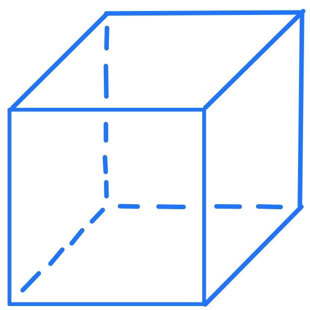
$\mathbb{T}^2 = [0,2\pi]^2$ with opposite sides identified





$\mathbb{T}^3 = [0,2\pi]^3$ with opposite faces identified

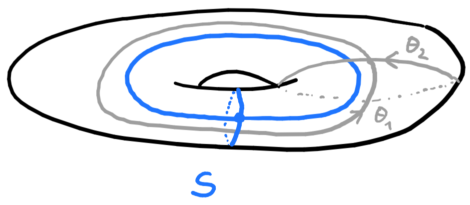




S = all faces of cube

$$S = \{ (\theta_1, \dots, \theta_n) \in \mathbb{T}^n \mid \theta_i = 0 \text{ for some } i \in \{1, \dots, n\} \}$$

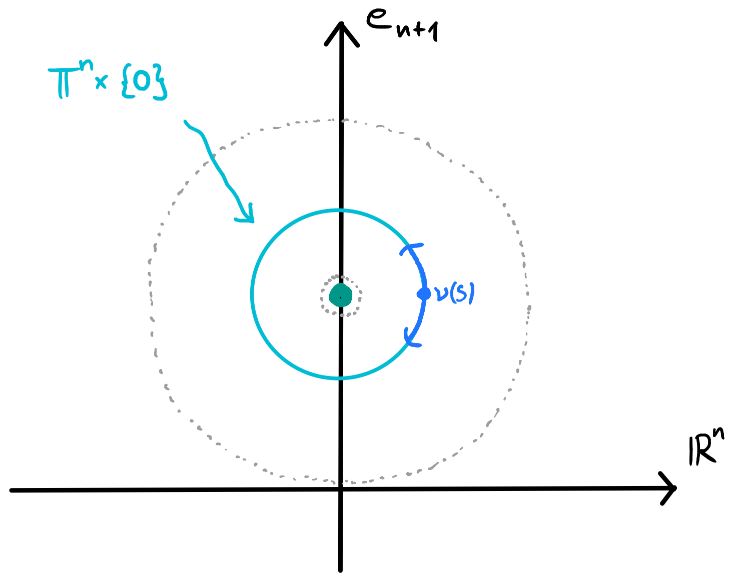
$\theta_1 \in [0,2\pi]/_{0\sim 2\pi}$, $\theta_n \in [0,2\pi]/_{0\sim 2\pi}$



Take the (explicit) standard embedding from earlier

$$\mathbb{T}^n \times (-1,1) \hookrightarrow \mathbb{R}^{n+1}$$

$$(\theta, t) \mapsto (f_1(\theta,t), \dots, f_n(\theta,t), f_{n+1}(\theta,t))$$

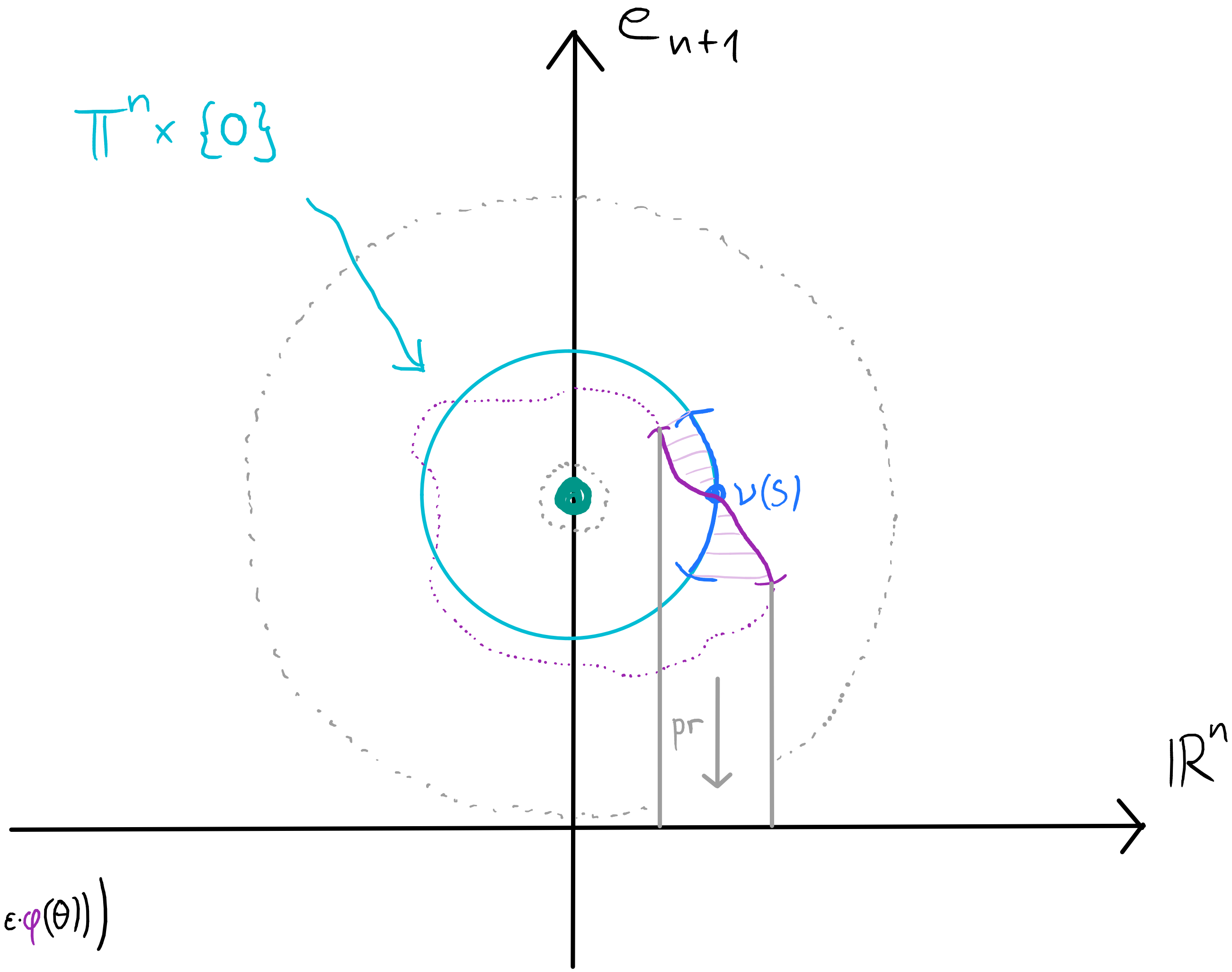


would like to perturb $\mathbb{T}^n \times \{0\}$ in the normal t-direction

$$(f_1(\theta,t), \dots, f_n(\theta,t), f_{n+1}(\theta,t)) \overset{\text{perturb}}{\rightsquigarrow} (f_1(\theta, \varepsilon\cdot\varphi(\theta)), \dots, f_n(\theta, \varepsilon\cdot\varphi(\theta)), f_{n+1}(\theta, \varepsilon\cdot\varphi(\theta))) \overset{\text{project}}{\rightsquigarrow} (f_1(\theta, \varepsilon\cdot\varphi(\theta)), \dots, f_n(\theta, \varepsilon\cdot\varphi(\theta)))$$

so that projecting to first n coordinates $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$

is an immersion on a $\underbrace{\text{regular neighborhood of } S}_{\mathbb{T}^n - \mathbb{D}^n}$



$$\left(f_1(\theta,t), \dots, f_n(\theta,t), f_{n+1}(\theta,t) \right)$$

$$\overset{\text{perturb}}{\rightsquigarrow} \left(f_1(\theta, \varepsilon\cdot\varphi(\theta)), \dots, f_n(\theta, \varepsilon\cdot\varphi(\theta)), f_{n+1}(\theta, \varepsilon\cdot\varphi(\theta)) \right)$$

$$\overset{\text{project}}{\rightsquigarrow} \left(f_1(\theta, \varepsilon\cdot\varphi(\theta)), \dots, f_n(\theta, \varepsilon\cdot\varphi(\theta)) \right)$$

Take $\varphi : \mathbb{T}^n \longrightarrow \mathbb{R}$ and $\varepsilon > 0$ small

$$\varphi(\vec{\theta}) = \frac{\sin\theta_1 \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_n}{2^n} + \frac{\sin\theta_2 \cdot \ldots \cdot \sin\theta_n}{2^{n-1}} + \dots + \frac{\sin\theta_{n-1} \cdot \sin\theta_n}{2^2} + \frac{\sin\theta_n}{2}$$

$$\left(f_1(\theta,t), \dots, f_n(\theta,t), f_{n+1}(\theta,t) \right)$$

perturb

$$\rightsquigarrow \left(f_1(\theta, \varepsilon\cdot\varphi(\theta)), \dots, f_n(\theta, \varepsilon\cdot\varphi(\theta)), f_{n+1}(\theta, \varepsilon\cdot\varphi(\theta)) \right)$$

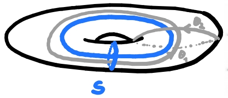
project

$$\rightsquigarrow \left(f_1(\theta, \varepsilon\cdot\varphi(\theta)), \dots, f_n(\theta, \varepsilon\cdot\varphi(\theta)) \right)$$

$\varphi : \mathbb{T}^n \longrightarrow \mathbb{R}$ and $\varepsilon > 0$ small

$$\varphi(\vec{\theta}) = \frac{\sin\theta_1 \cdot \sin\theta_2 \cdot \ldots \cdot \sin\theta_n}{2^n} + \frac{\sin\theta_2 \cdot \ldots \cdot \sin\theta_n}{2^{n-1}} + \ldots + \frac{\sin\theta_{n-1} \cdot \sin\theta_n}{2^2} + \frac{\sin\theta_n}{2}$$

$S = \{ (\underset{[0,2\pi]/_{0\sim 2\pi}}{\theta_1}, \dots, \underset{[0,2\pi]/_{0\sim 2\pi}}{\theta_n}) \in \mathbb{T}^n \mid \theta_i = 0 \text{ for some } i \in \{1,\dots,n\} \}$



Proof. Using elementary properties of determinants, we compute:

$$\det\left(\frac{\partial f_i}{\partial \theta_j} + \varepsilon \frac{\partial f_i}{\partial t}\frac{\partial \varphi}{\partial \theta_j} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}} \overset{\substack{\text{Laplace}\\ \text{last column}}}{=} \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} + \varepsilon \dfrac{\partial f_i}{\partial t}\dfrac{\partial \varphi}{\partial \theta_j} & 0 \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 1 \end{array} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}} = \quad -\varepsilon\cdot\frac{\partial f_i}{\partial t}$$

$$= \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon\dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 1 \end{array} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}} \overset{\substack{\text{multilinear}\\ \text{last row}}}{=} \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon\dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 0 \end{array} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}} + \det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & -\varepsilon\dfrac{\partial f_i}{\partial t} \\ \hline 0 & 1 \end{array} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}}$$

By construction, f_i involves only $\theta_1, \dots, \theta_i$ and $\dfrac{\partial f_i}{\partial \theta_i}$ has a factor of $\sin\theta_i$. Thus, on S the upper left hand corner of the second matrix is triangular with at least one zero on the diagonal. We have

$$\det\left(\frac{\partial f_i}{\partial \theta_j} + \frac{\partial f_i}{\partial t}\frac{\partial \varphi}{\partial \theta_j} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}} = -\varepsilon\det\left(\begin{array}{c|c} \dfrac{\partial f_i}{\partial \theta_j} & \dfrac{\partial f_i}{\partial t} \\ \hline \dfrac{\partial \varphi}{\partial \theta_j} & 0 \end{array} \right)\Bigg|_{\substack{\theta\in S\\ t=\varepsilon\varphi}}$$

Notice that $f_{n+1}(\theta, 0) = 2^n\varphi(\theta)$ and that $\dfrac{\partial f_{n+1}}{\partial t}$ is identically zero on S. Thus, if the above determinant is evaluated at $\theta\in S$, $t = 0$ it is $\left(\dfrac{-\varepsilon}{2^n}\right)$ times the determinant of the Jacobian of the standard embedding. It is therefore nonsingular when evaluated at $\theta \in S$, $t = \varepsilon\varphi$ for sufficiently small ε. This completes the proof.

[Barden, 1969]

$h_0 = h$. If W is a subset of U, a deformation $\Phi: P \times I \to E(U; M) \times I$ is **modulo** W if $\Phi_t(h) \mid W = h \mid W$ for all $h \in P$ and $t \in I$.

Suppose that $\Phi: P \times I \to E(U, M) \times I$ and $\psi: Q \times I \to E(U, M) \times I$ are deformations of subsets of $E(U, M)$, and suppose that $\Phi_1(P) \subset Q$. Then, the **composition of** ψ **with** Φ denoted by $\psi * \Phi: P \times I \to E(U, M) \times I$ is defined by

$$\psi * \Phi(h, t) = \begin{cases} (\Phi_{2t}(h), t) & \text{if } t \in [0, \frac{1}{2}], \\ (\psi_{2t-1}\Phi_1(h), t) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

We shall denote the cube $\{x \in E^n \mid |x_i| \leqslant r, 1 \leqslant i \leqslant n\}$ by I_r^n. We regard S^1 as the space obtained by identifying the endpoints of $[-4, 4]$ and we let $p: E^1 \to S^1$ denote the natural covering projection, that is, $p(x) = (x + 4)_{(\mathrm{mod}\ 8)} - 4$. Let T^n be the n-fold product of S^1. Then, I_r^n can be regarded as a subset of T^n for $r < 4$. Let $p^n: E^n \to T^n$ be the product covering projection and let $p^{k,n}: I_1^k \times E^n \to I_1^k \times T^n$ be the map $1_{I_1^k} \times p^n$. These maps will each be denoted by p when there is no possibility of confusion.

Let B^n be the closed unit n-ball in E^n and let S^{n-1} be its boundary as usual. We regard $S^{n-1} \times [-1, 1]$ as a subset of E^n by identifying (x, t) with $(1 + t/2) \cdot x$.

With the above discussions, definitions and notation out of the way, we are ready to start formulating some lemmas preliminary to the proofs of the main results of this section.

A discussion of, and a geometrical proof of, our first lemma will be postponed until the end of this section. (An **immersion** of one space into another is a continuous map which is locally an embedding.)

Immersion Lemma 5.6.1. *There is an immersion* $\alpha: T^n - B^n \to E^n$ *of the punctured torus into* E^n.

For a picture of α in the case $n = 2$, see Fig. **5.6.2**.

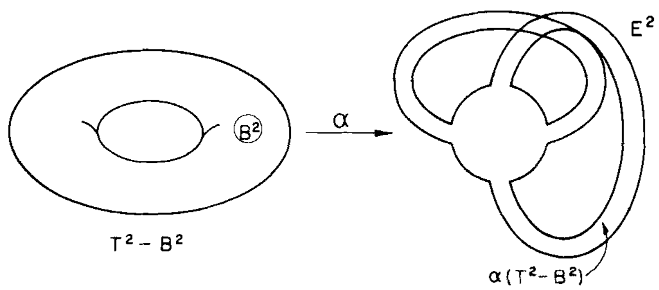


Figure 5.6.2

We are now ready to give a proof of Immersion Lemma **5.6.1**. The proof presented here was communicated to this author by R. D. Edwards. It was originated by Barden [2] and was formulated in the following picturesque form by Siebenmann. (Immersion Lemma **5.6.1** also follows from [Hirsch, 1].)

Proof of Immersion Lemma 5.6.1. We will work with the following inductive statement which is stronger than Lemma **5.6.1**. α is the immersion $\mathbb{T}_0^n \hookrightarrow \mathbb{R}^n$ we are looking for

> n-Dimensional Inductive Statement: *There exists an immersion f of $T^n \times I$ into $E^n \times I$ such that $f \mid T_0^n \times I$ is a product map, $f = \alpha \times 1$, where T_0^n is T^n minus an n-cell.*

We adopt the following notation for this proof: Let $I = [-1, 1] = J$, $J^n = (J)^n$, $S^1 = I \cup_\partial J$, $T^n = (S^1)^n$ and $T_0^n = T^n - \operatorname{Int} J^n$. It is easy to see that $E^n \times S^1$ can be regarded as a subset of E^{n+1} where the I-fibers of $E^n \times I$ are straight and vertical in E^{n+1} (see Fig. **5.6.9**).

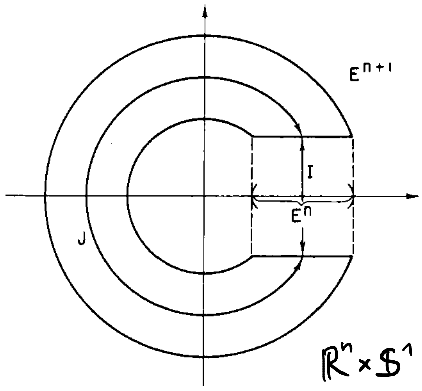
$\mathbb{T}^n \times \mathbb{I} \overset{f}{\hookrightarrow} \mathbb{R}^n \times \mathbb{I}$

$f| = \alpha \times \mathrm{id}$ product

$\mathbb{T}^n - (n\text{-cell}) \times \mathbb{I} \hookrightarrow \mathbb{R}^n \times \mathbb{I}$

$\parallel$

$\mathbb{T}_0^n$



$\mathbb{R}^n \times S^1 \subset \mathbb{R}^{n+1}$

Figure 5.6.9

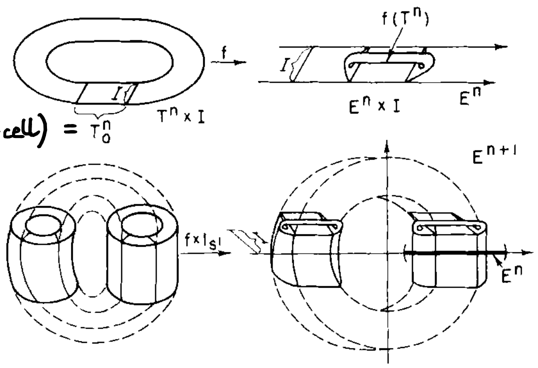
Assume that f and α are given by the inductive statement in dimension n. It is a simple matter to extend f to an immersion of $T^{n+1} \times I$ into $E^{n+1} \times I$, that is, just let

$$f \times 1_{S^1}: T^n \times S^1 \times I \to E^n \times S^1 \times I \subset E^{n+1} \times I$$

be the extension (see Fig. **5.6.10**). However, $f \times 1_{S^1}$ is not a product on $T_0^{n+1} \times I$, but merely on $T_0^n \times S^1 \times I$. The way to correct this is to conjugate $f \times 1_{S^1}$ with a 90° rotation (on the $I \times I$ factor) of the missing plug $(T_0^{n+1} \times I) - (T_0^n \times S^1 \times I) = \operatorname{Int} J^n \times I \times I$. The fact that $f \times 1_{S^1} \mid T_0^n \times I^2$ is a product in the I^2 factor allows one to do this.

$f \restriction \mathbb{T}_0^n \times I$ is a product

$\mathbb{T}^n - (n\text{-cell}) = T_0^n$



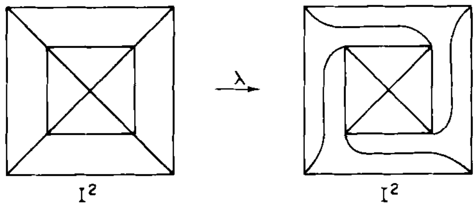
$\mathbb{T}^n \times S^1 \times \mathbb{I} \longrightarrow \mathbb{E}^n \times S^1 \times \mathbb{I} \subset$

Figure 5.6.10

product on $\mathbb{T}_0^n \times S^1 \times \mathbb{I}$

want: product on $\mathbb{T}_0^{n+1} \times \mathbb{I}$

Assume without loss of generality that $f(T^n \times [-\frac{1}{2}, \frac{1}{2}]) \subset E^n \times [-\frac{2}{3}, \frac{2}{3}]$. Let λ be a homeomorphism of I^2 that is the identity on Bd I^2 and is a $\pi/2$-rotation on $[-\frac{2}{3}, \frac{2}{3}] \times [-\frac{2}{3}, \frac{2}{3}]$ (see Fig. **5.6.11**). Extend λ via the identity to a homeomorphism $\bar{\lambda}$: $S^1 \times I \to S^1 \times I$ (see Fig. **5.6.12**).



rotation in small square

Figure 5.6.11

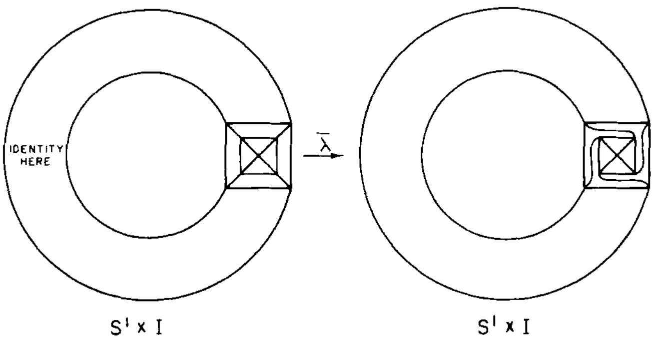


Figure 5.6.12

$\mathbb{T}^n \times \mathbb{S}^1 \times \mathbb{I}$

Consider now the following immersion h of $T^{n+1} \times I$ into $E^{n+1} \times I$,

$$h = (1_{E^n} \times \bar{\lambda}^{-1})(f \times 1_{S^1})(1_{T^n} \times \bar{\lambda}).$$

If we let $g = h \mid T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$, then it can easily be checked that g is a product on $(T_0{}^n \times S^1) \cup (J^n \times [-\frac{1}{2}, \frac{1}{2}])$ which is a deformation retract of $T_0^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$. Thus, without loss of generality we can assume that g is a product on $T_0^{n+1} \times [\frac{1}{2}, \frac{1}{2}]$ (see Figs. **5.6.13** and **5.6.14**).

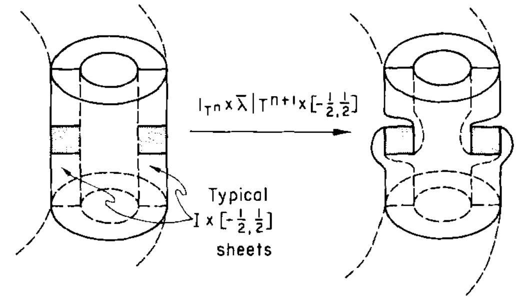


Figure 5.6.13

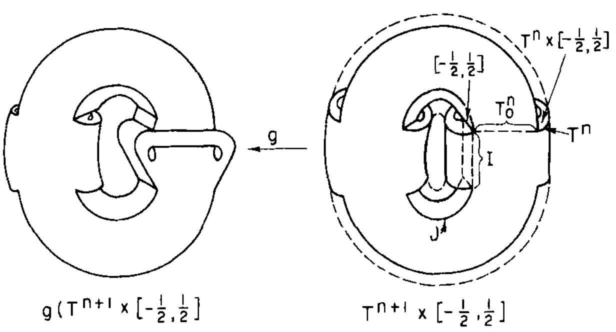


Figure 5.6.14

It is now easy to see that such a g gives rise to an immersion as desired in the theorem. This would be a trivial matter of reparametrizing the I coordinate if we knew that $g(T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]) \subset E^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$. To get such an inclusion, one can shrink T_0^{n+1} a little, with the help of an interior collar, to T_1^{n+1}, and using the fact that $g \mid T_0^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$ is a product, isotop $g(T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}])$ into $E^{n+1} \times [-\frac{1}{2}, \frac{1}{2}])$ keeping $g \mid T_1^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$ fixed.

Let us conclude this section with a couple of remarks concerning how the preceding results on local contractibility relate to codimension zero taming. First note that if M^n and $\tilde{M}^n$ are PL manifolds and if $h: M \to \tilde{M}$ is a topological homeomorphism which can be approximated arbitrarily closely by PL homeomorphisms, then the results on local contractibility imply that h is ϵ-tame. For instance, it follows from Theorem **4.11.1** that stable homeomorphisms of E^n are ϵ-tame. (For an example of another use of this observation, see Theorem 1 of [Cantrell and Rushing, 1].) A strong form of the *hauptvermutung* for PL manifolds (Question **1.6.5**) is just the following codimension zero taming question: Can every topological homeomorphism of a PL manifold M^n onto a PL manifold $\tilde{M}^n$ be ϵ-tamed? By using some of the techniques presented in this section as well as some work of Wall, it has recently been established by Kirby and Siebenmann that this codimension-zero taming theorem holds for many manifolds and fails for others.

Barden's inductive statement:

(c) [Barden]

(Bard_n) $\exists$ an immersion $\mathbb{T}^n \times \mathbb{I} \overset{f}{\looparrowright} \mathbb{R}^n \times \mathbb{I}$

such that restriction $f| = \alpha \times id_{\mathbb{I}}$ is a product map.

$$\underbrace{\mathbb{T}^n - (n\text{-cell})}_{\mathbb{T}^n_0} \times \mathbb{I} \looparrowright \mathbb{R}^n \times \mathbb{I}$$

Then $\alpha : \mathbb{T}^n_0 \looparrowright \mathbb{R}^n$ is the desired immersion.

(Bard_1) $\exists$ an immersion $\mathbb{T}^1 \times \mathbb{I} \xhookrightarrow{f} \mathbb{R}^1 \times \mathbb{I}$

such that restriction $f\restriction = \alpha \times \mathrm{id}$ is a product map.

$$\mathbb{T}^1 - (1\text{-cell}) \times \mathbb{I} \hookrightarrow \mathbb{R}^1 \times \mathbb{I}$$

$\mathbb{S}^1 = \mathbb{J} \cup_\partial \mathbb{I}$

$\mathbb{T}^n - (n\text{-cell}) = (\mathbb{S}^1)^{\times n} - \mathrm{int}(\mathbb{J}^n)$

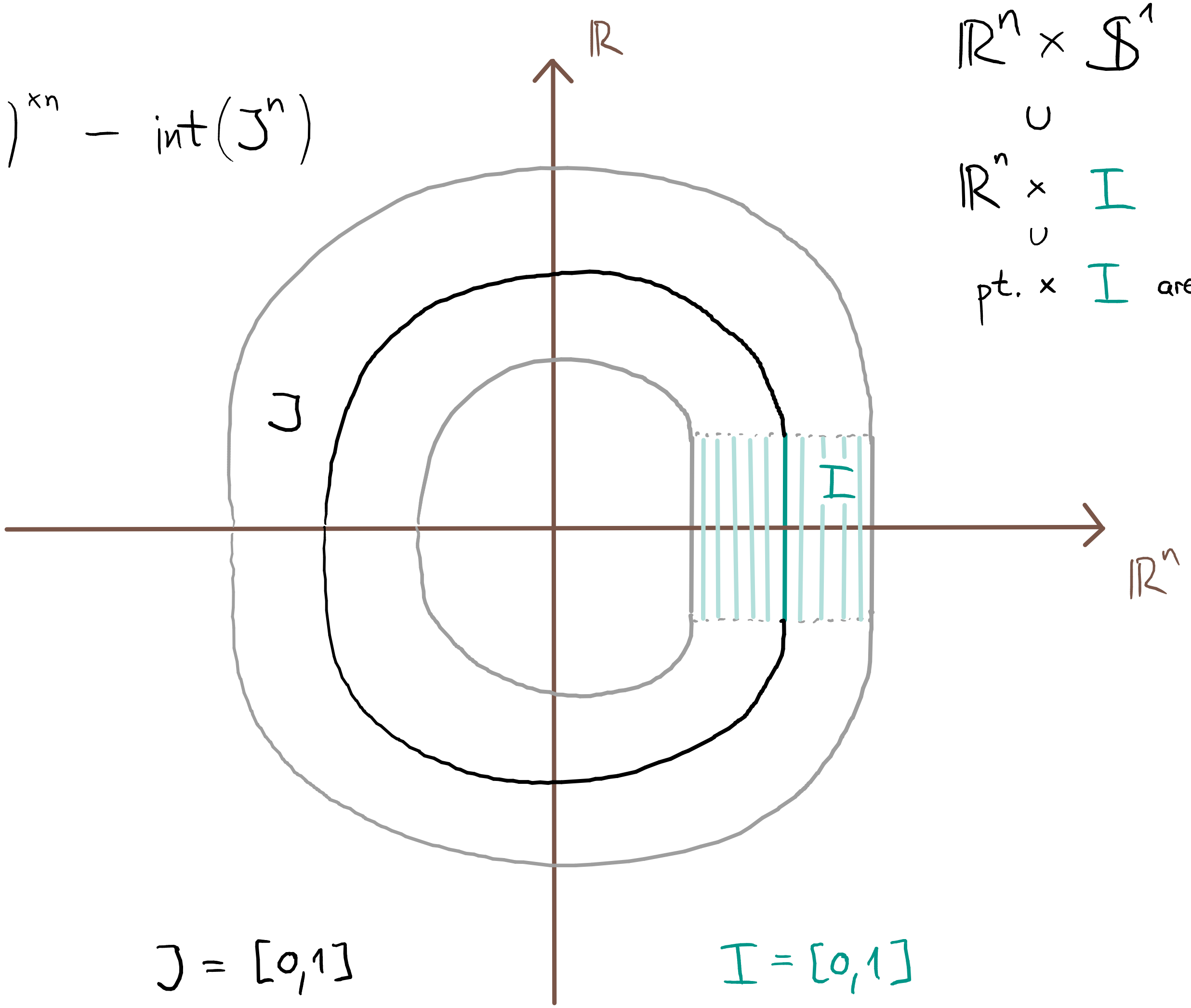
$\mathbb{R}^n \times \mathbb{S}^1 \hookrightarrow \mathbb{R}^{n+1}$

$\cup$

$\mathbb{R}^n \times \mathbb{I}$

$\cup$

pt. $\times$ $\mathbb{I}$ are straight & vertical in $\mathbb{R}^{n+1}$



$\mathbb{J} = [0,1]$ $\mathbb{I} = [0,1]$

$\boxed{(\mathrm{Bard}_n)} \rightsquigarrow \boxed{(\mathrm{Bard}_{n+1})}$

$\exists$ an immersion $\mathbb{T}^n \times \mathbb{I} \overset{f}{\looparrowright} \mathbb{R}^n \times \mathbb{I}$

such that restriction $f| = \alpha \times id$ is a product map.

$\mathbb{T}^n - (n\text{-cell}) \times \mathbb{I} \looparrowright \mathbb{R}^n \times \mathbb{I}$

$$\mathbb{T}^n \times \mathbb{S}^1 \times \mathbb{I} \xrightarrow{f \times id_{\mathbb{S}^1}} \mathbb{R}^n \times \mathbb{S}^1 \times \mathbb{I} \subset \mathbb{R}^{n+1} \times \mathbb{I}$$

Product on $(\mathbb{T}^n - (n\text{-cell})) \times \mathbb{S}^1 \times \mathbb{I}$

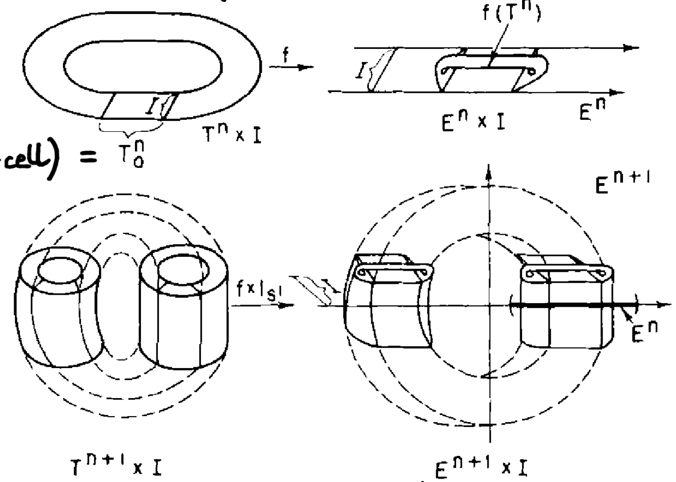
Want: Product on $(\mathbb{T}^{n+1} - (n+1)\text{-cell}) \times \mathbb{I}$

Assume that f and α are given by the inductive statement in dimension n. It is a simple matter to extend f to an immersion of $T^{n+1} \times I$ into $E^{n+1} \times I$, that is, just let

$$f \times 1_{S^1}: T^n \times S^1 \times I \to E^n \times S^1 \times I \subset E^{n+1} \times I$$

be the extension (see Fig. **5.6.10**). However, $f \times 1_{S^1}$ is not a product on $T_0^{n+1} \times I$, but merely on $T_0{}^n \times S^1 \times I$. The way to correct this is to conjugate $f \times 1_{S^1}$ with a 90° rotation (on the $I \times I$ factor) of the missing plug $(T_0^{n+1} \times I) - (T_0{}^n \times S^1 \times I) = \mathrm{Int}\, J^n \times I \times I$. The fact that $f \times 1_{S^1} \mid T_0{}^n \times I^2$ is a product in the I^2 factor allows one to do this.

$f|_{\mathbb{T}_0^n \times \mathbb{I}}$ is a product



$\mathbb{T}^n - (n\text{-cell}) = \mathbb{T}_0^n$

$\mathbb{T}^n \times \mathbb{S}^1 \times \mathbb{I} \longrightarrow \mathbb{E}^n \times \mathbb{S}^1 \times \mathbb{I} \subset E^{n+1} \times I$

Figure 5.6.10

$f\restriction \mathbb{T}_0^n \times \mathbb{I}$ is a product $\alpha \times id_{\mathbb{I}}$

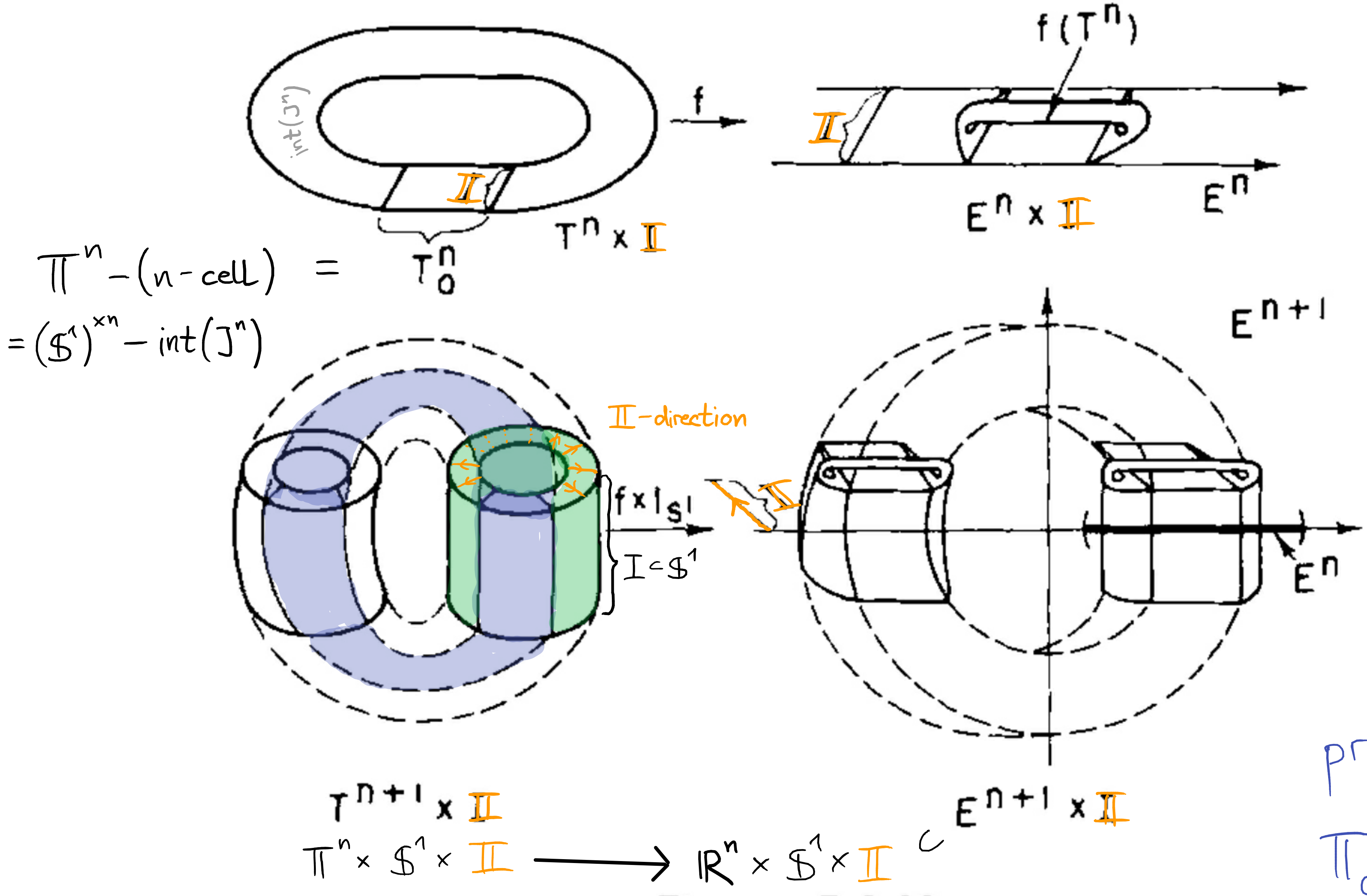


Figure 5.6.10

product on

$\mathbb{T}_0^n \times \mathbb{S}^1 \times \mathbb{I}$

<u>want:</u> product on

$\mathbb{T}_0^{n+1} \times \mathbb{I}$

missing piece

$$\left(\mathbb{T}_0^{n+1} \times \mathbb{I}\right) - \left(\mathbb{T}_0^n \times \mathbb{S}^1 \times \mathbb{I}\right) = int(J^n) \times I \times \mathbb{I}$$

$f\times \mathrm{id}_{S^1}$ is a product on $\mathbb{T}^n_0 \times S^1 \times \mathbb{I}$

Fix this by conjugating $f \times \mathrm{id}_{S^1}$

$$f\times \mathrm{id}_{S^1} : \mathbb{T}^n \times \underbrace{S^1 \times \mathbb{I}} \longrightarrow \mathbb{R}^n\times S^1\times\mathbb{I} \subset \mathbb{R}^{n+1}\times\mathbb{I}$$

with a 90° rotation on the $I \times \mathbb{I}$ factor

of the missing piece

$$(\mathbb{T}^{n+1}_0 \times \mathbb{I}) - (\mathbb{T}^n_0\times S^1\times\mathbb{I}) = \mathrm{int}(J^n)\times I\times\mathbb{I}$$

We use:

$f\times\mathrm{id}_{S^1}\restriction_{\mathbb{T}^n_0\times I\times\mathbb{I}}$

is a product on $I\times\mathbb{I}$

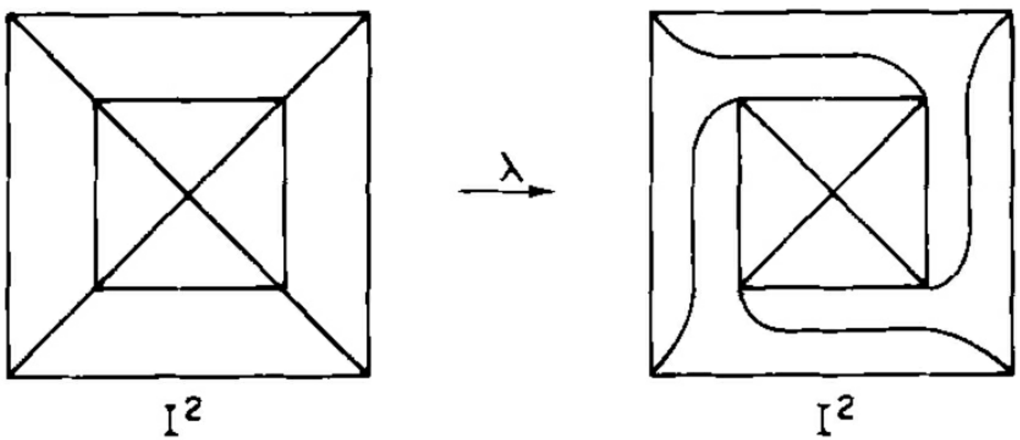


Figure 5.6.11

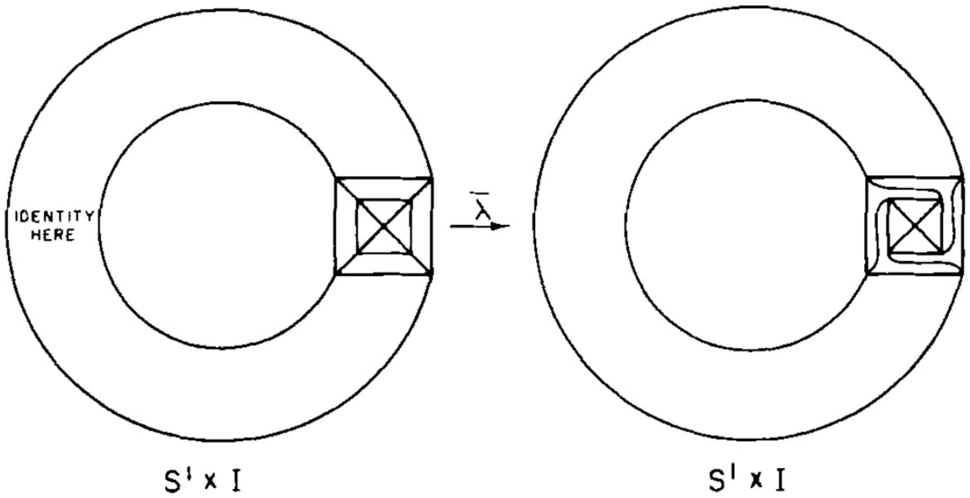


Figure 5.6.12

$\mathbb{T}^n \times \mathbb{S}^1 \times \mathbb{I}$

Consider now the following immersion h of $T^{n+1} \times I$ into $E^{n+1} \times I$,

$$h = (1_{E^n} \times \bar{\lambda}^{-1})(f \times 1_{S^1})(1_{T^n} \times \bar{\lambda}).$$

If we let $g = h \mid T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$, then it can easily be checked that g is a product on $(T_0{}^n \times S^1) \cup (J^n \times [-\frac{1}{2}, \frac{1}{2}])$ which is a deformation retract of $T_0^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$. Thus, without loss of generality we can assume that g is a product on $T_0^{n+1} \times [\frac{1}{2}, \frac{1}{2}]$ (see Figs. **5.6.13** and **5.6.14**).

$$\boxed{\begin{array}{l} h: \mathbb{T}^n \times \mathbb{S}^1 \times \mathbb{I} \looparrowright \mathbb{R}^{n+1} \times \mathbb{I} \\ \\ h = (id_{\mathbb{R}^n} \times \bar{\lambda}^{-1})(f \times id_{\mathbb{S}^1})(id_{\mathbb{T}^n} \times \bar{\lambda}) \end{array}}$$

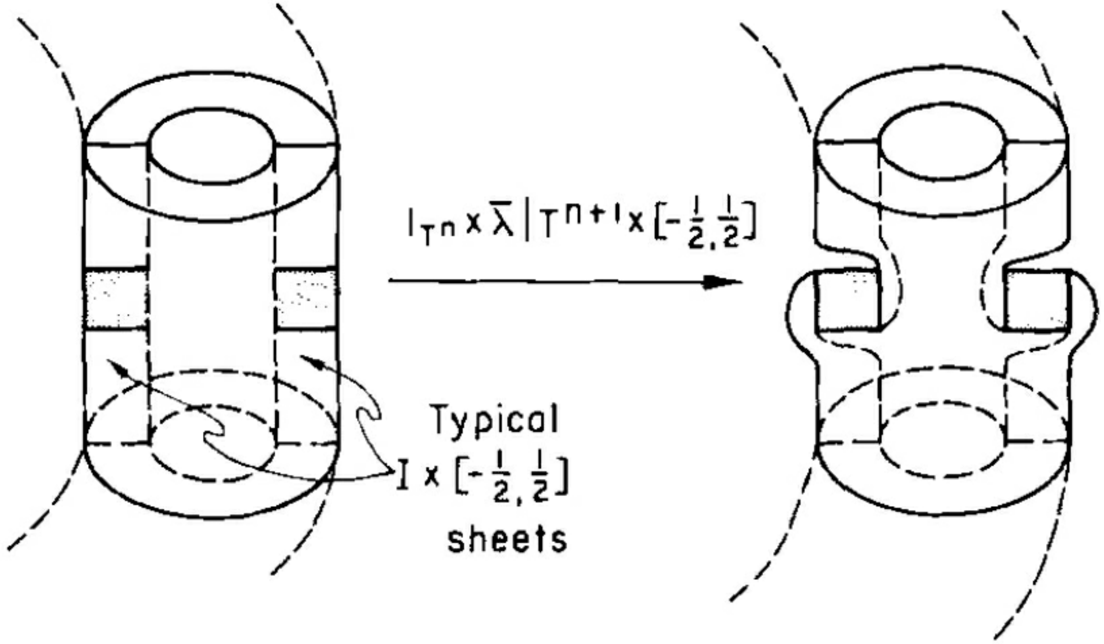


Figure 5.6.13

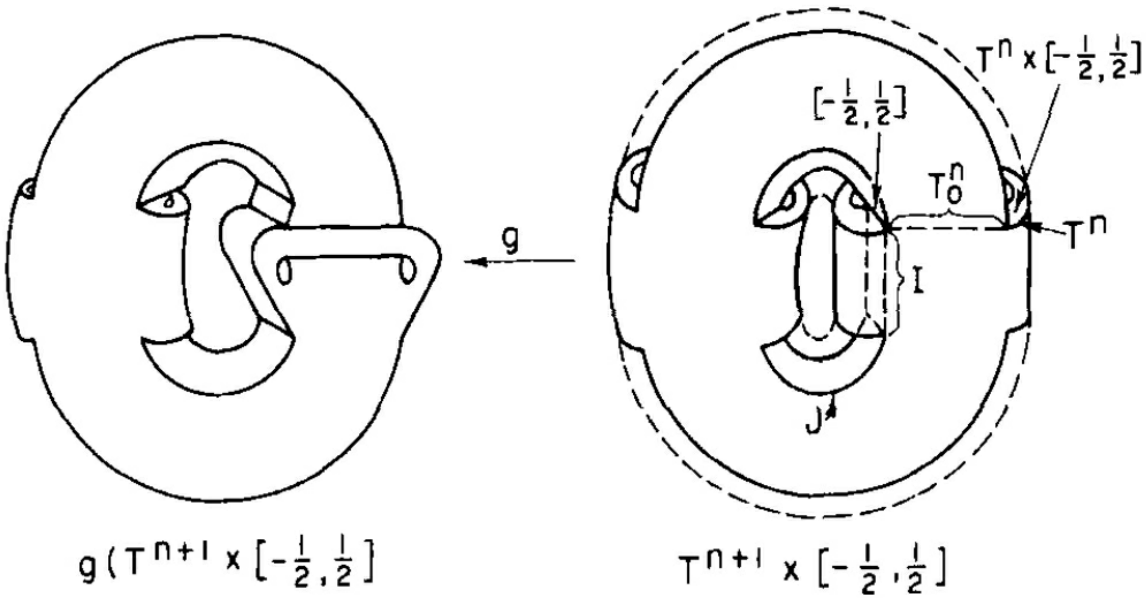


Figure 5.6.14

Product on $(\mathbb{T}^n - (n\text{-cell})) \times \mathbb{S}^1 \times \mathbb{I}$

$$\begin{array}{ccc} \mathbb{T}^n \times \mathbb{S}^1 \times \mathbb{I} & \xrightarrow{f \times id_{\mathbb{S}^1}} & \mathbb{R}^n \times \mathbb{S}^1 \times \mathbb{I} \\ & & \cap \\ & & \mathbb{R}^{n+1} \times \mathbb{I} \end{array}$$

It is now easy to see that such a g gives rise to an immersion as desired in the theorem. This would be a trivial matter of reparametrizing the I coordinate if we knew that $g(T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]) \subset E^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$. To get such an inclusion, one can shrink T_0^{n+1} a little, with the help of an interior collar, to T_1^{n+1}, and using the fact that $g \mid T_0^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$ is a product, isotop $g(T^{n+1} \times [-\frac{1}{2}, \frac{1}{2}])$ into $E^{n+1} \times [-\frac{1}{2}, \frac{1}{2}])$ keeping $g \mid T_1^{n+1} \times [-\frac{1}{2}, \frac{1}{2}]$ fixed.