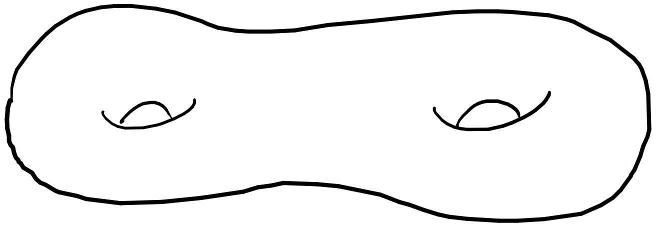
Top Flavours 2021, 17–18 June 2021, 25 min talk

Group trisections and smoothly knotted surfaces

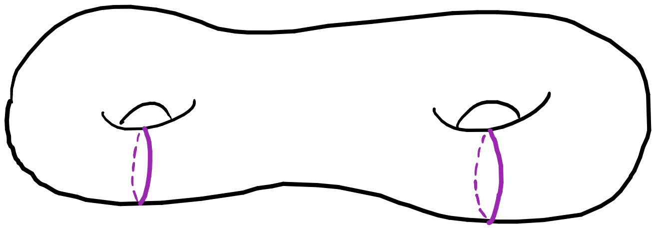
with Sarah Blackwell, Rob Kirby, Michael Klug and Vincent Longo

Benjamin Matthias Ruppik, 3rd year PhD student at the Max-Planck-Institute for Mathematics, Bonn

Handlebodies:



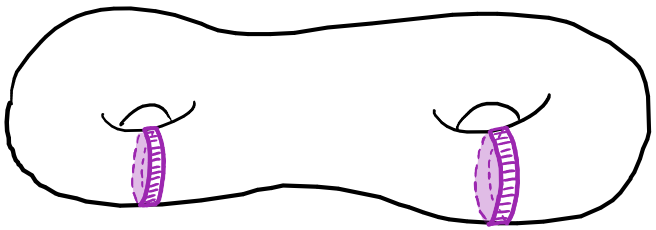
surface Σ_g



cut system of a handlebody:

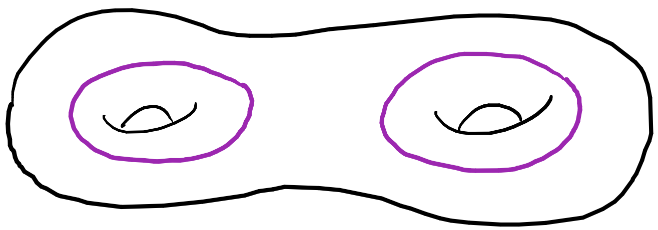
curves on Σ_g

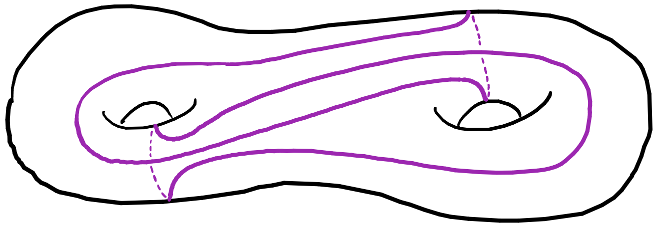
attach 2-handles along the curves

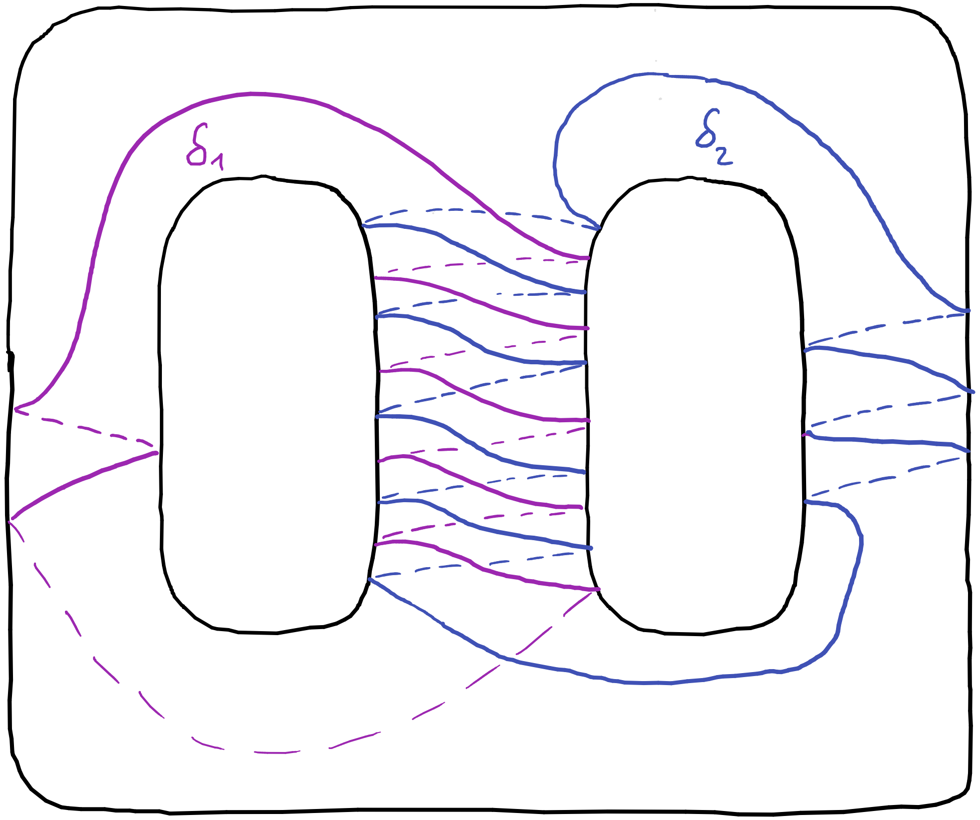


fill 2-sphere boundaries with 3-balls

Can you see the handlebodies?



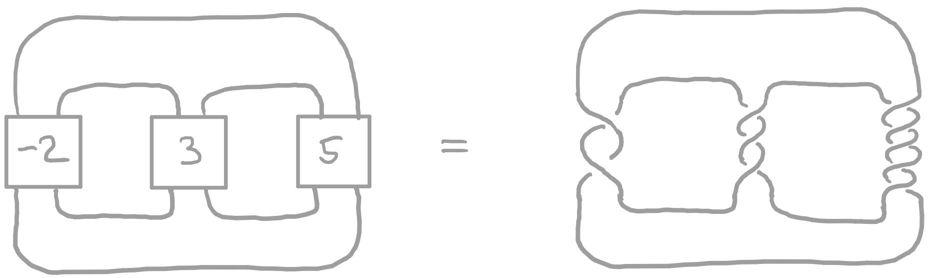


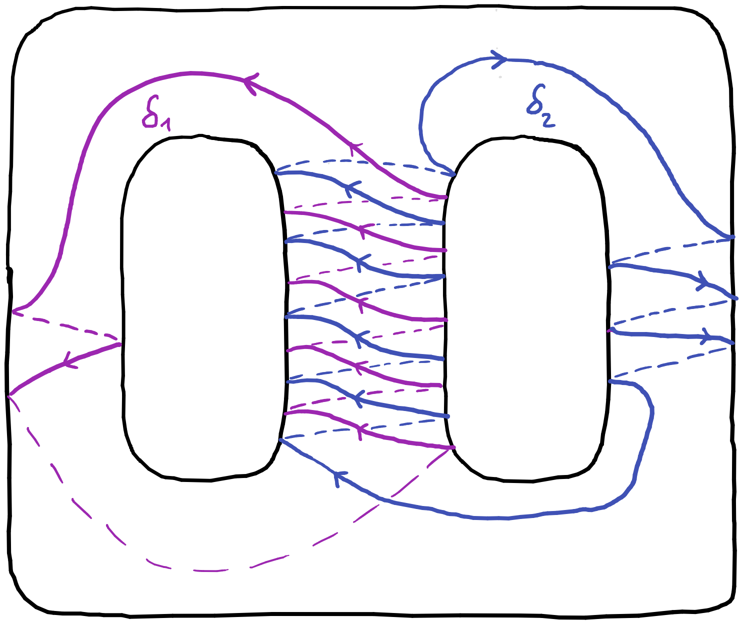


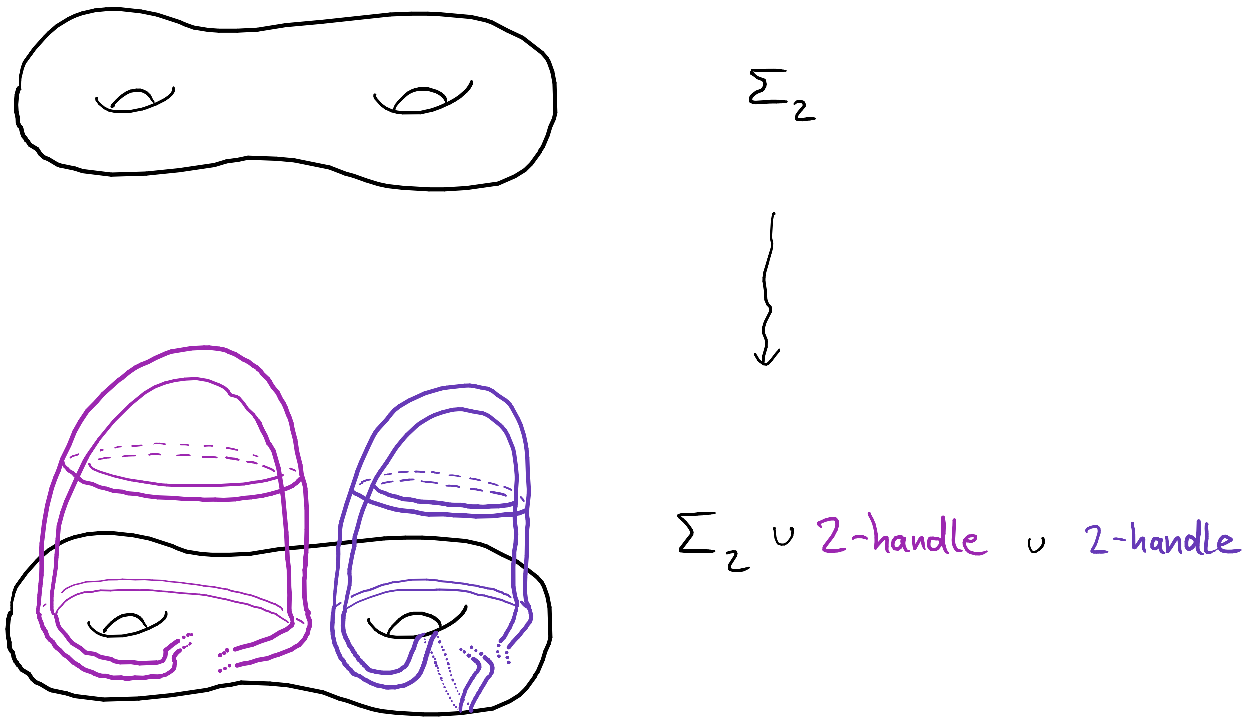
Side remark: This is one of the handlebodies in a genus 2 Heegaard diagram for the 3-mfld. P = Poincaré homology sphere

P = double branched cover $\Sigma_2(K)$ of S^3 branched over

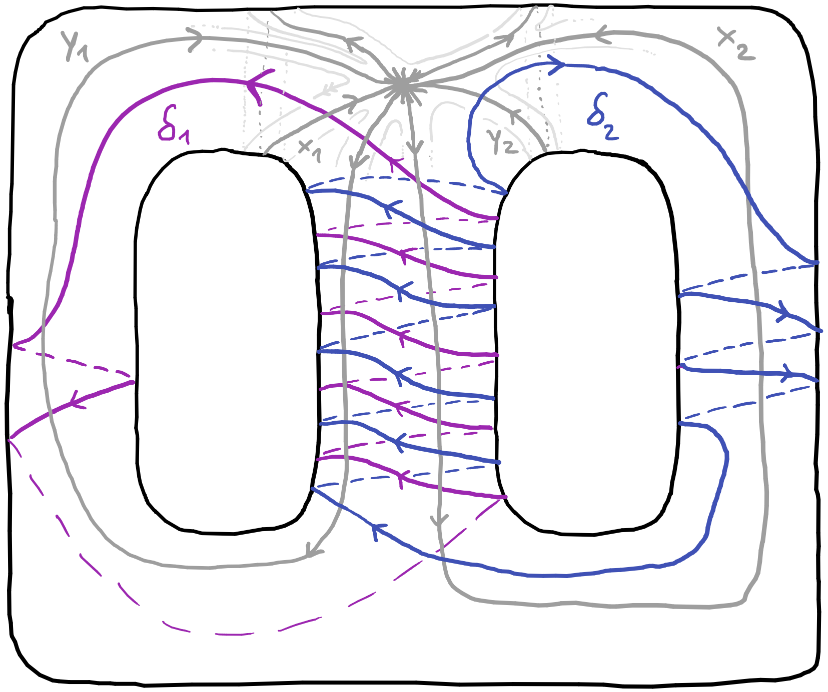
$K = (-2,3,5)$ Pretzel knot



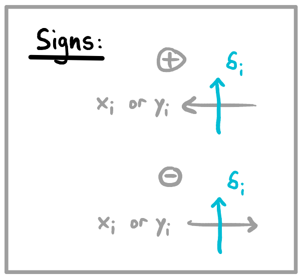

δ_1
δ_2


Σ_2
$\Sigma_2 \cup$ 2-handle $\cup$ 2-handle

Topology



Algebra



$$\pi_1(\text{surface}) \twoheadrightarrow \pi_1(\text{handlebody})$$

$$\langle x_1, y_1, x_2, y_2 \mid x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} \rangle \twoheadrightarrow \langle d_1, d_2 \rangle$$

$$x_1 \longmapsto d_1^{-1}$$

$$y_1 \longmapsto (d_1 d_2)^5 \cdot d_1^{-2}$$

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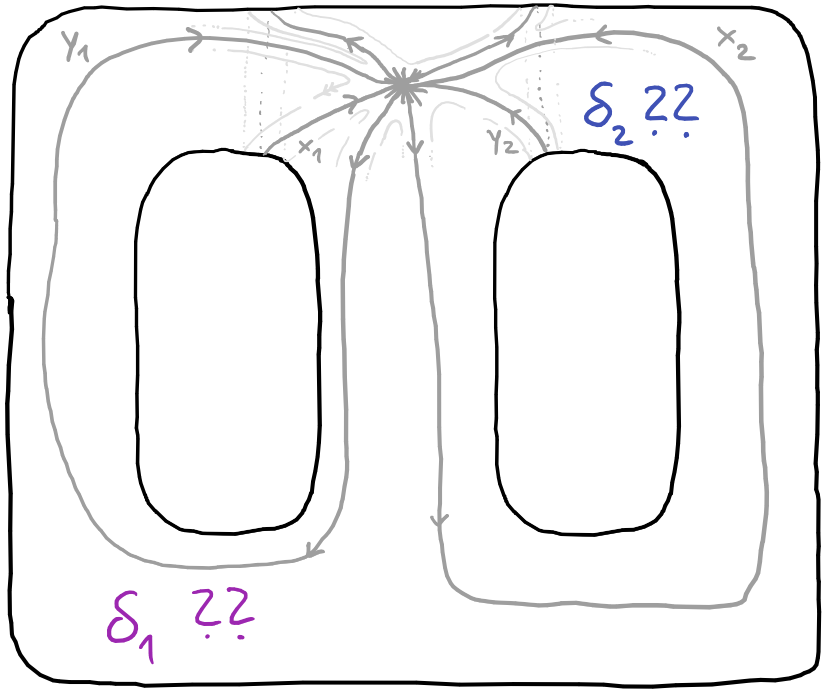

Topology

Algebra

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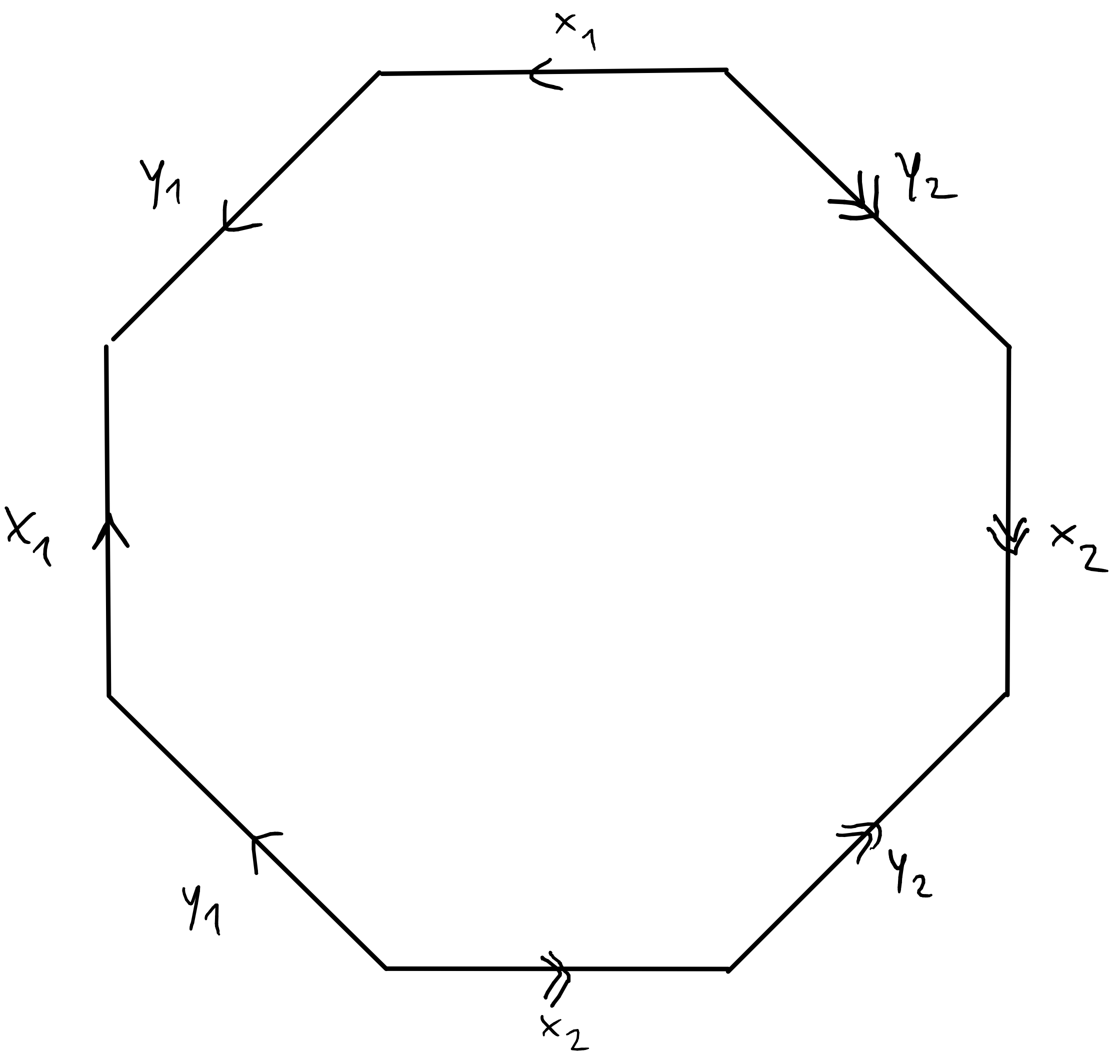
Surface relation: $x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2} (d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^3][d_2][d_2^{-3} (d_1 d_2)^{-5}][d_2^{-1}]$$



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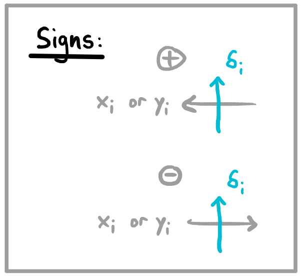
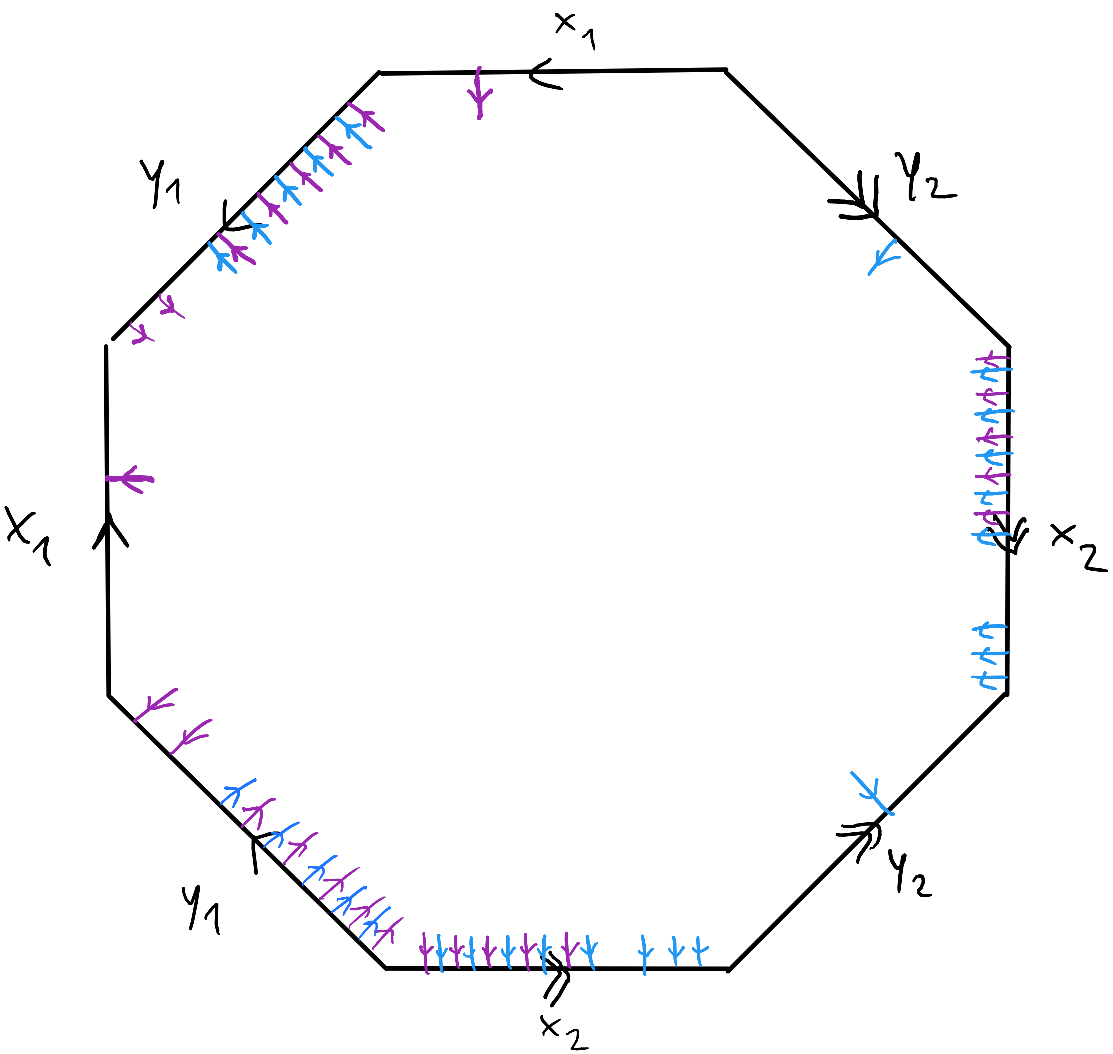


Colour coding: d_1, d_2

Surface relation: $\quad x_1 y_1 x_1^{-1} y_1^{-1} \; x_2 y_2 x_2^{-1} y_2^{-1} = 1$

$$\big\downarrow$$

$$[d_1^{-1}][(d_1 d_2)^5 d_1^{-2}][d_1][d_1^{2}(d_1 d_2)^{-5}][(d_1 d_2)^5 d_2^{3}][d_2][d_2^{-3}(d_1 d_2)^{-5}][d_2^{-1}]$$



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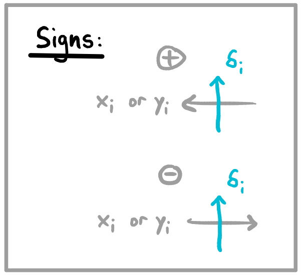

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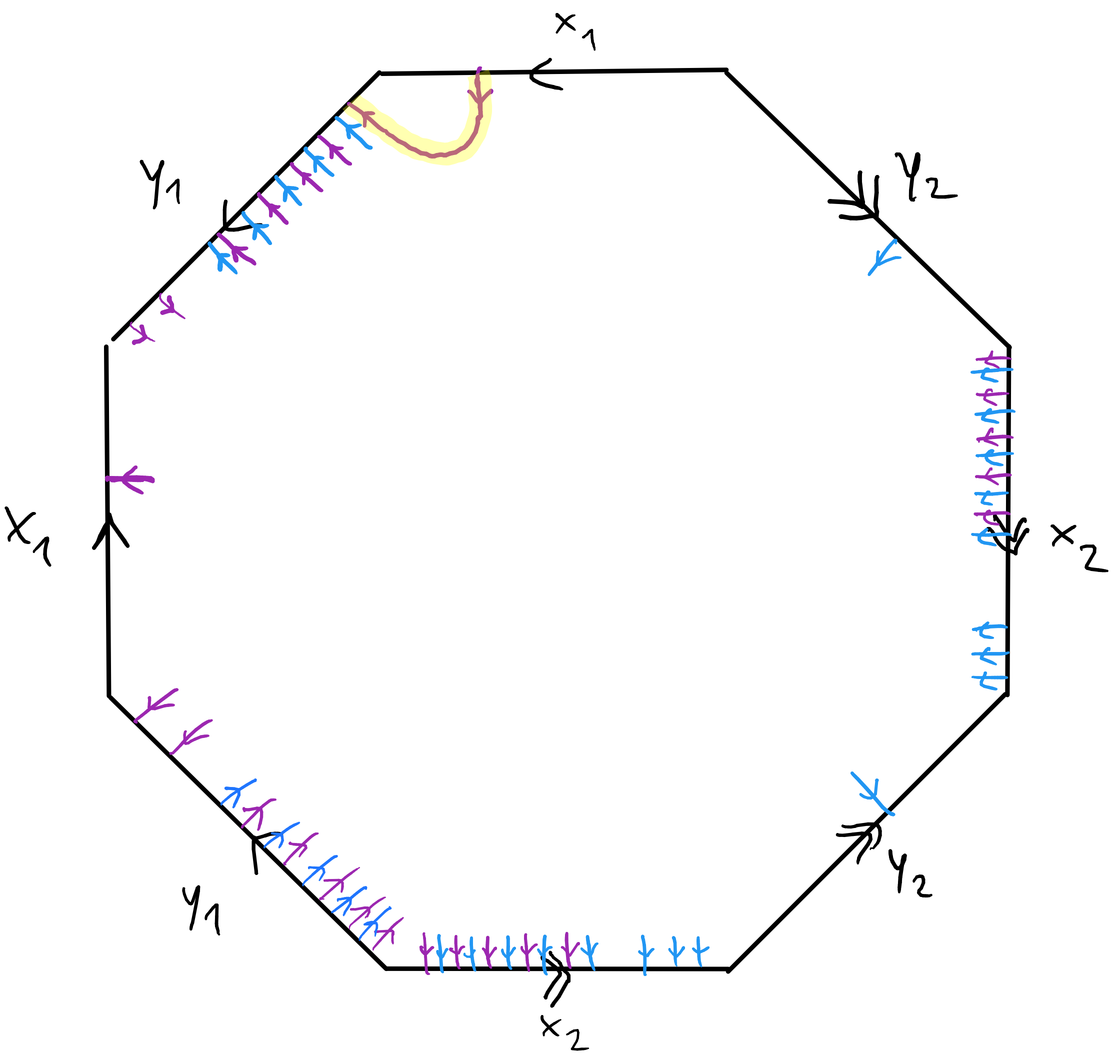

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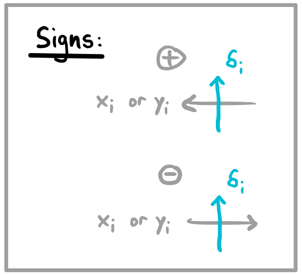

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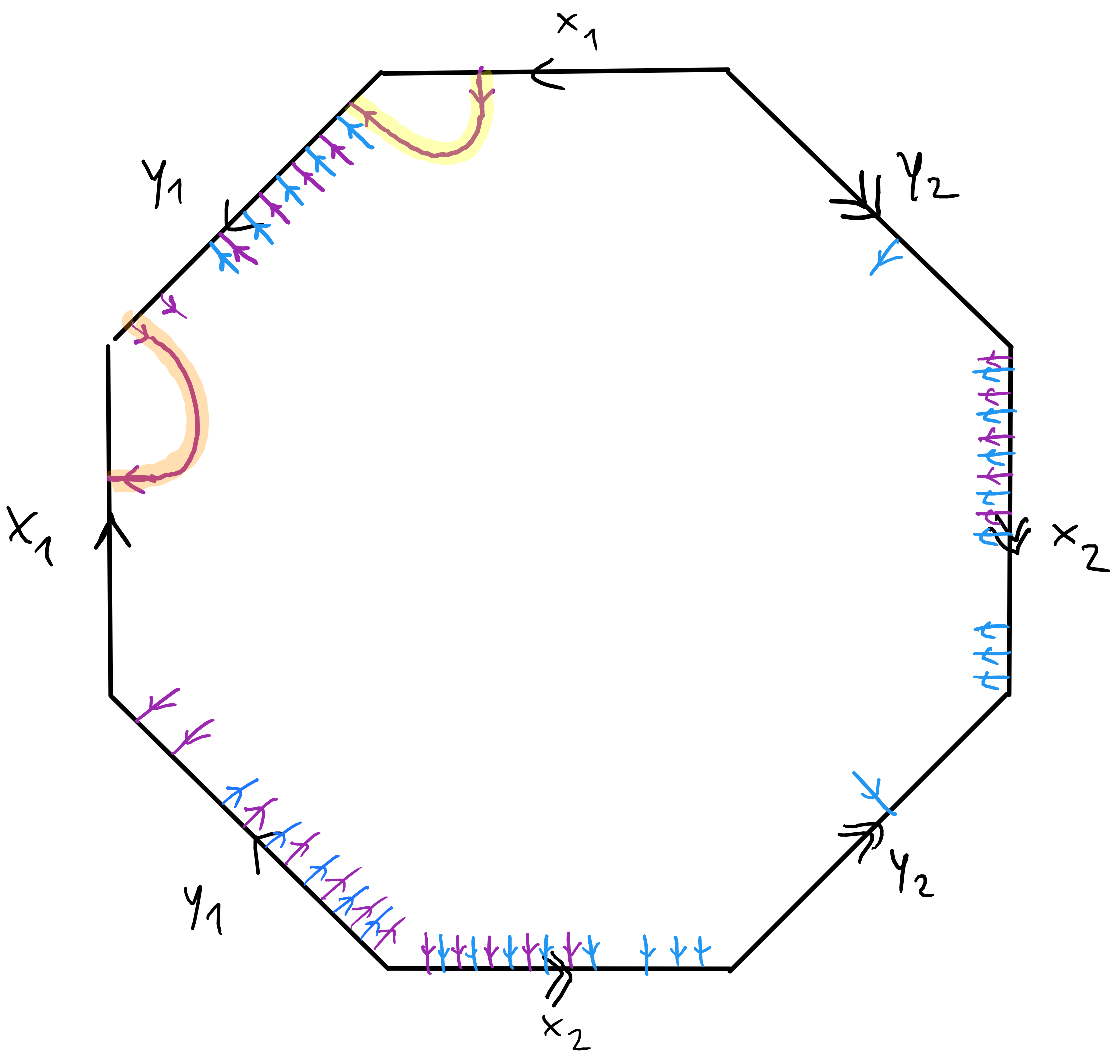

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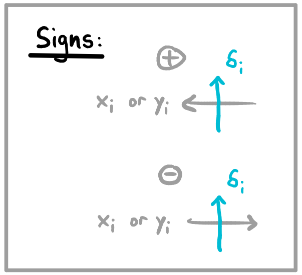

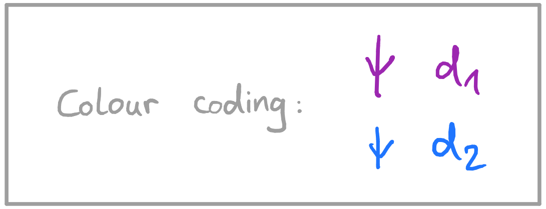

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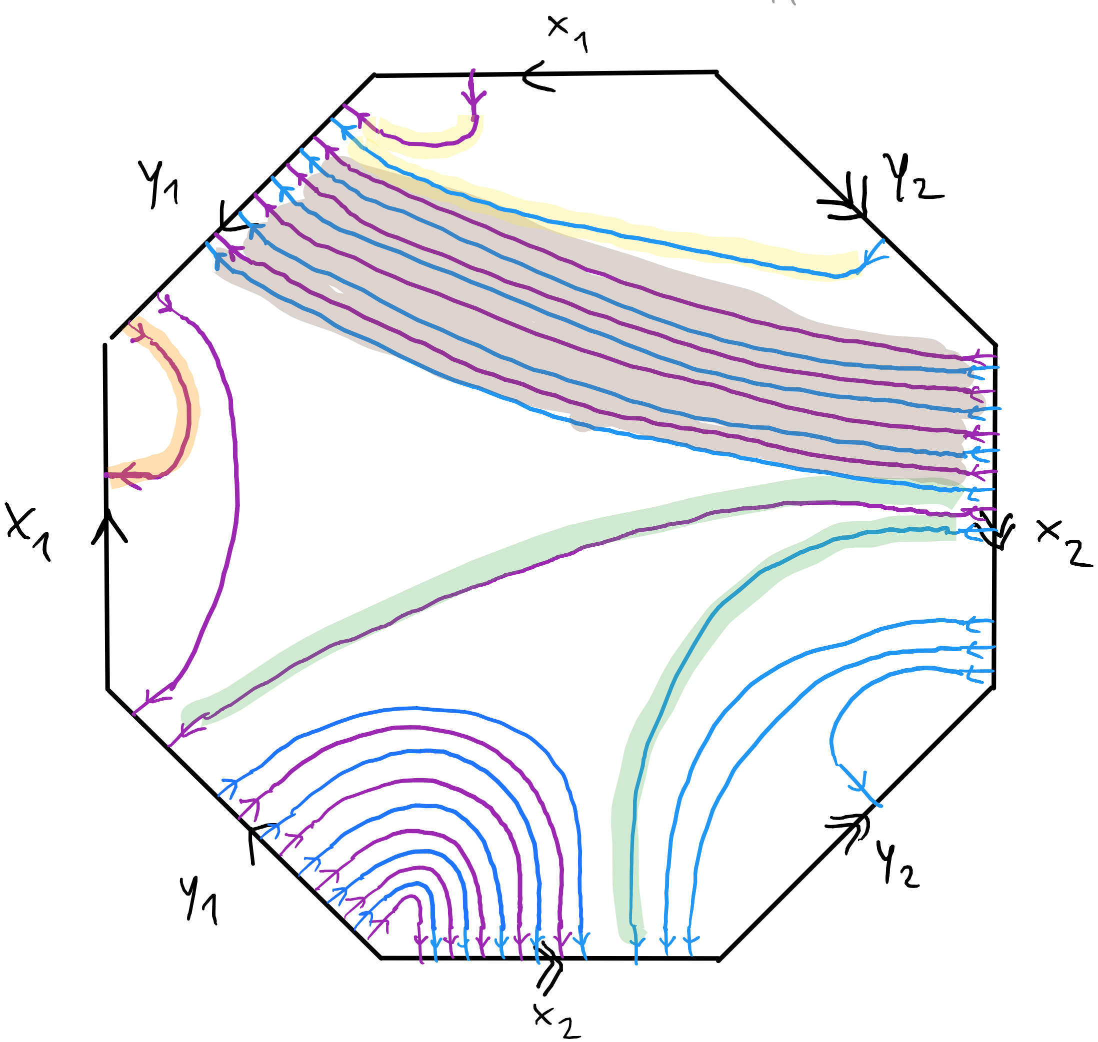
$$\underset{x_1}{[d_1^{-1}]}\underset{y_1}{[(d_1 d_2)^5 d_1^{-2}]}\underset{x_1^{-1}}{[d_1]}\underset{y_1^{-1}}{[d_1^{2}(d_1 d_2)^{-5}]}\underset{x_2}{[(d_1 d_2)^5 d_2^3]}\underset{y_2}{[d_2]}\underset{x_2^{-1}}{[d_2^{-3}(d_1 d_2)^{-5}]}\underset{y_2^{-1}}{[d_2^{-1}]}$$



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Signs:

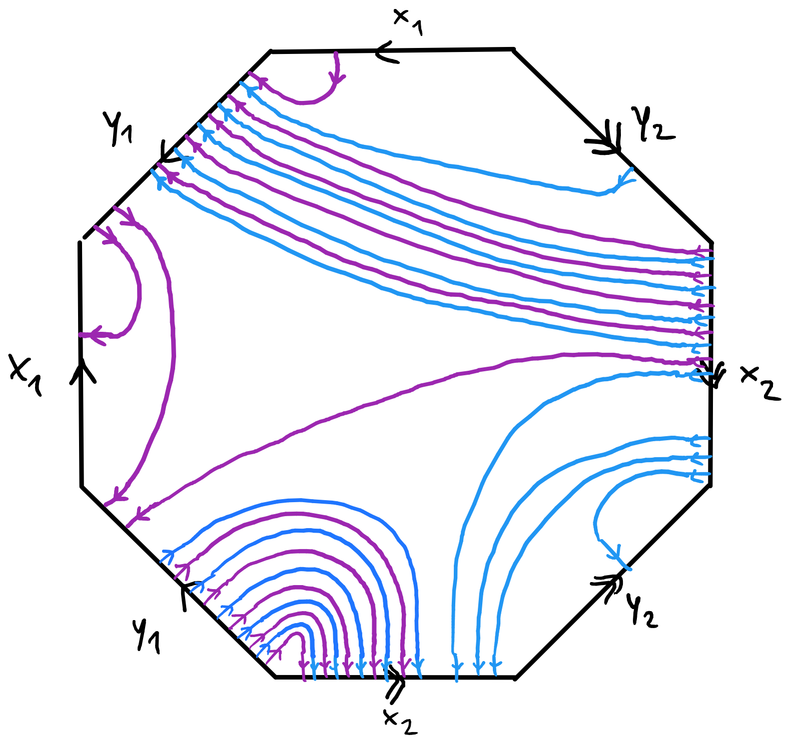
$\oplus$ δ_i, x_i or y_i

$\ominus$ δ_i, x_i or y_i

Colour coding: $\downarrow$ d_1 $\quad$ $\downarrow$ d_2

Topology

Algebra



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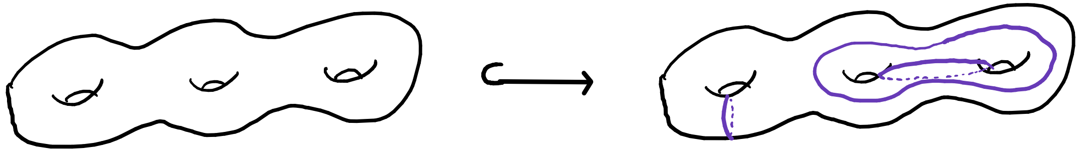
From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} \twoheadrightarrow F_g$

surface group $\twoheadrightarrow$ free group

is uniquely realized geometrically by a handlebody.

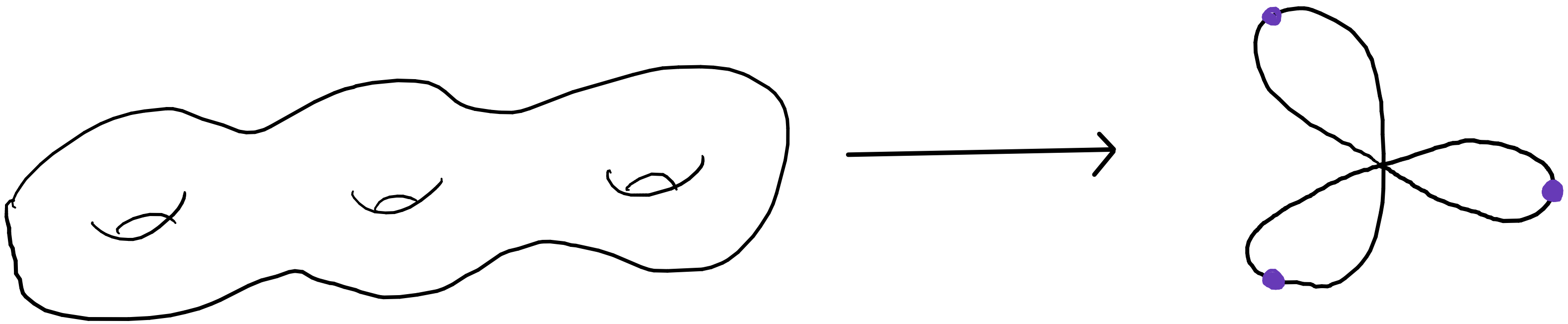


Folklore proof sketch:

Homomorphism $\pi_1(\Sigma_g) \overset{\varphi}{\twoheadrightarrow} F_g$

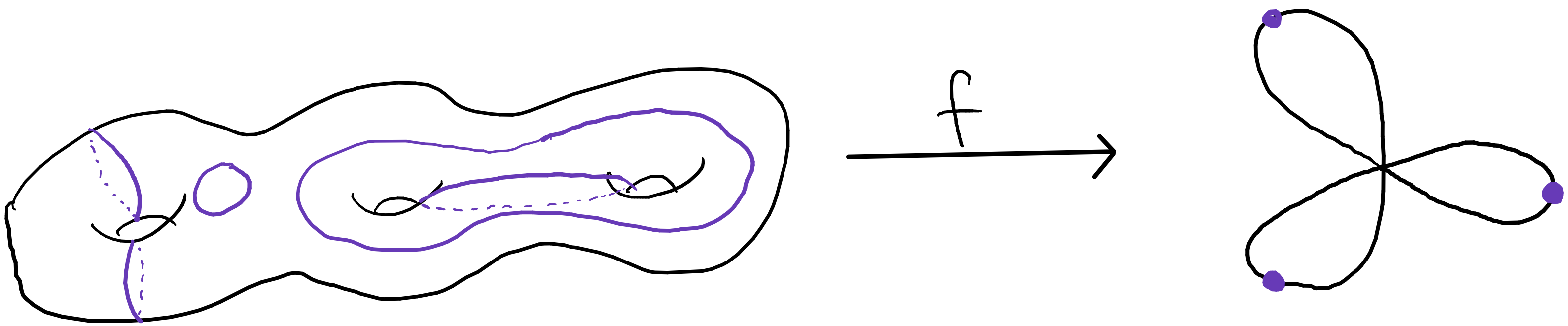
determines a unique map up to homotopy

$$\begin{tikzcd} \Sigma_g \arrow[r, "f"] \arrow[d, phantom, "\simeq" rotate=90] & \bigvee^g S^1 \arrow[d, phantom, "\simeq" rotate=90] \\ K(\pi_1(\Sigma_g),1) & K(F_g,1) \end{tikzcd}$$



make map transverse to north poles

$$\Sigma_g \xrightarrow{f} \bigvee^g S^1$$



look at preimage

f^{-1}(North poles)

make map transverse to north poles

Collection of simple closed curves in Σ_g, contains a cut system

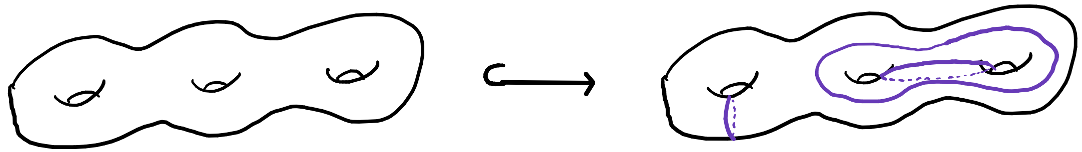
$\square$ (Folklore)

From algebra to topology

Folklore result: Any epimorphism $\pi_1(\Sigma_g) \xrightarrow{\varphi} \twoheadrightarrow F_{r_g}$

surface group $\twoheadrightarrow$ free group

is realized geometrically by a handlebody (uniquely) ...



[Blackwell-Kirby-Klug-Longo-R, 2021]

... which can be computed algorithmically.

Group trisections of a finitely presented group G:

Commutative cube

$$\begin{tikzcd}
 & \color{red}{Fr_g} \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_{k_1} \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \color{blue}{Fr_g} \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & Fr_{k_3} \arrow[rr, two heads] & & G \\
\color{green}{Fr_g} \arrow[ur, two heads] \arrow[rr, two heads] & & Fr_{k_2} \arrow[ur, two heads] &
\end{tikzcd}$$

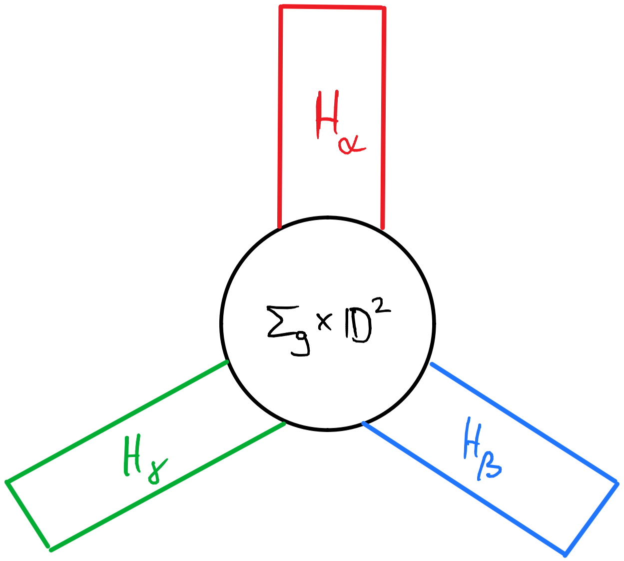
s.th. all maps are surjective

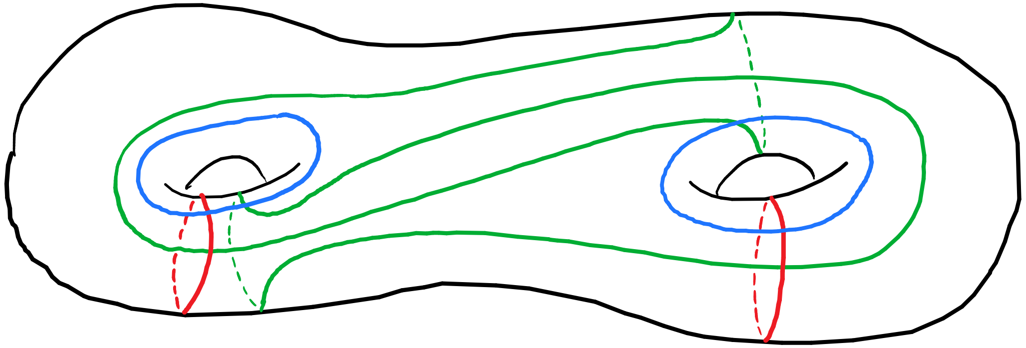
and all faces are push-outs

[Abrams, Gay, Kirby]

Group trisections of closed 4-manifolds:

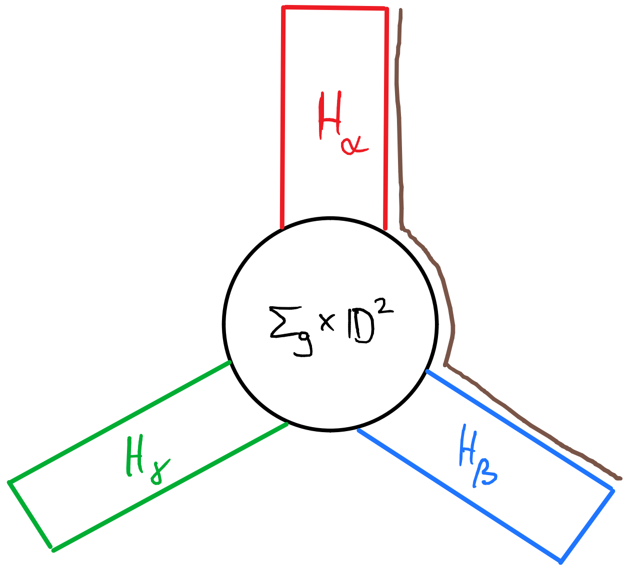
The handlebody-story three times

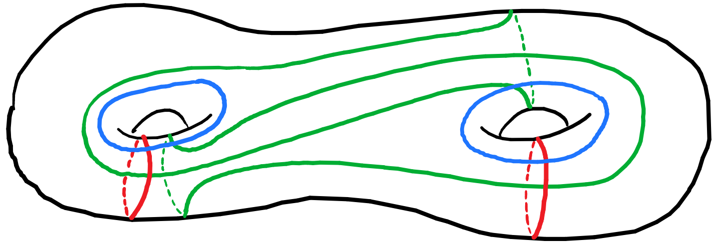




$$
\begin{tikzcd}
 & \pi_1(H_\alpha) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(H_\beta) \\
\pi_1(H_\gamma) &
\end{tikzcd}
$$

[Abrams, Gay, Kirby]





$$
\begin{tikzcd}
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\pi_1(H_\gamma) & &
\end{tikzcd}
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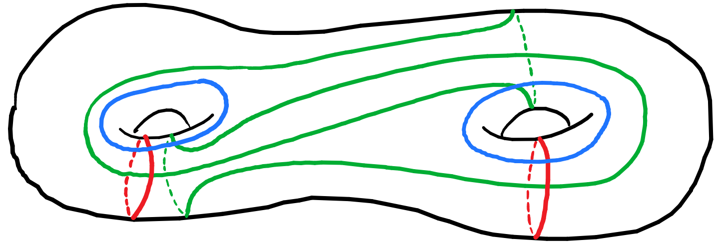
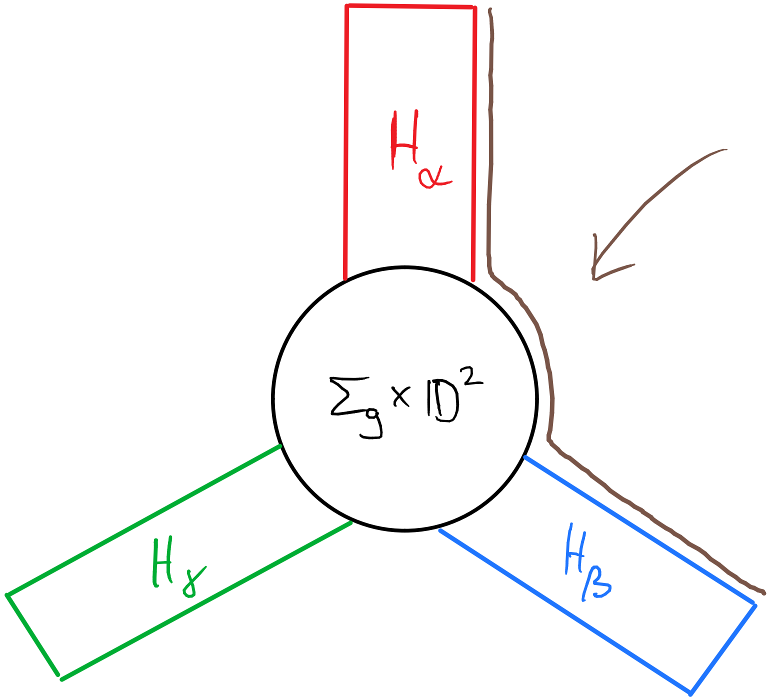
[Abrams, Gay, Kirby]





from our algebra assumption:

this is a closed 3-manifold M

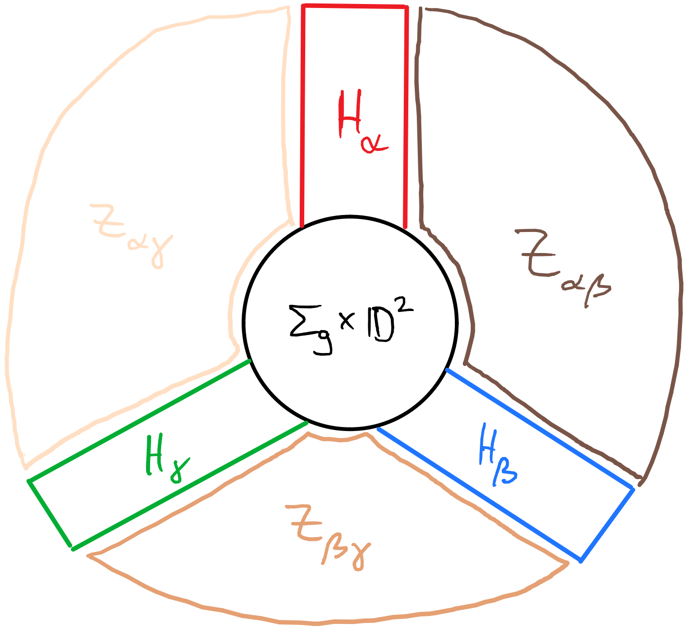
with $\pi_1(M) \cong Fr_k$ free

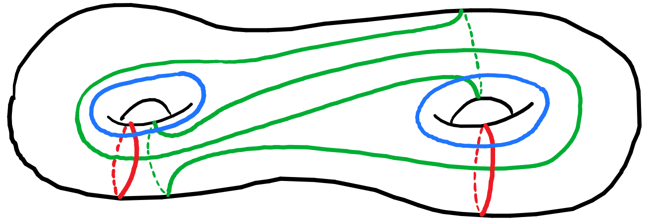
Kneser's thm. + 3D Poincaré conj.

$$\Longrightarrow \qquad M \cong \#^k S^1 \times S^2$$

[Abrams, Gay, Kirby]

[Laudenbach-Poenaru] allows us to fill the sectors uniquely with $\natural^{k_i} S^1 \times D^3$

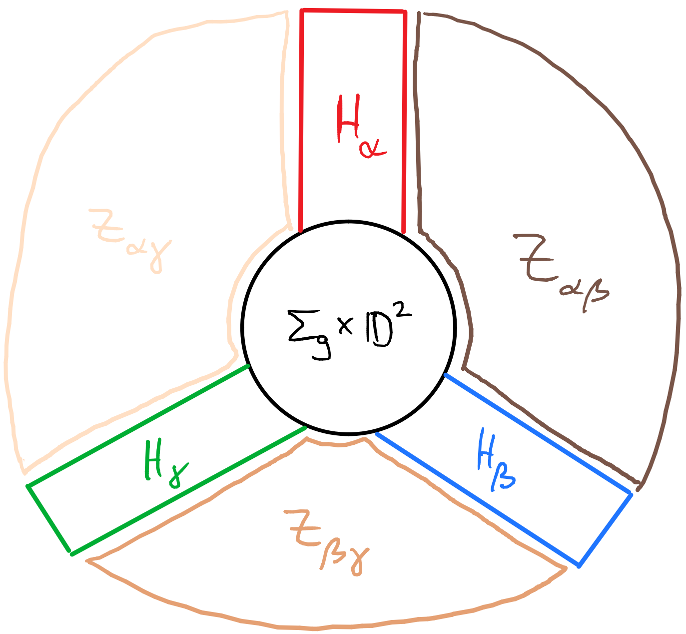




We can do this for all pairs of handlebodies

$$\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[r, two heads] \arrow[dd, two heads] & \pi_1(H_\alpha \cup_\Sigma H_\beta) \cong \pi_1(Z_{\alpha\beta}) \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[r, two heads] \arrow[ddd, two heads] & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(H_\alpha \cup_\Sigma H_\gamma) \cong \pi_1(Z_{\alpha\gamma}) & \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[r, two heads] & \pi_1(H_\beta \cup_\Sigma H_\gamma) \cong \pi_1(Z_{\beta\gamma}) &
\end{tikzcd}$$

[Abrams, Gay, Kirby]



4 – manifold M^4 with $\pi_1(M^4) \cong G$

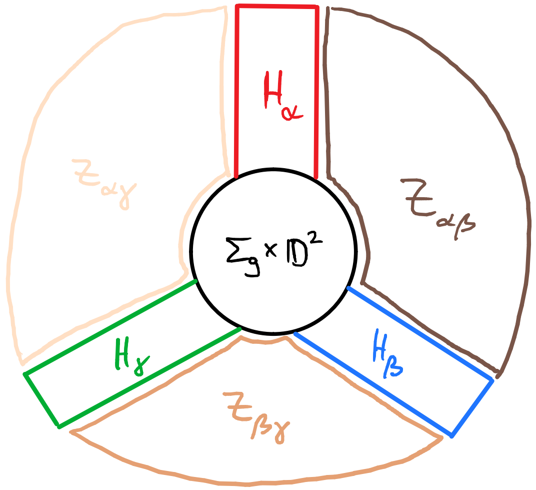
and group trisection corresponding to

the cube below

$$
\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(Z_{\alpha\beta}) \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(Z_{\alpha\gamma}) \arrow[rr, two heads] & & \pi_1(M) \cong G \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \pi_1(Z_{\beta\gamma}) \arrow[ur, two heads] &
\end{tikzcd}
$$

[Abrams, Gay, Kirby]

$\left\{ \begin{array}{l} \text{(based, parameterized)} \\ \textbf{trisections} \\ \text{of a 4-manifold } X^4 \end{array} \right.$



$\Big\}$

take π_1 of pieces $\downarrow$

$\boxed{\begin{array}{c} 1:1 \\ \text{[Abrams, Gay, Kirby]} \end{array}}$

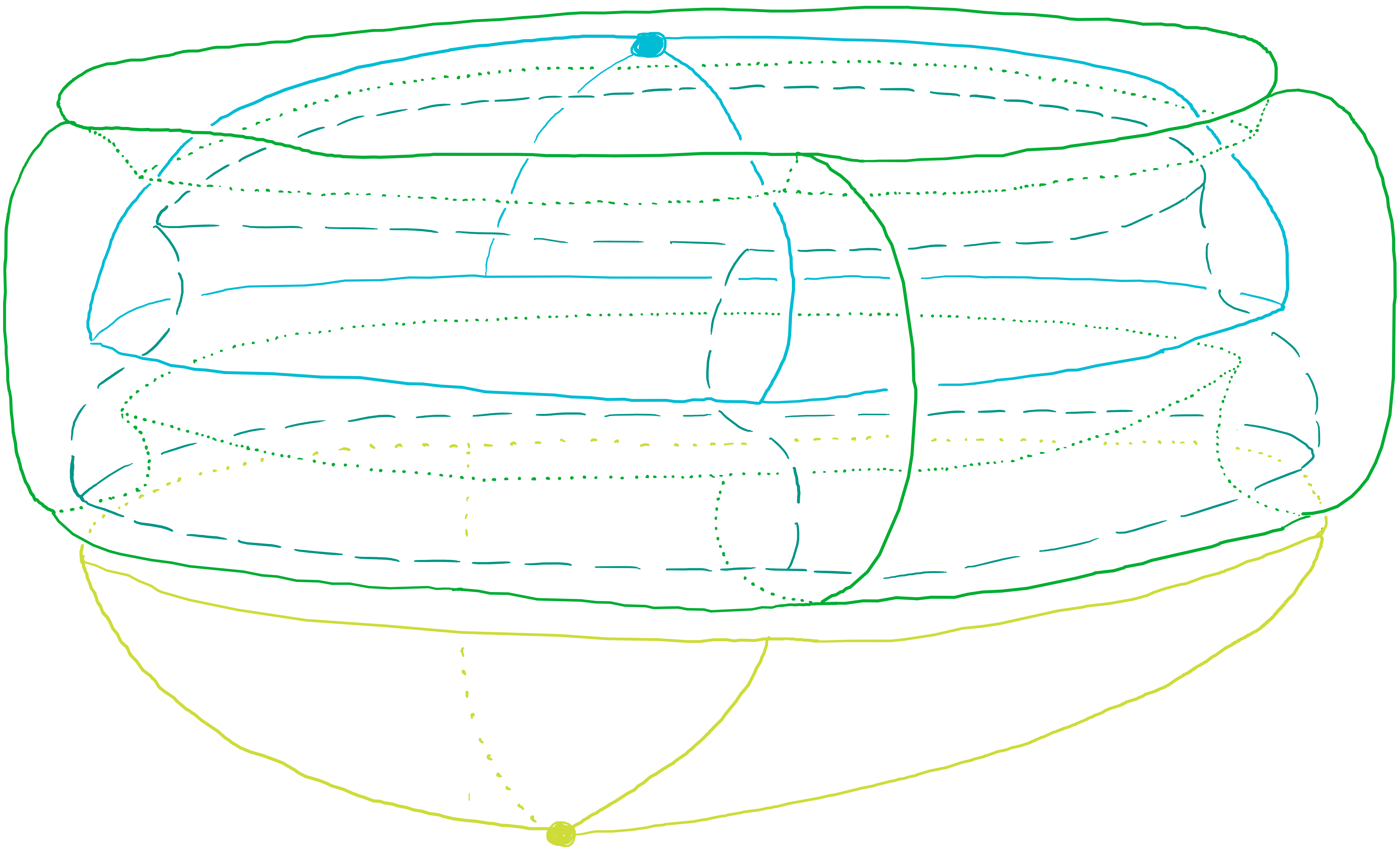
$\uparrow$ the previously explained construction

$\left\{ \begin{array}{l} \text{group} \\ \textbf{trisections} \\ \text{of } \pi_1(X,*) \end{array} \right.$

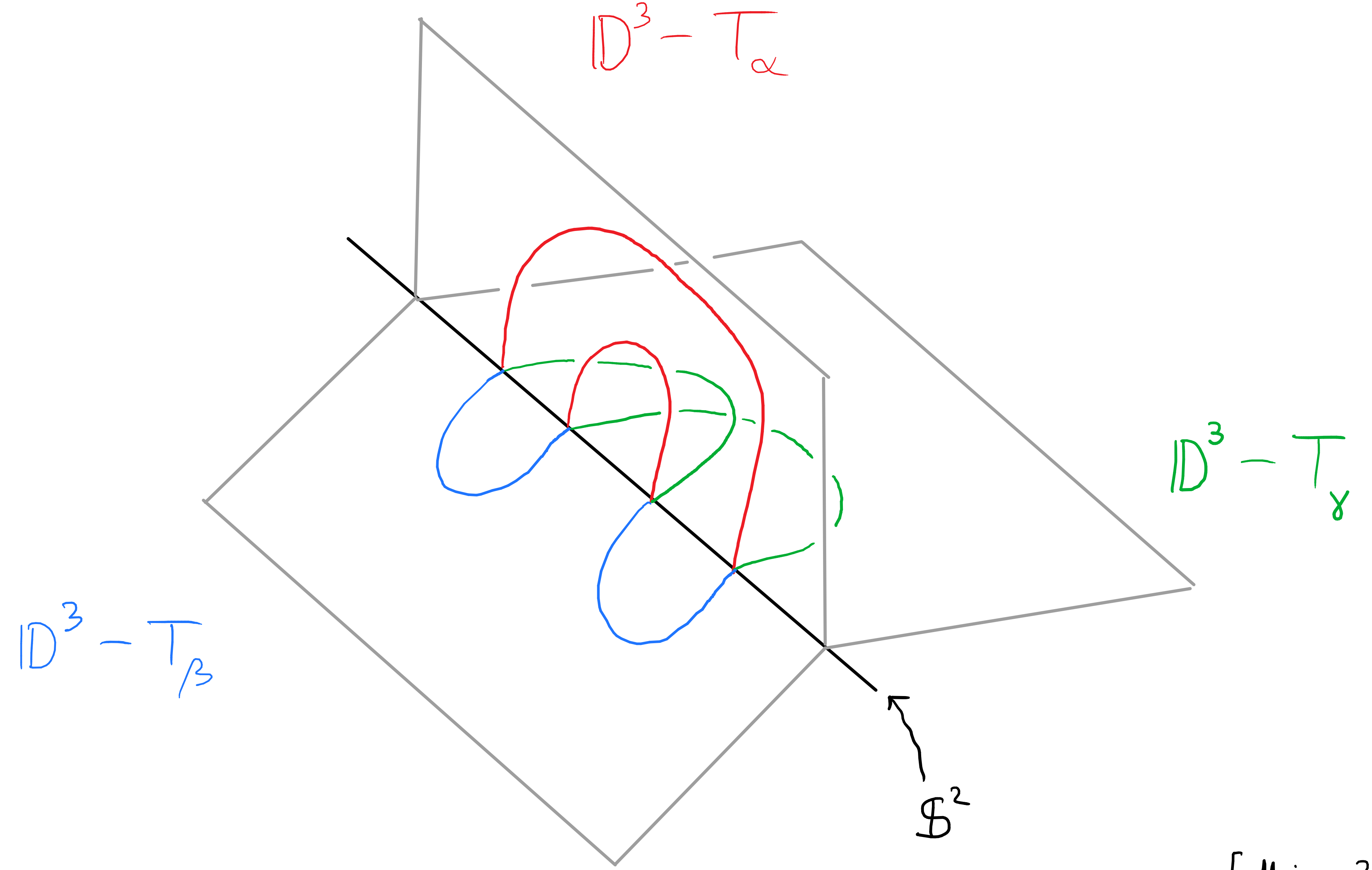
$$
\begin{tikzcd}
 & \pi_1(H_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(Z_{\alpha\beta}) \arrow[dd, two heads] \\
\pi_1(\Sigma_g) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(H_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(Z_{\alpha\gamma}) \arrow[rr, two heads] & & \pi_1(X) \\
\pi_1(H_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \pi_1(Z_{\beta\gamma}) \arrow[ur, two heads] &
\end{tikzcd}
$$

$\Big\}$

Spun trefoil - a knotted surface in S^4

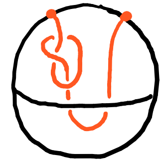


Bridge-trisected surfaces in the 4-sphere

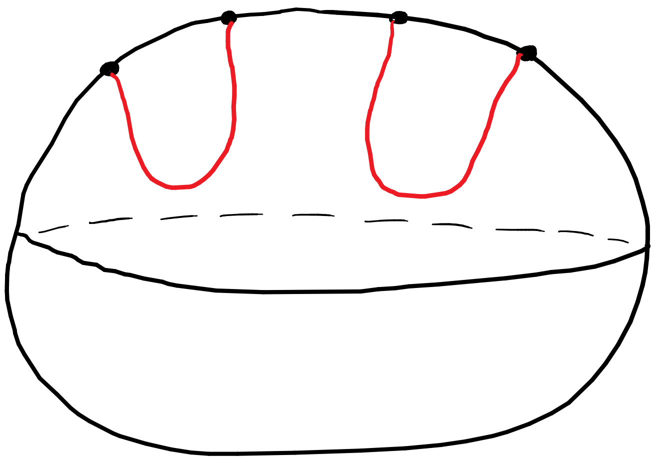


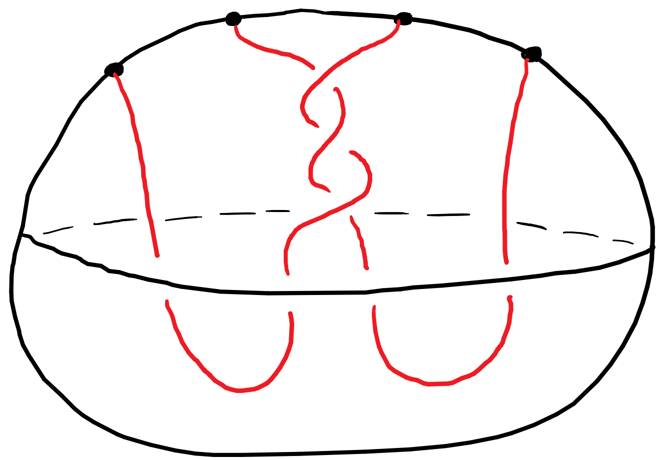
[Meier, Zupan]

Trivial tangles in 3-balls (and in handlebodies)

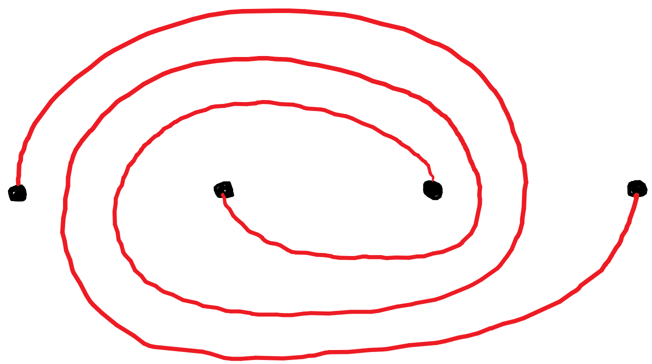


not a trivial tangle

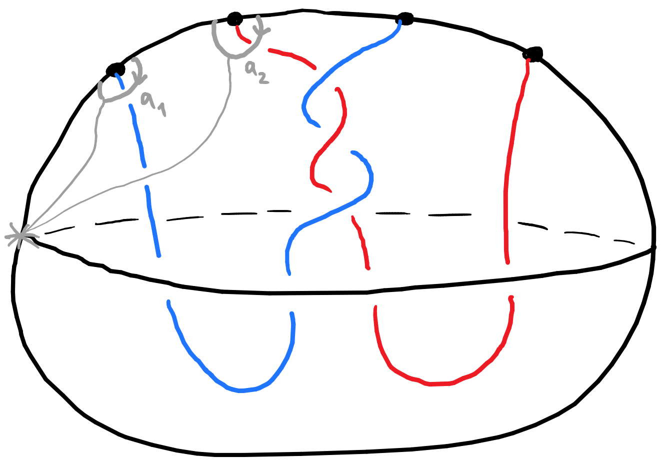


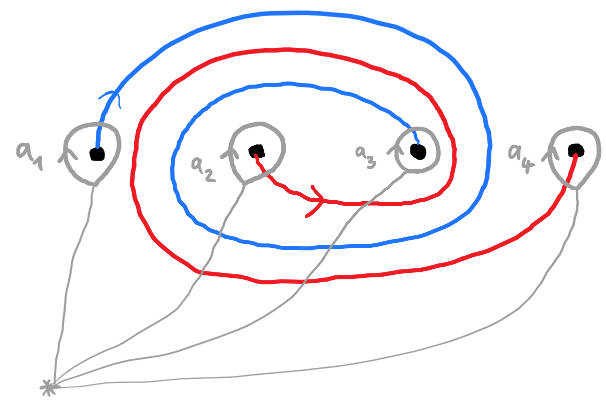


We like to draw the "shadows" of the tangles on a punctured plane:

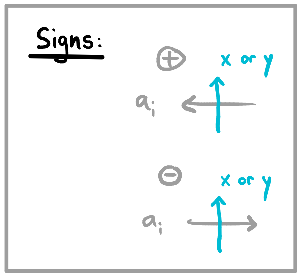


Topology





Algebra



$$\pi_1\left(\mathbb{S}^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1\left(\mathbb{D}^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

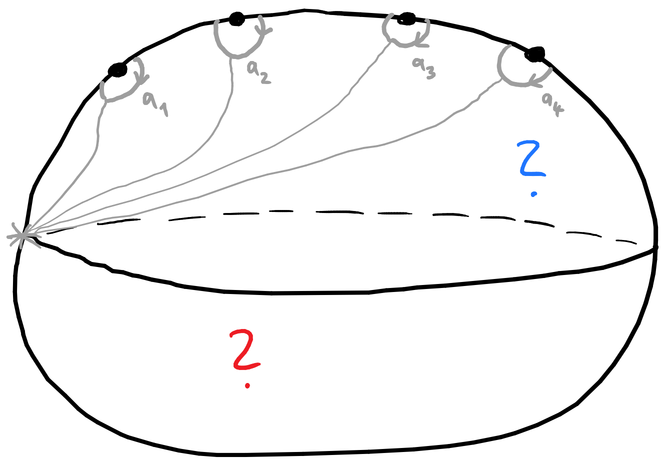
$$a_1 \longmapsto x^{-1}$$

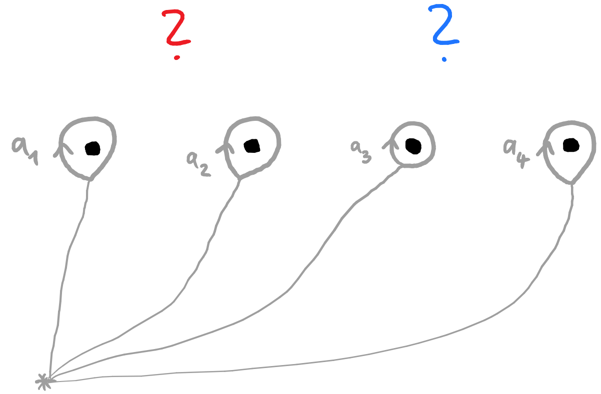
$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

$$a_4 \longmapsto y$$

Topology





Algebra

$$\pi_1\left(\mathbb{S}^2 - \left\{\begin{smallmatrix}\text{4 bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1\left(\mathbb{D}^3 - \text{tangle}\right)$$

$$\langle a_1, a_2, a_3, a_4 \mid a_1 \cdot a_2 \cdot a_3 \cdot a_4 \rangle \twoheadrightarrow \langle x, y \rangle$$

$$a_1 \longmapsto x^{-1}$$

$$a_2 \longmapsto y\, x^{-1}\, y^{-1}\, x\, y^{-1}$$

$$a_3 \longmapsto y\, x^{-1}\, y\, x\, y^{-1}\, x\, y^{-1}$$

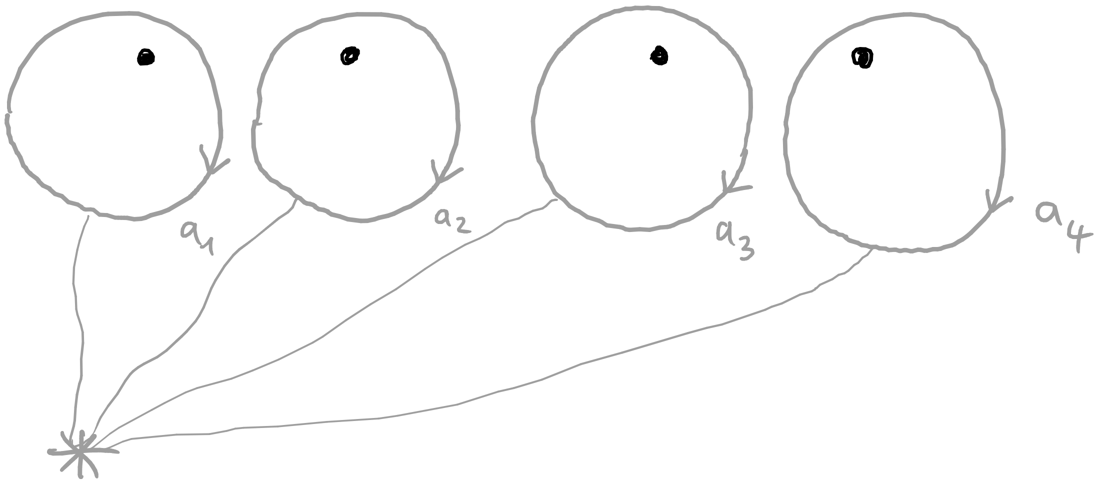
$$a_4 \longmapsto y$$

Punctured Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

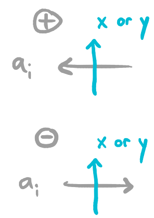
$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge} \\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto yxy^{-1}$$
$$a_2 \longmapsto yx^{-1}y^{-1}$$
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$
$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



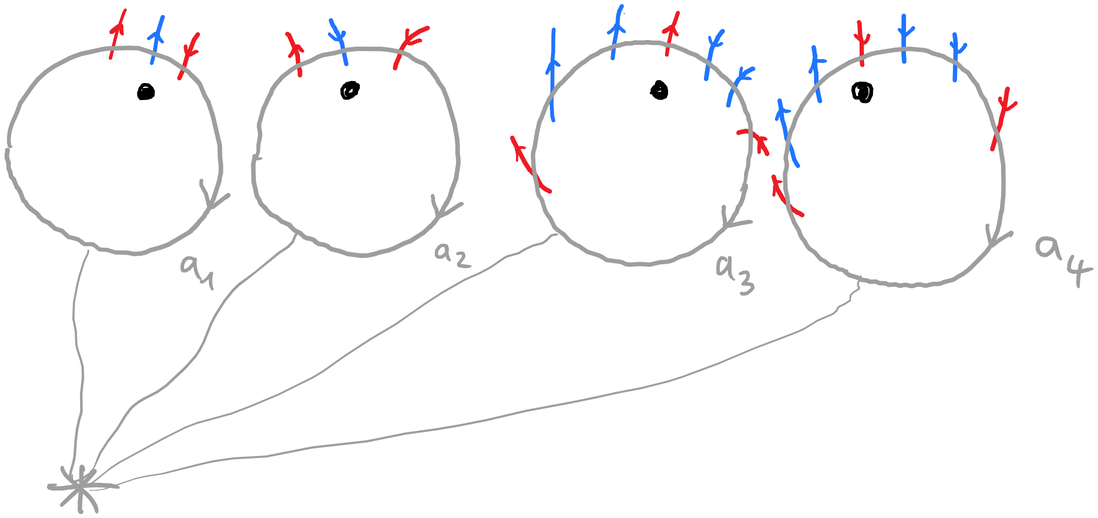
Colour coding: x y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

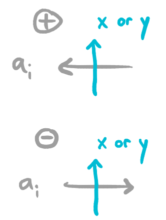
$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



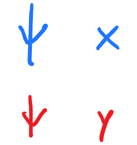
$$\pi_1(S^2 - \{\text{4 bridge points}\}) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto yxy^{-1}$$
$$a_2 \longmapsto yx^{-1}y^{-1}$$
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$
$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



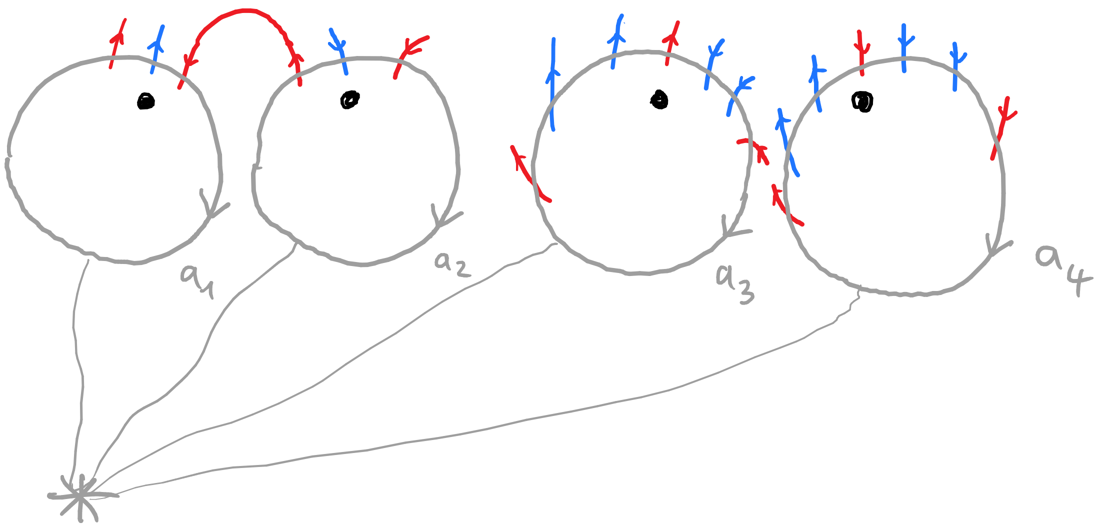
Colour coding:



Surface relation: $\quad a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$

$$\downarrow$$

$$[y x y^{-1}][y x^{-1} y^{-1}][y x x y x^{-1} x^{-1} y^{-1}][y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{smallmatrix}4 \text{ bridge}\\ \text{points}\end{smallmatrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

$$a_1 \longmapsto y x y^{-1}$$
$$a_2 \longmapsto y x^{-1} y^{-1}$$
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$
$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:

$\oplus$ x or y a_i

$\ominus$ x or y a_i

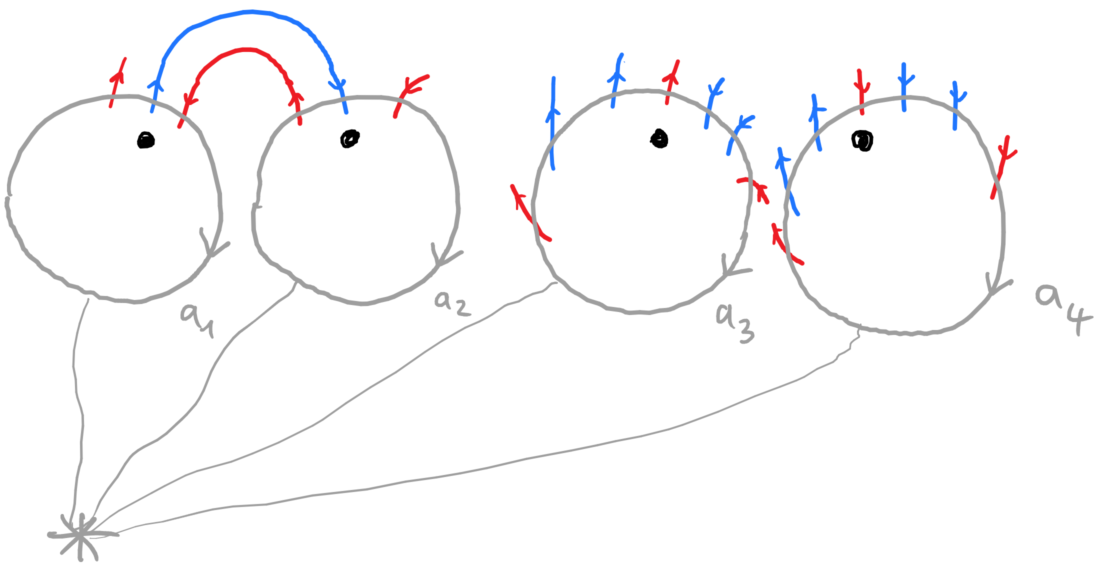
Colour coding: x y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

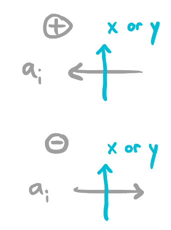
$$[y\,\cancel{x}\,\cancel{y^{-1}}]\,[\cancel{y}\,\cancel{x^{-1}}\,y^{-1}]\,[y x x y x^{-1} x^{-1} y^{-1}]\,[y x x y^{-1} x^{-1} x^{-1} y^{-1}]$$



$$\pi_1\left(S^2 - \left\{ \begin{smallmatrix} 4 \text{ bridge} \\ \text{points} \end{smallmatrix} \right\}\right) \longrightarrow \pi_1\left(D^3 - \text{tangle}\right)$$

$$a_1 \longmapsto y x y^{-1}$$
$$a_2 \longmapsto y x^{-1} y^{-1}$$
$$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$$
$$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$$

Signs:



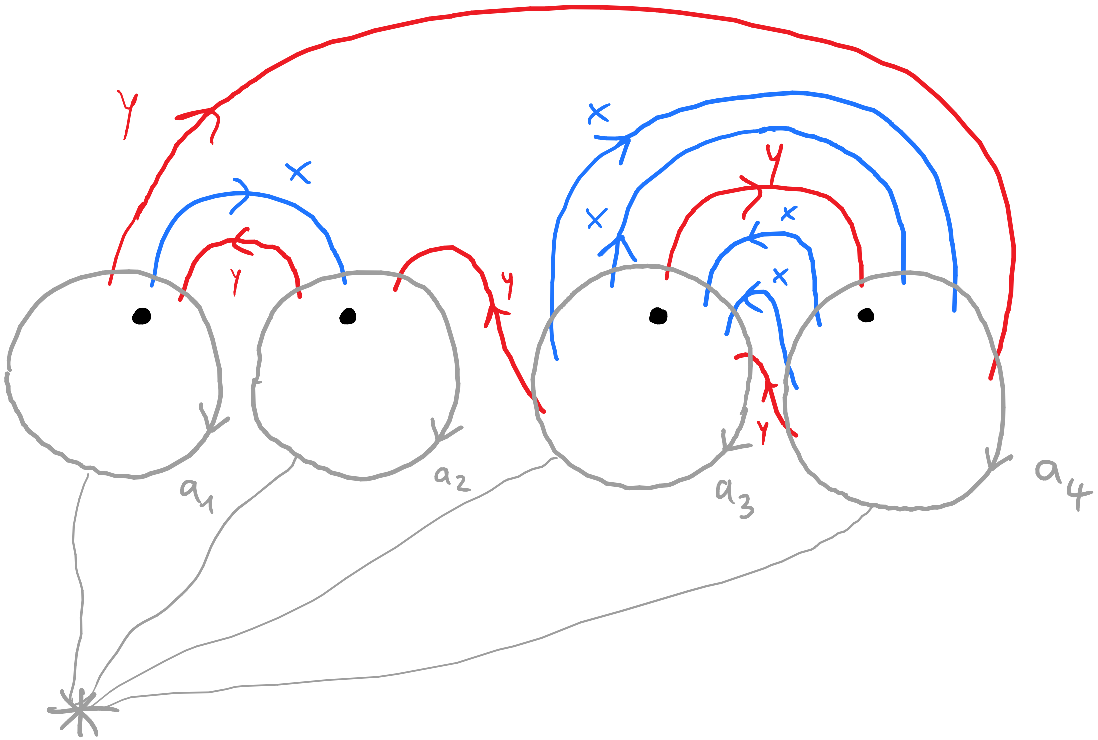
Colour coding: x y

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

$$[\cancel{y}\cancel{x}\cancel{y^{-1}}][\cancel{y}\cancel{x^{-1}}\cancel{y^{-1}}][\cancel{y}\cancel{x}\cancel{x}\cancel{y}\cancel{x^{-1}}\cancel{x^{-1}}\cancel{y^{-1}}][\cancel{y}\cancel{x}\cancel{x}\cancel{y^{-1}}\cancel{x^{-1}}\cancel{x^{-1}}\cancel{y^{-1}}]$$



$\pi_1 \left(S^2 - \{ \text{4 bridge points} \} \right) \longrightarrow \pi_1 \left(D^3 - \text{tangle} \right)$

$a_1 \longmapsto y x y^{-1}$

$a_2 \longmapsto y x^{-1} y^{-1}$

$a_3 \longmapsto y x x y x^{-1} x^{-1} y^{-1}$

$a_4 \longmapsto y x x y^{-1} x^{-1} x^{-1} y^{-1}$

Signs:

$\oplus$ x or y, a_i

$\ominus$ x or y, a_i

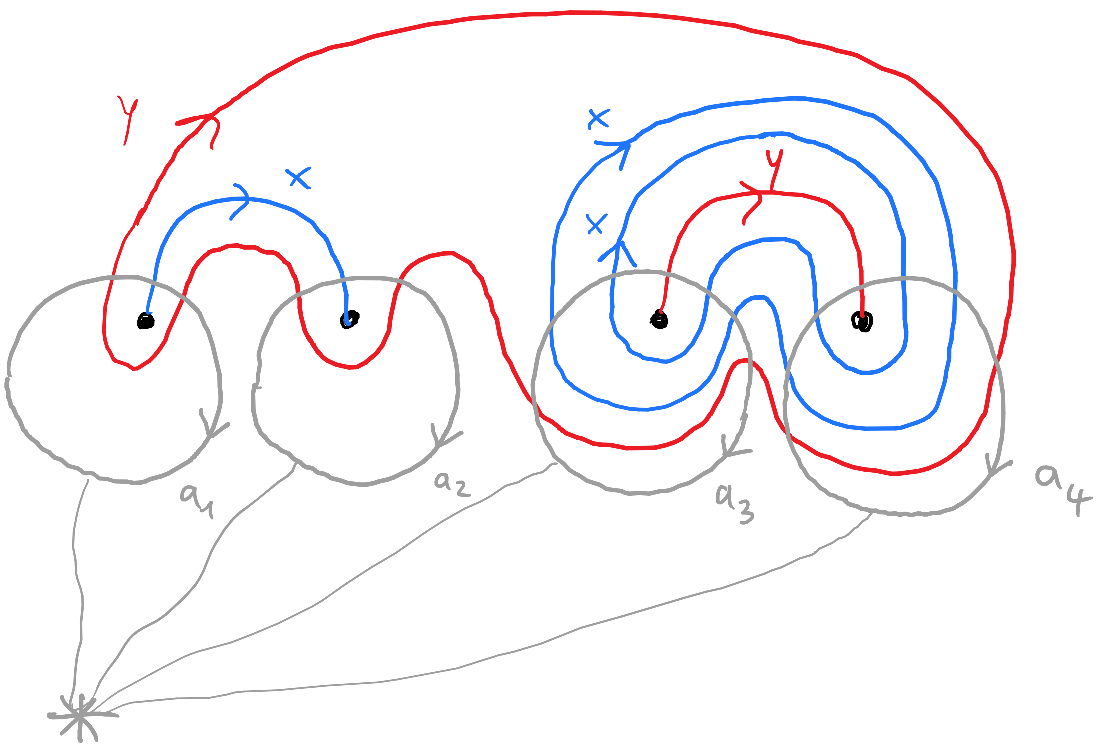
Colour coding: x (blue), y (red)

Surface relation:

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 = 1$$

$$\downarrow$$

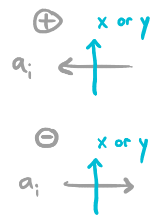
$$[yxy^{-1}][yx^{-1}y^{-1}][yxxyx^{-1}x^{-1}y^{-1}][yxxy^{-1}x^{-1}x^{-1}y^{-1}]$$



$$\pi_1\left(S^2 - \left\{\begin{matrix}4 \text{ bridge}\\ \text{points}\end{matrix}\right\}\right) \longrightarrow \pi_1(D^3 - \text{tangle})$$

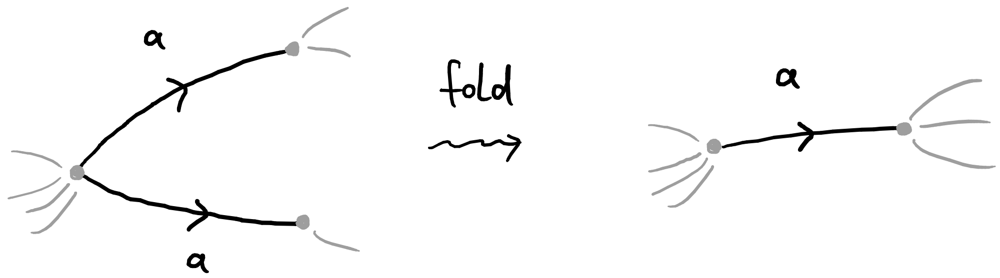
$$a_1 \longmapsto yxy^{-1}$$
$$a_2 \longmapsto yx^{-1}y^{-1}$$
$$a_3 \longmapsto yxxyx^{-1}x^{-1}y^{-1}$$
$$a_4 \longmapsto yxxy^{-1}x^{-1}x^{-1}y^{-1}$$

Signs:



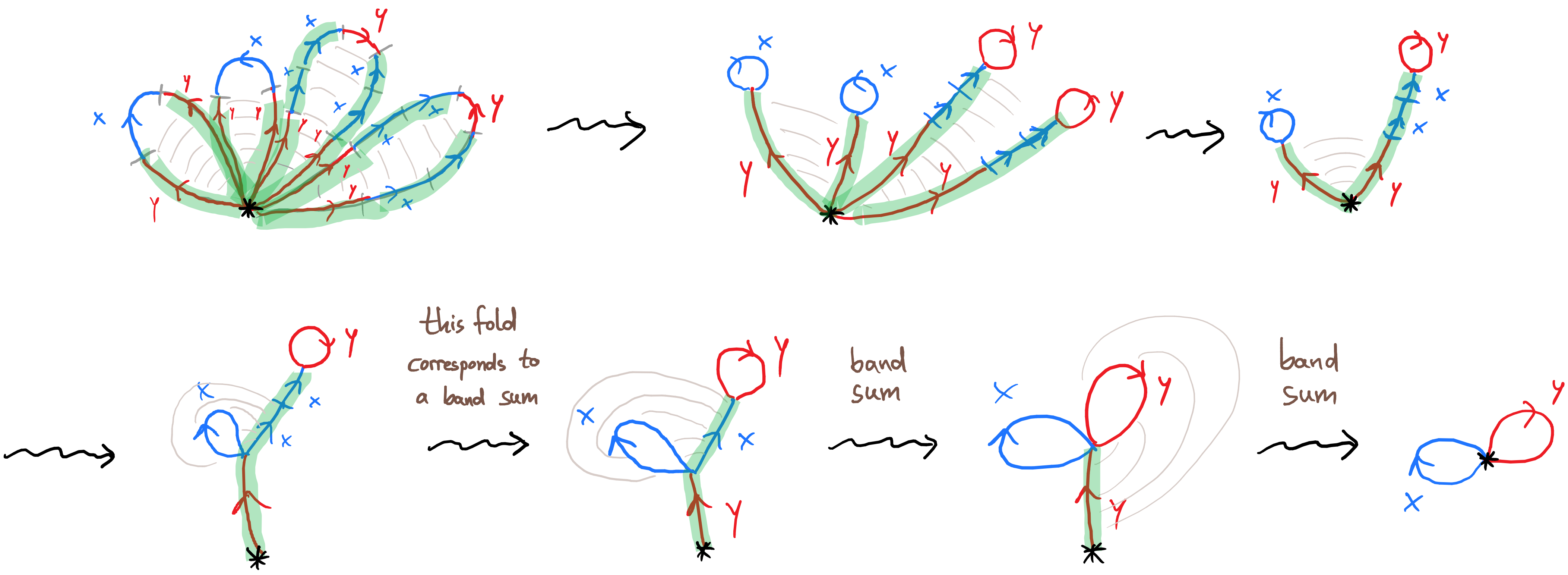
Colour coding: x y

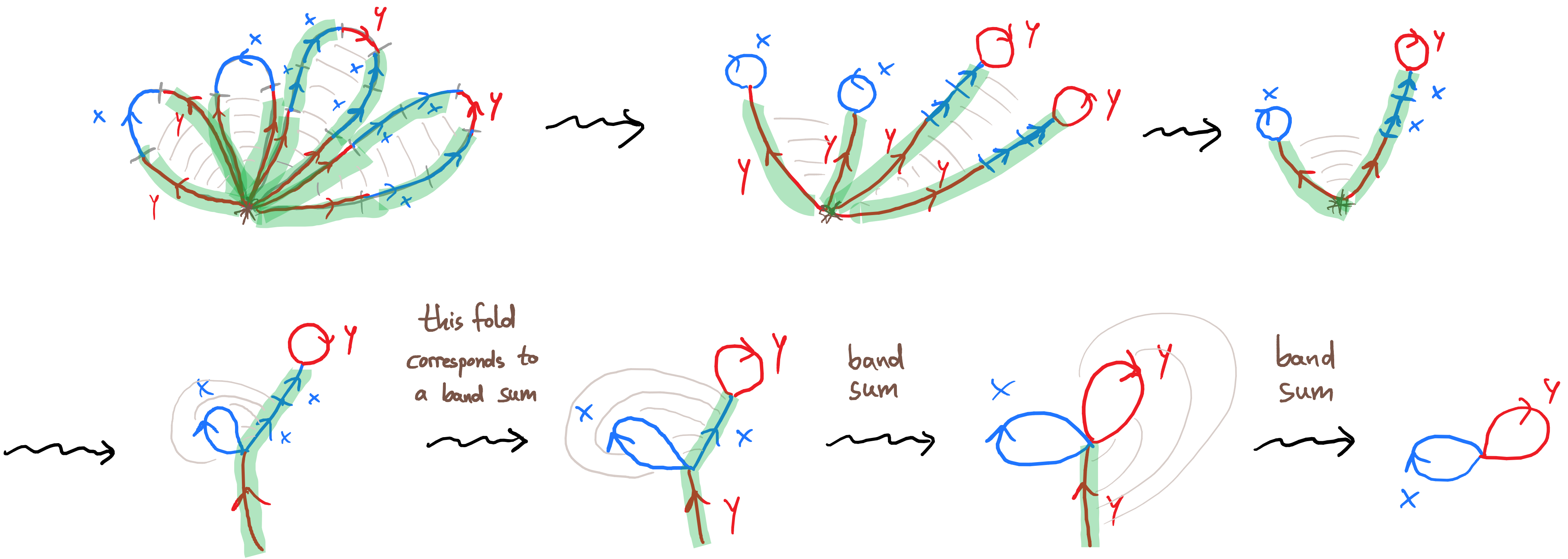
If there are closed circle components, we use band sums guided by <u>Stallings folding</u>

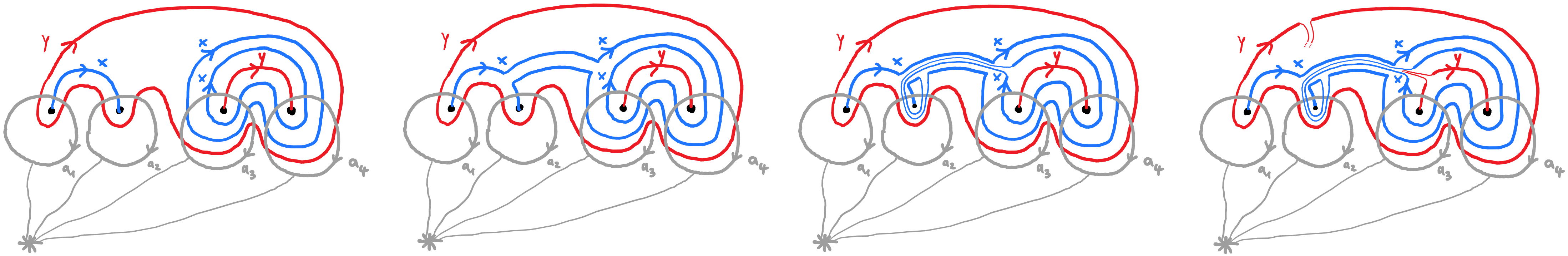


Sequence of folds which show that

$\langle yxy^{-1},\ yx^{-1}y^{-1},\ yxxyx^{-1}x^{-1}y^{-1},\ yxxy^{-1}x^{-1}x^{-1}y^{-1} \rangle$ generates the free group $\langle x, y \rangle$

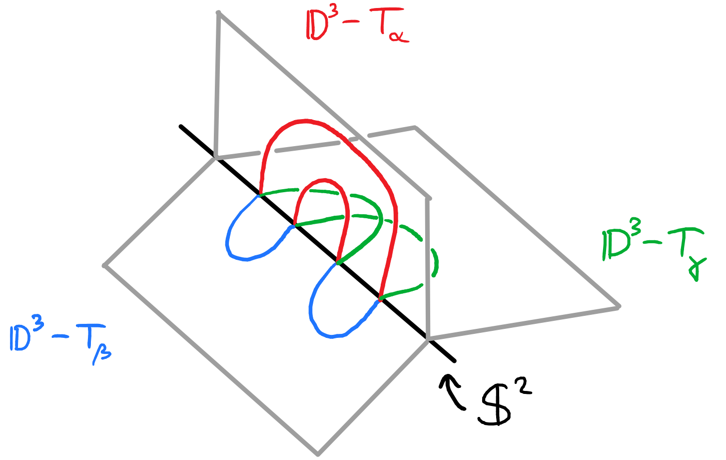



x
y
this fold
corresponds to
a band sum
band
sum
band
sum


y
x
a1
a2
a3
a4

(based, parameterized)

bridge trisections of a smoothly knotted surface $K^2 \subset S^4$



take π_1 of pieces

[Blackwell-Kirby-Klug-Longo-R, 2021]

trisected knotted surface group $\pi_1(S^4-K)$

$$\begin{tikzcd}
 & \pi_1(D^3-T_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(S^3-L_{\alpha\beta}) \arrow[dd, two heads] \\
\pi_1\left(S^2-\left\{\substack{26\text{ bridge}\\ \text{points}}\right\}\right) \arrow[ur] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(D^3-T_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \pi_1(S^3-L_{\alpha\gamma}) \arrow[rr, two heads] & & \pi_1(S^4-K) \\
\pi_1(D^3-T_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \pi_1(S^3-L_{\beta\gamma}) \arrow[ur, two heads] &
\end{tikzcd}$$

free

free

free

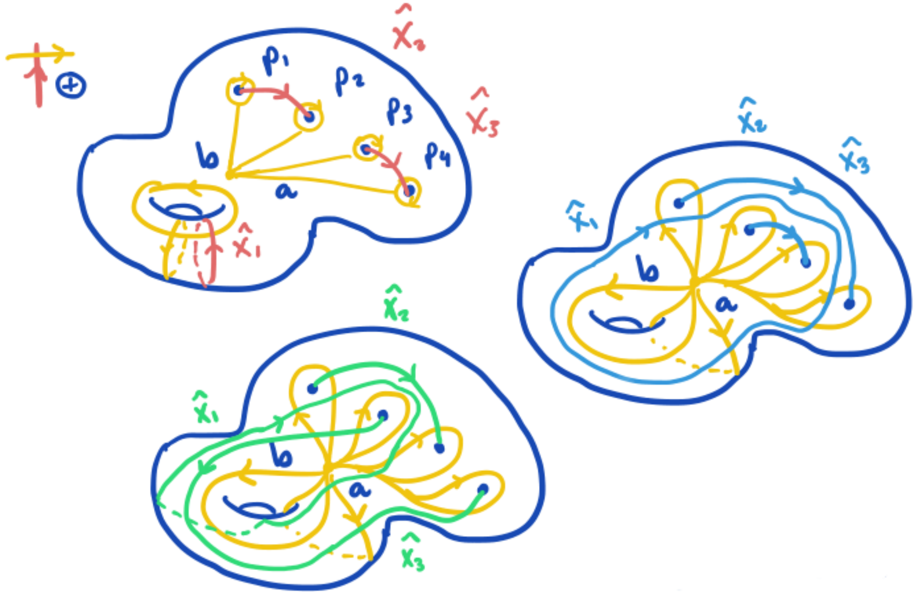
all faces are push-outs & conditions apply

We take inspiration from:

- [Stallings: How not to prove the Poincaré conjecture (1965)]
- [Jaco: Heegaard splittings and splitting homomorphisms (1968)]

  [Jaco: Stable equivalence of splitting homomorphisms (1970)]
- [Abrams, Gay, Kirby: Group trisections and smooth 4-manifolds (2018)]

Thanks!



$a \mapsto 1$	$a \mapsto x_1$	$a \mapsto \bar{x}_1 x_3$
$b \mapsto x_1$	$b \mapsto 1$	$b \mapsto x_1$
$p_1 \mapsto x_2$	$p_1 \mapsto \bar{x}_1 x_2 x_1$	$p_1 \mapsto x_3 \bar{x}_1 x_2 x_1 \bar{x}_3$
$p_2 \mapsto \bar{x}_2$	$p_2 \mapsto x_3$	$p_2 \mapsto x_3$
$p_3 \mapsto x_3$	$p_3 \mapsto \bar{x}_3$	$p_3 \mapsto \bar{x}_1 \bar{x}_2 x_1$
$p_4 \mapsto \bar{x}_3$	$p_4 \mapsto \bar{x}_1 \bar{x}_2 x_1$	$p_4 \mapsto \bar{x}_1 \bar{x}_3 x_1$