2021-05-21

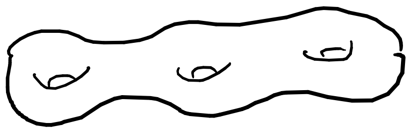
Graduate student conference in Algebra, Geometry, and Topology ; GTA: Philadelphia 2021

Unknotting 2-spheres in S^4 with Finger- & Whitney moves

with Jason Joseph, Michael Klug & Hannah Schwartz

Benjamin Matthias Ruppik, PhD student at the Max Planck Institute for Mathematics, Bonn

__Knotted 2-spheres:__ $S^2 \hookrightarrow S^4$ smooth embedding

__Knotted (orientable) surfaces:__  $= \Sigma_g \hookrightarrow S^4$

up to smooth ambient isotopy

There is a difference between $\dfrac{\text{topologically locally flat embedded surfaces}}{\text{topological isotopy}}$

"exotic knotting"

and $\dfrac{\text{smoothly embedded surfaces}}{\text{smooth isotopy}}$

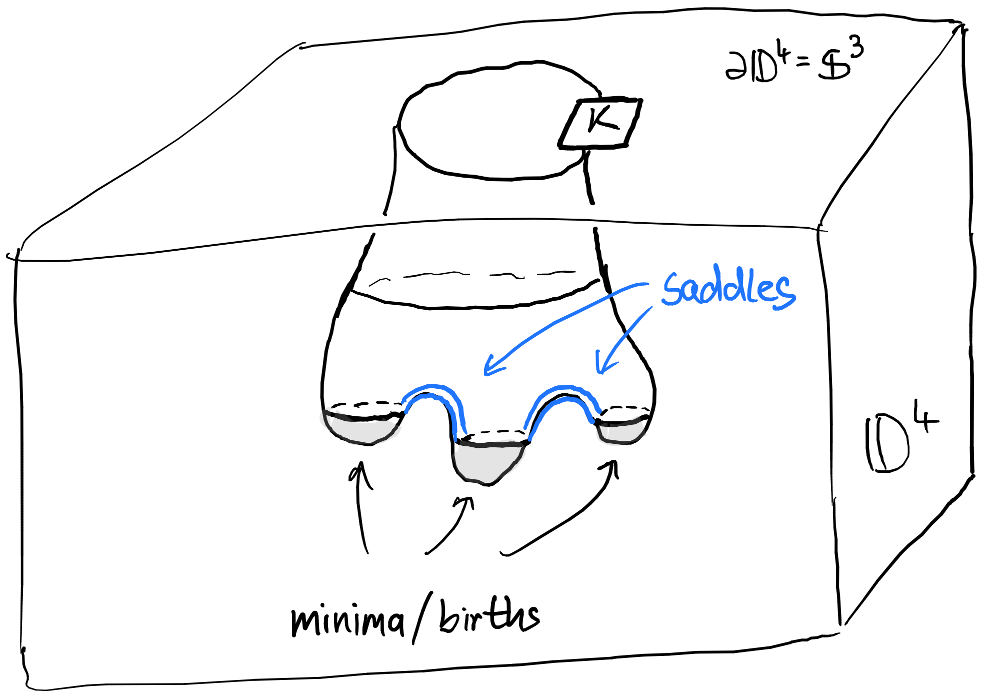
everything [the manifolds, embeddings, ...] is smooth in this talk

Classical ribbon knots

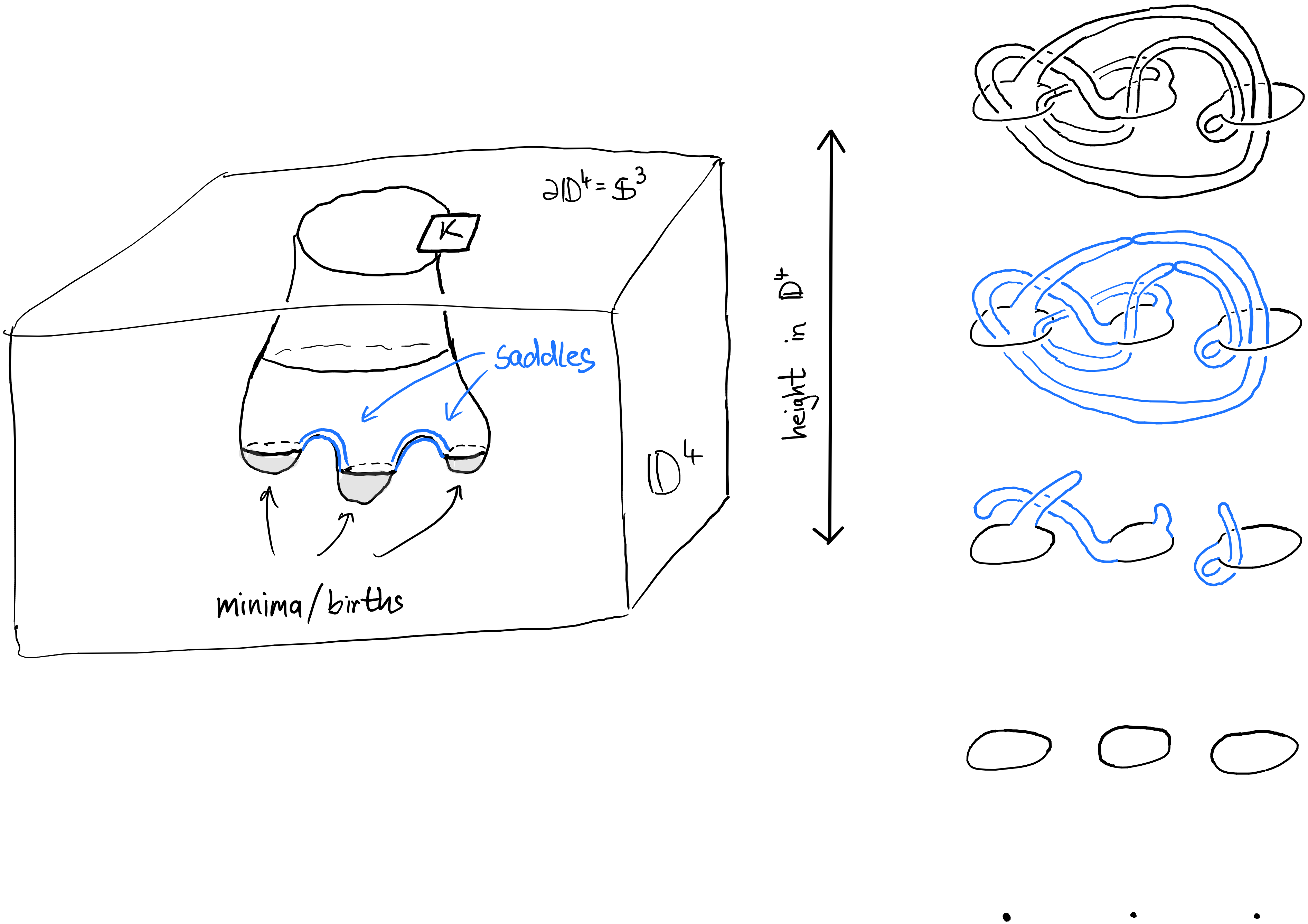
Start with unlink in S^3

Join components with fusion bands

Ribbon disk:



Describing knotted surfaces via movies

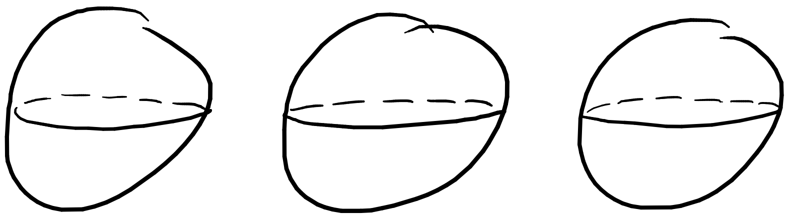


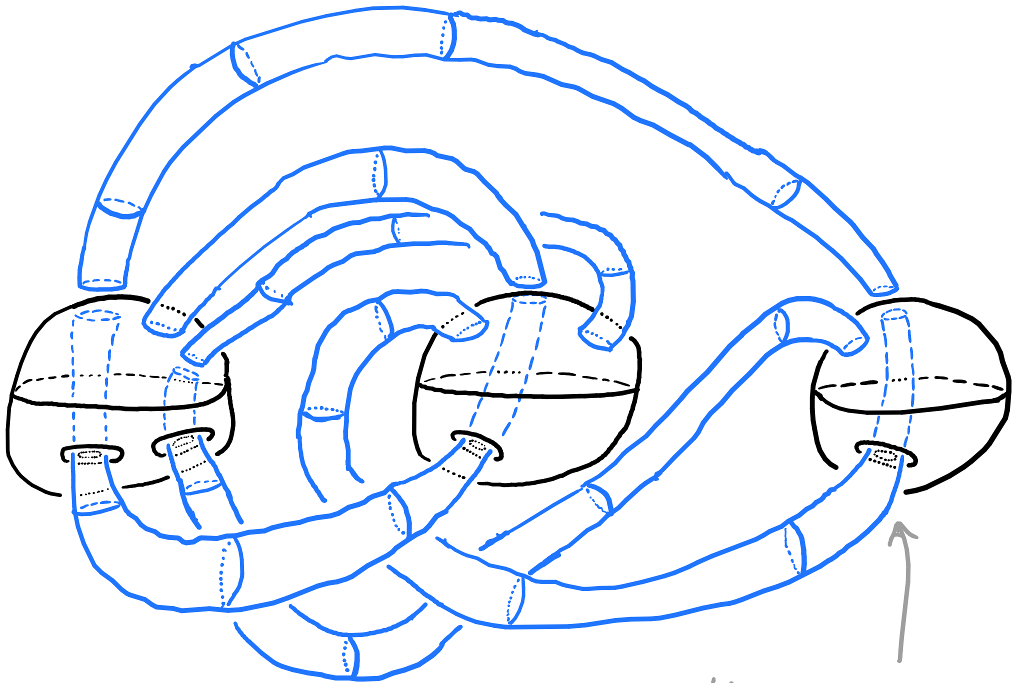
Ribbon 2-knots

↗ Satoh's tube map

Start with an unlink of 2-spheres in S^4

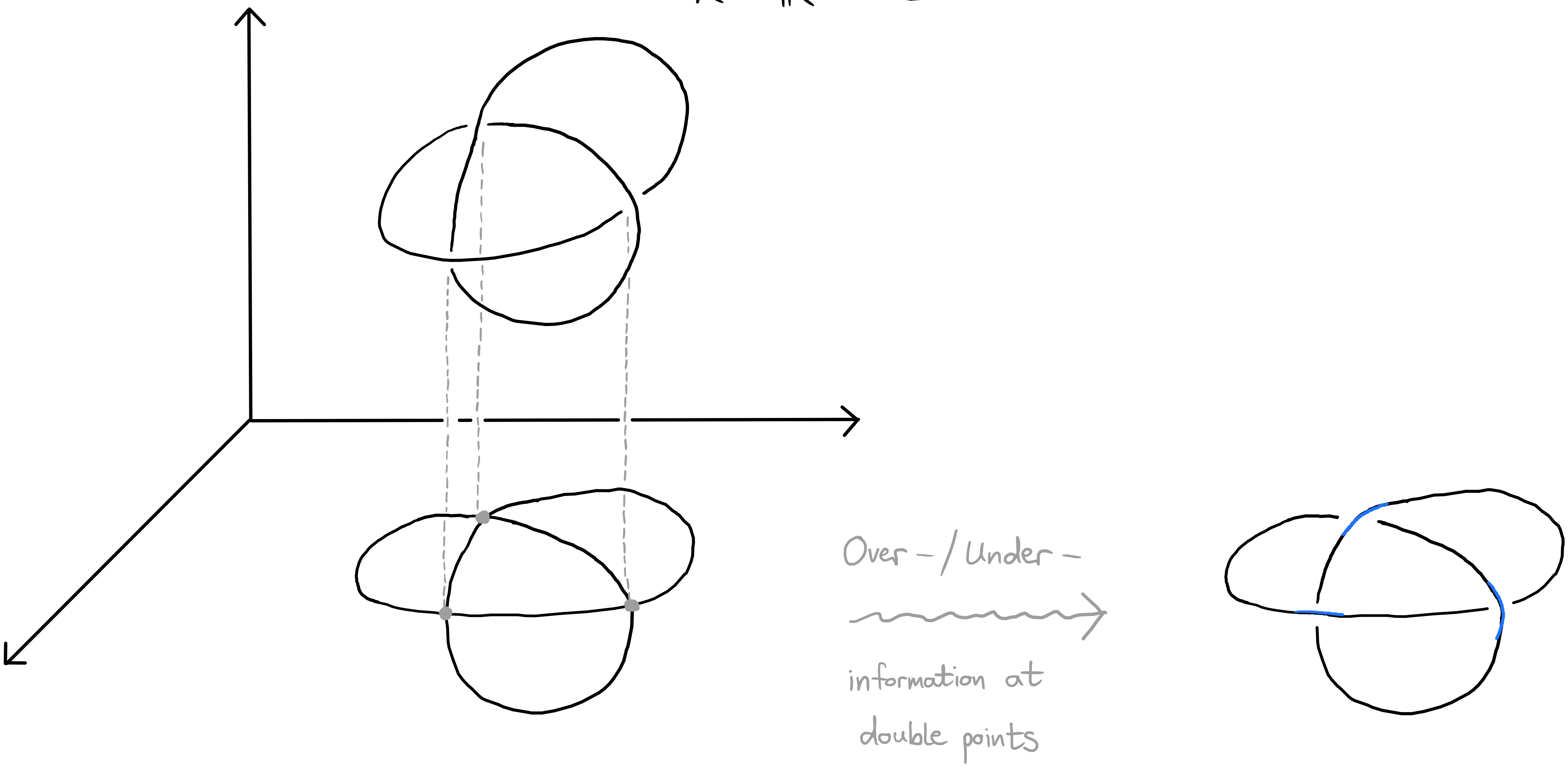
Attach fusion tubes



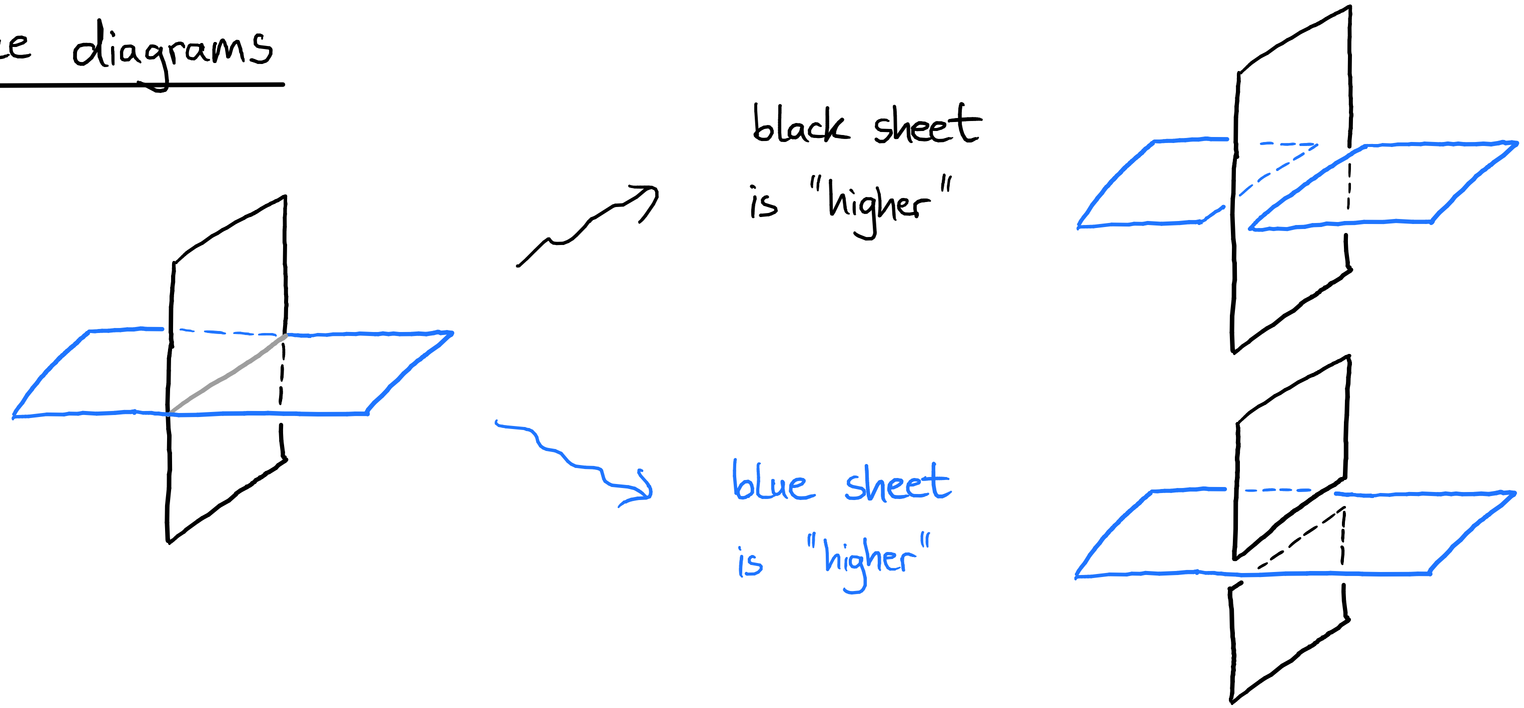


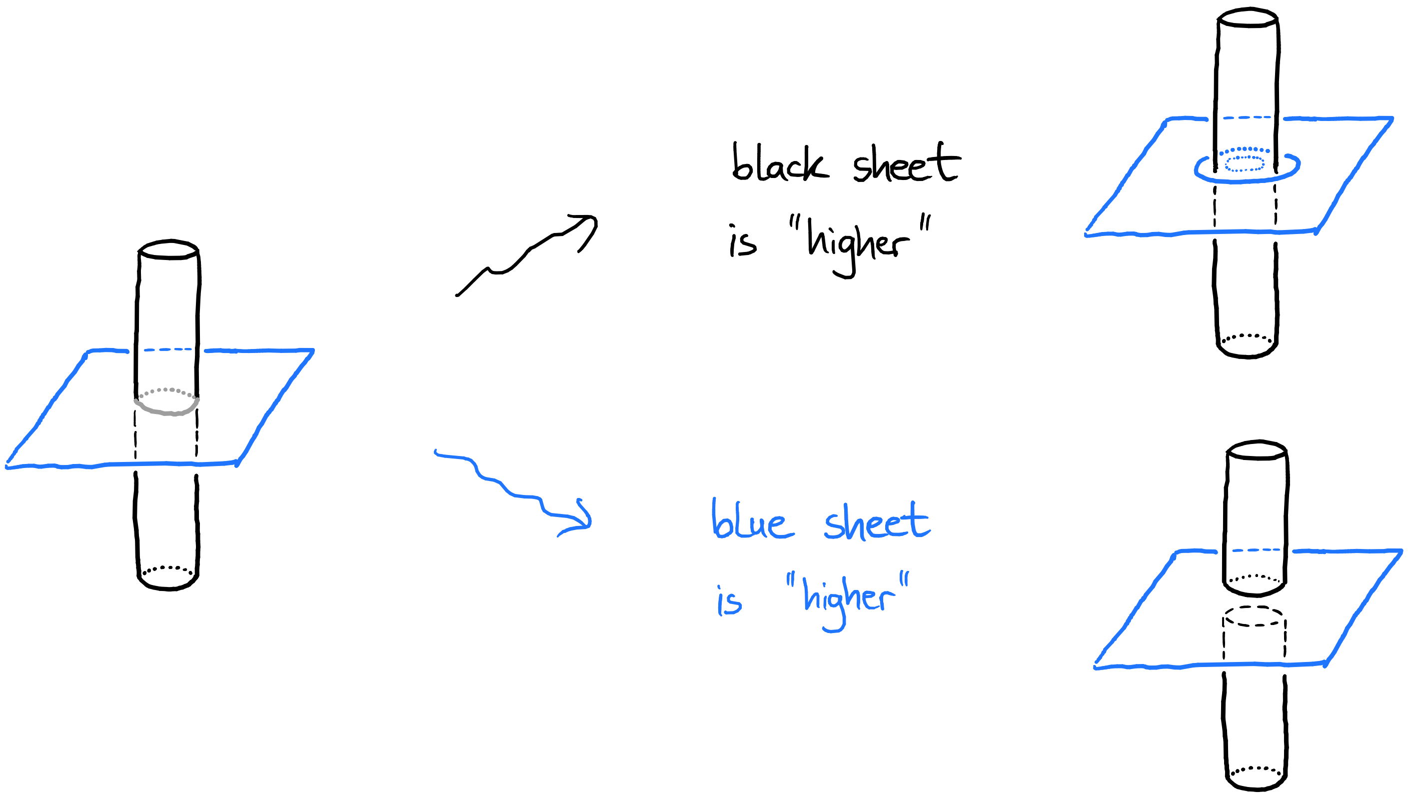
the blue tubes link with the black spheres

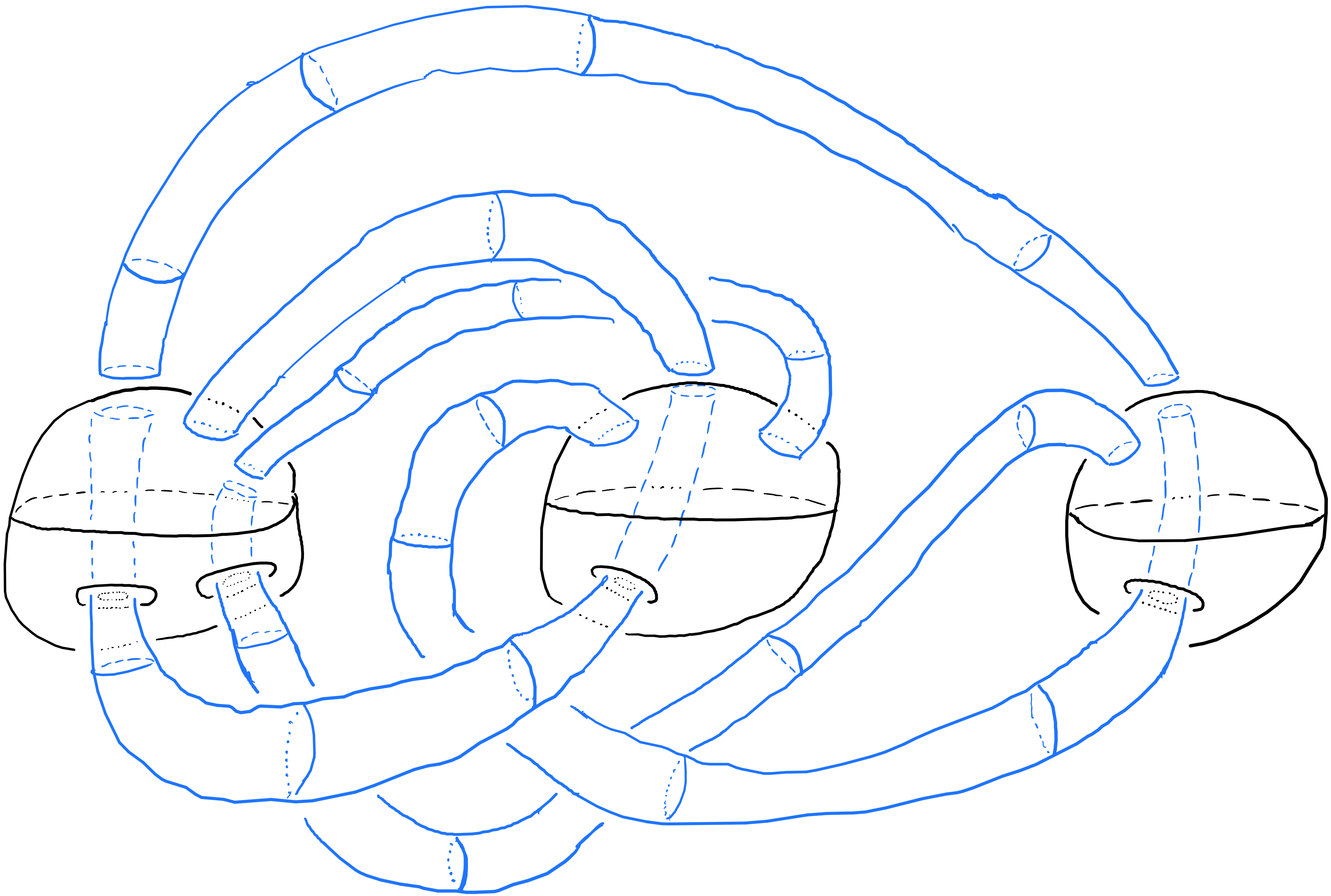
$k \subset \mathbb{R}^3 \subset S^3$



Broken surface diagrams

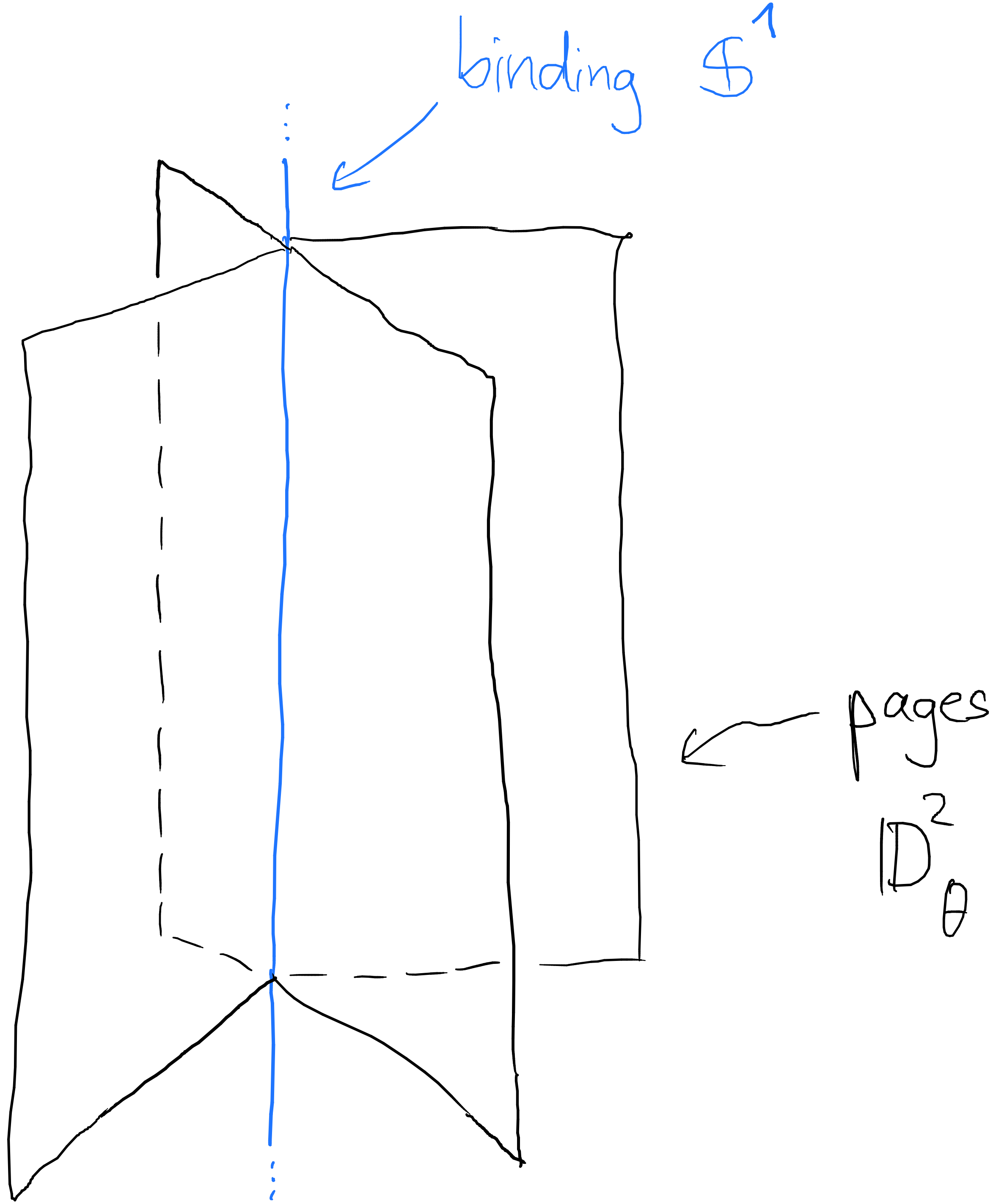






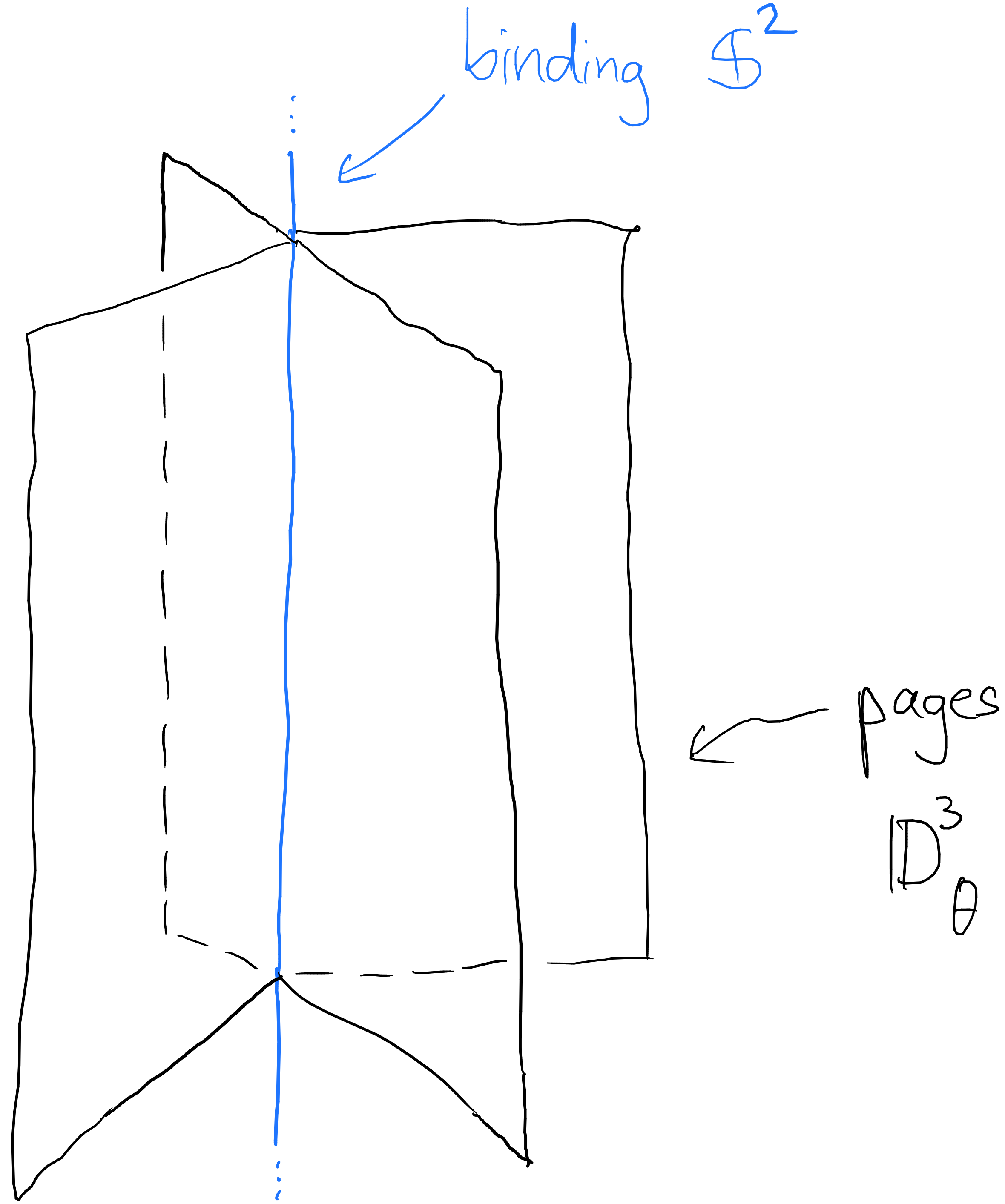
Spinning

open book decomposition of S^3 =



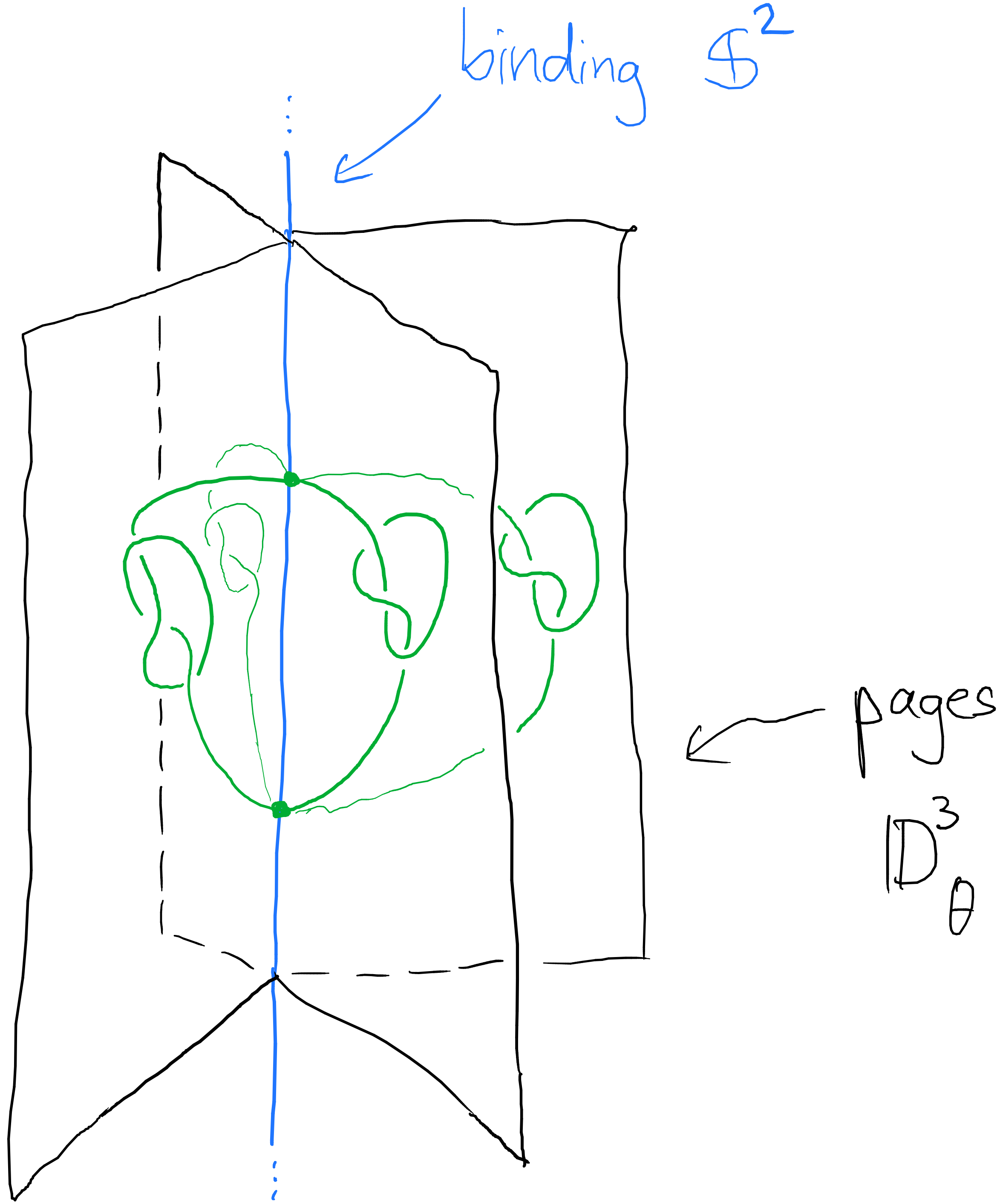
Spinning

open book decomposition of S^4 =

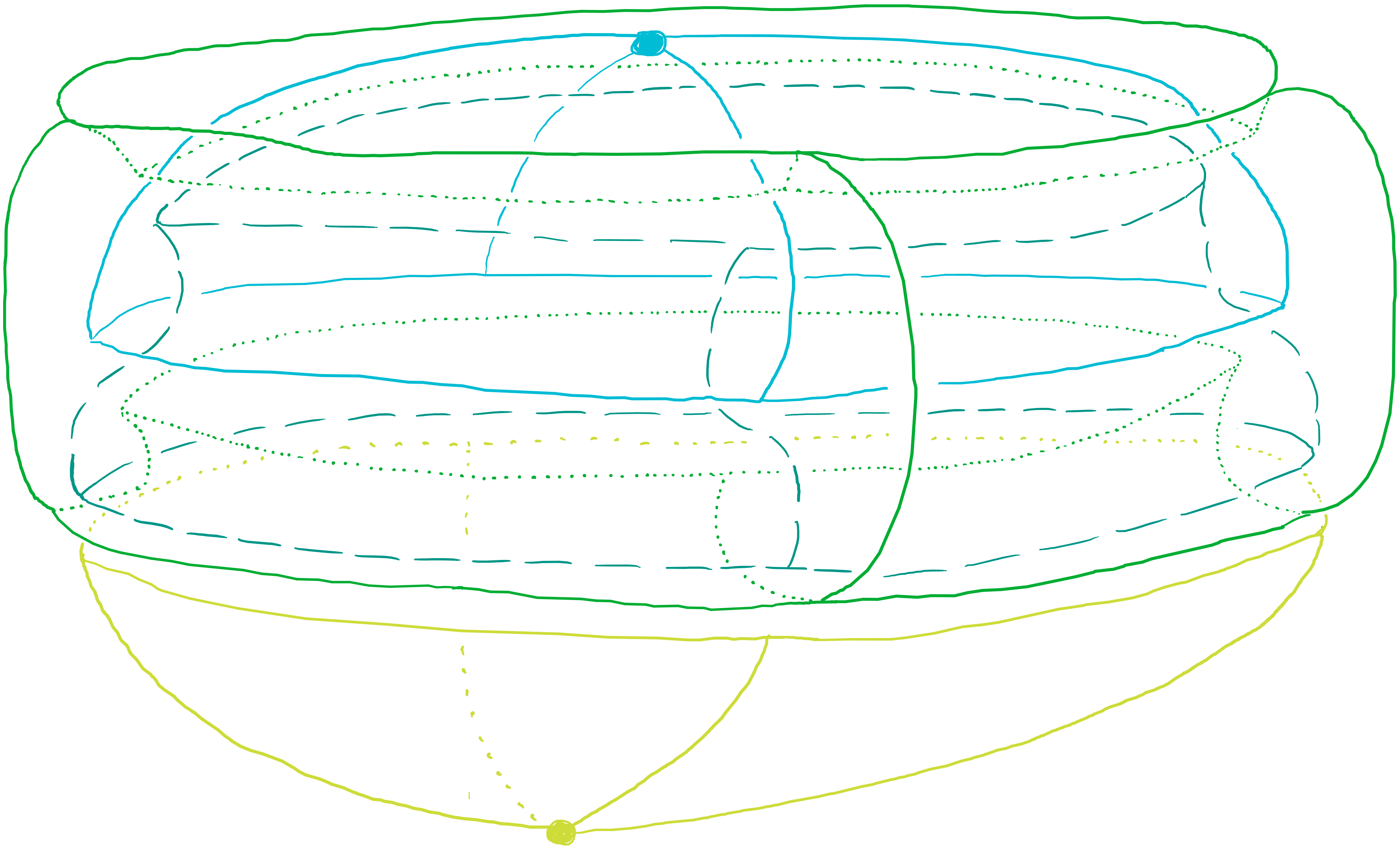


Spinning

open book decomposition of S^4 =

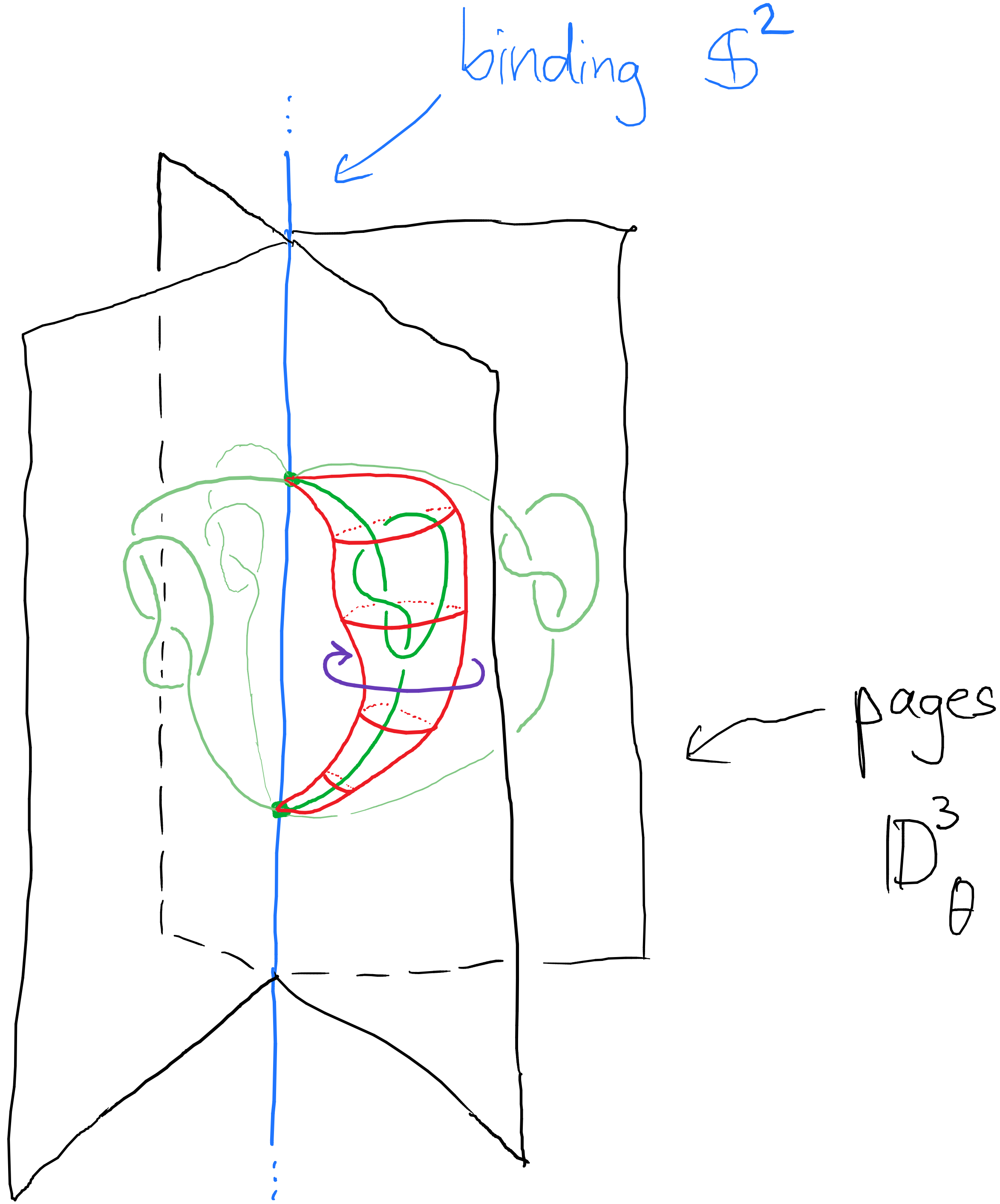


spun trefoil



Twist-Spinning

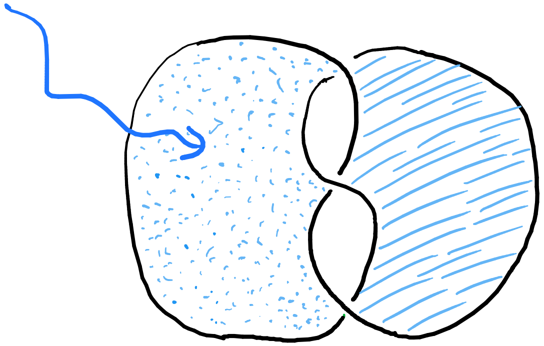
open book decomposition of $\mathbb{S}^4$ =



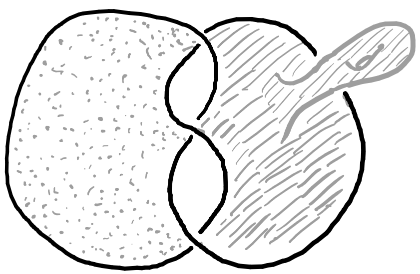
$\boxed{\begin{array}{l}\underline{\text{Idea:}}\ \text{Study codimension} = 2 \text{ knots} \\ \qquad\quad \text{via submanifolds that they bound}\end{array}}$

Just as knots $S^1 \hookrightarrow S^3$ bound

Seifert surfaces ...



(not unique $\rightsquigarrow$ S-equivalence)

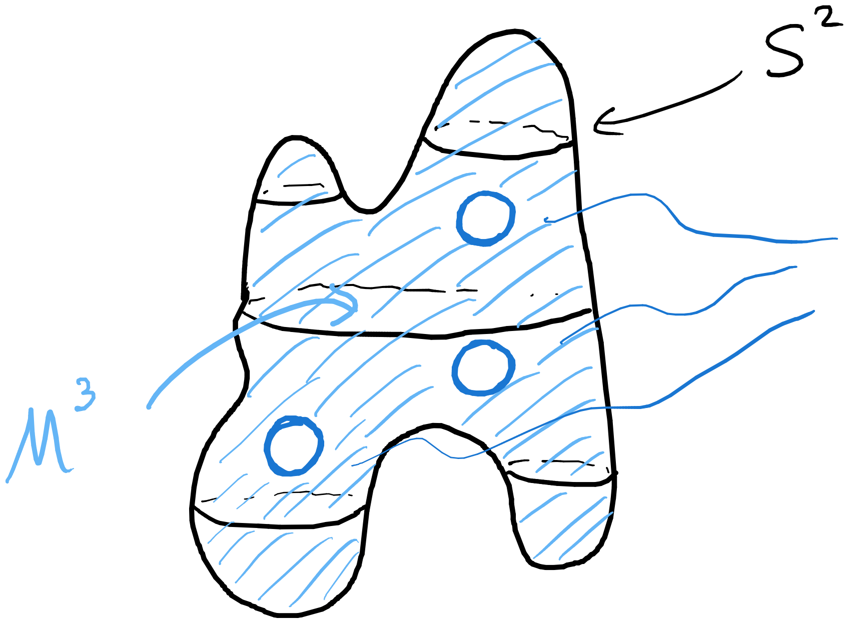


... knotted surfaces $\Sigma_g \overset{S}{\hookrightarrow} S^4$

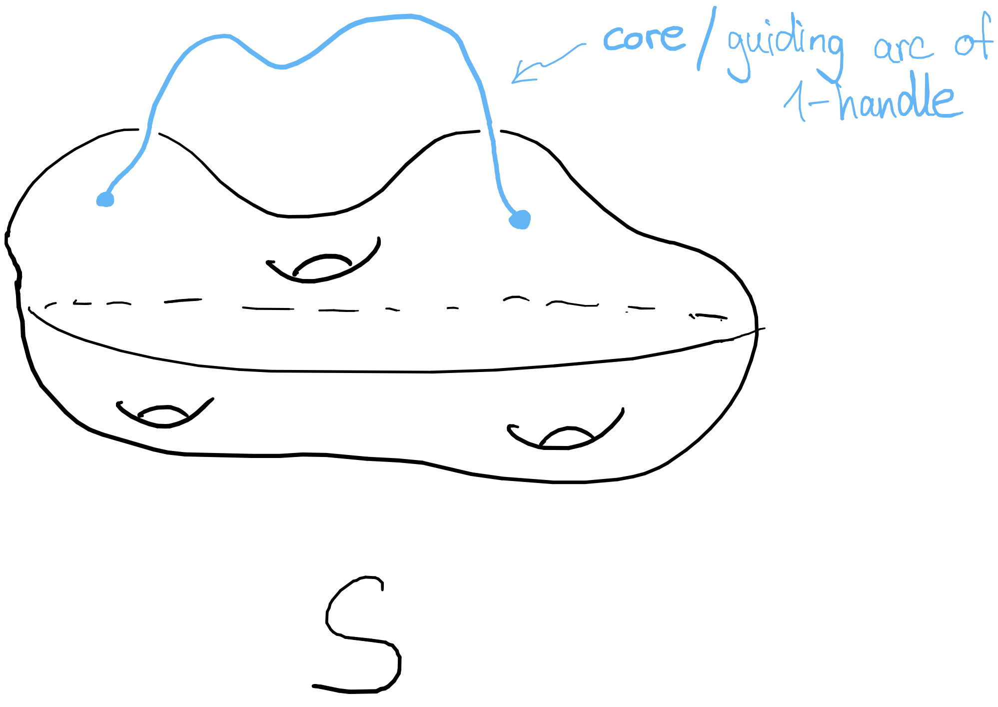
bound Seifert hypersurfaces /

Seifert solids

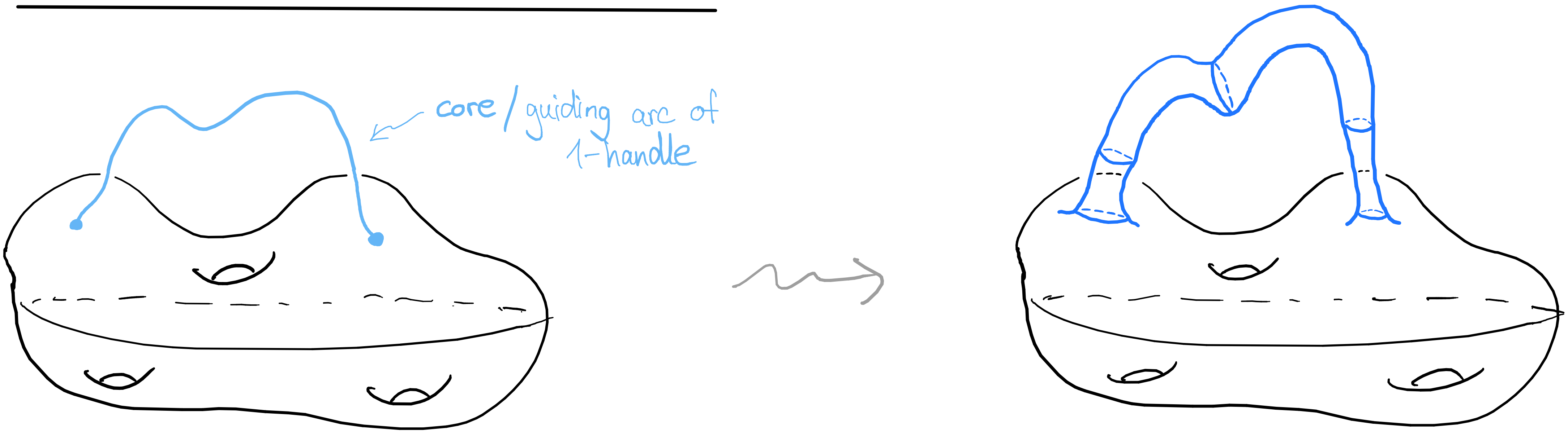
$\boxed{\text{oriented, smooth compact 3-mflds. } M^3 \hookrightarrow S^4 \text{ with } \partial M = S}$



1-handle stabilization of a surface



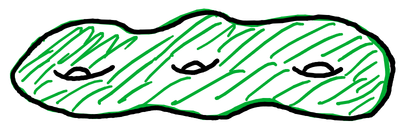
1-handle stabilization of a surface



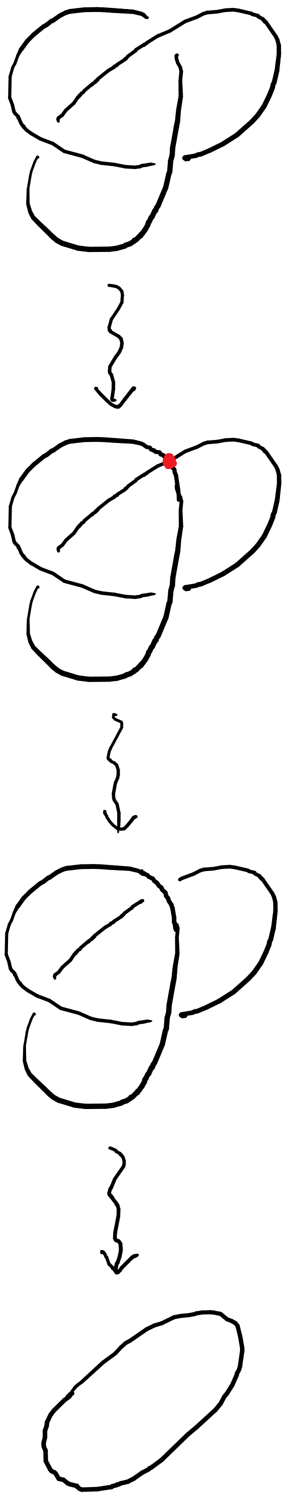
$S \qquad\qquad\qquad\qquad S + h^1$

Fact: Any surface $S \subset S^4$ can be unknotted with enough 1-handle stabilizations.

A surface $S : \Sigma_g \hookrightarrow S^4$ is unknotted if it bounds a handlebody



$\boxed{\text{\underline{Idea}: Study knots via regular homotopies to the unknot}}$

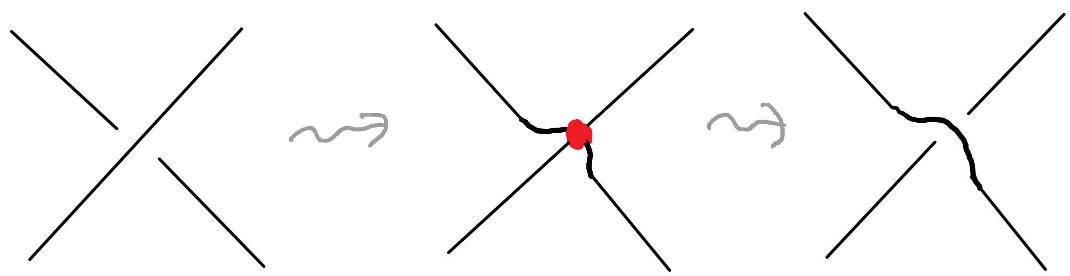


$\boxed{\text{Any knot } K \text{ in } S^3 \text{ is } \underline{\text{homotopic}} \text{ to unknot}}$

$\pi_1(S^3) = \{1\}$

(of course if K non-trivial, not isotopic to unknot)

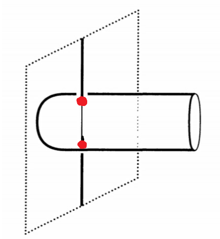
sequence of isotopies and <u>crossing changes</u>:



Unknotting by Finger- & Whitney moves:

2-knot S

Finger moves



[from Scorpan: The wild world of 4-manifolds]

immersed middle stage

Whitney moves

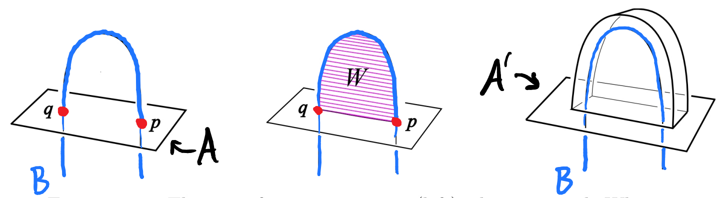
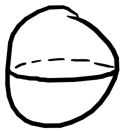


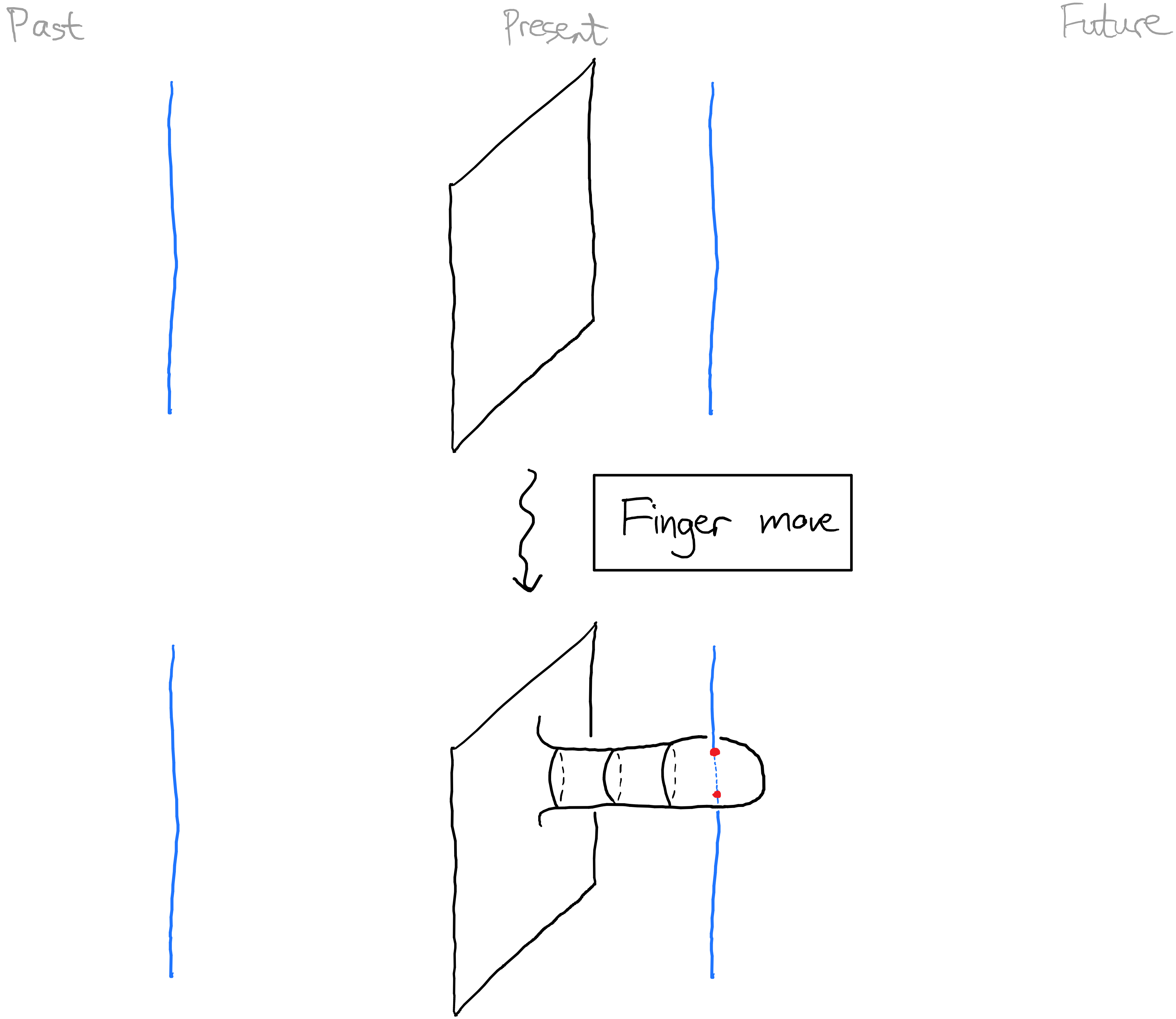
FIGURE 2.3. The pair of intersections p, q (left) admits a purple Whitney disk W (center) which guides a Whitney move eliminating p, q by adding a *Whitney bubble* to the horizontal sheet (right).

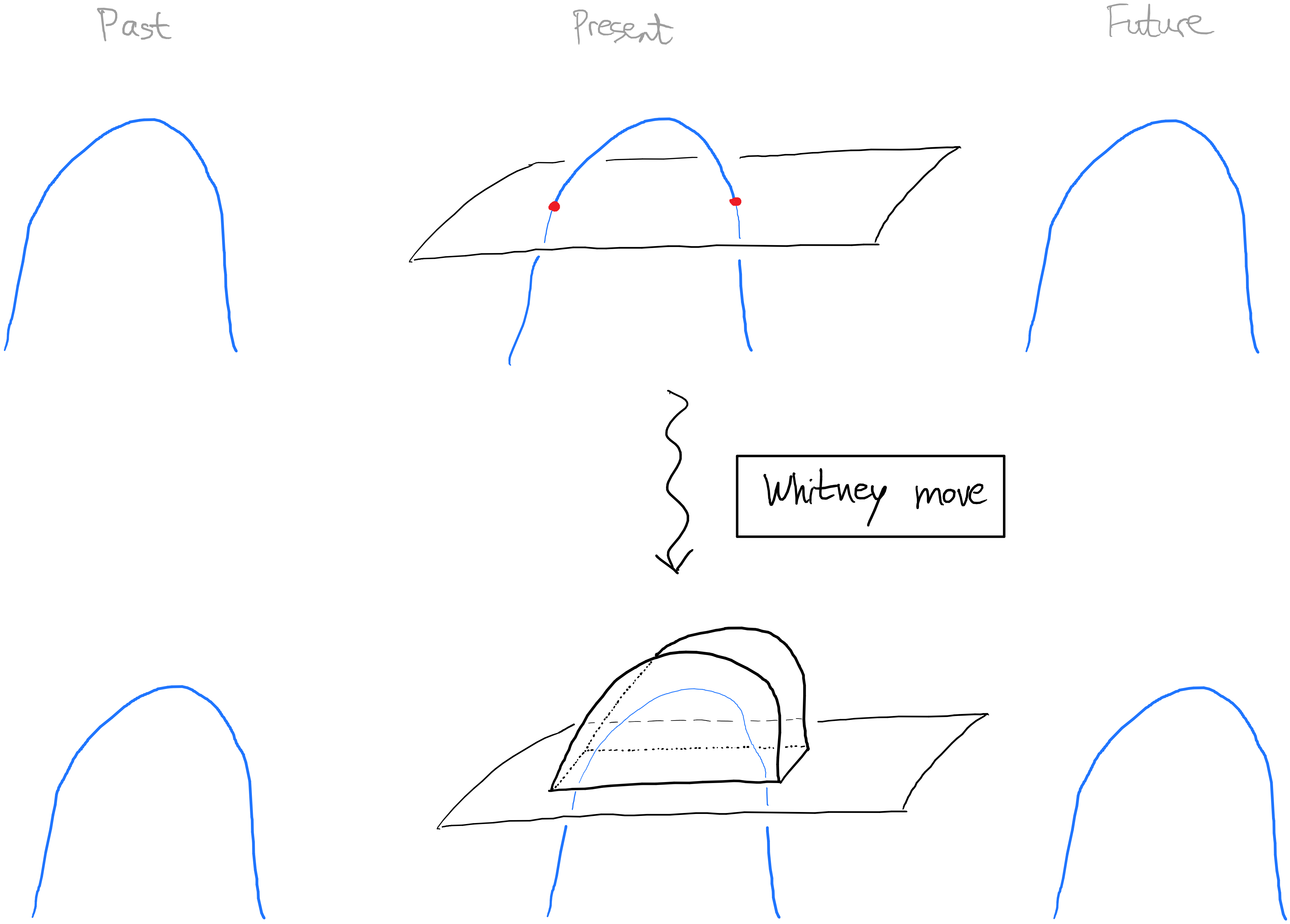
[picture borrowed from Schneiderman–Teichner]

unknot  $\subset S^4$

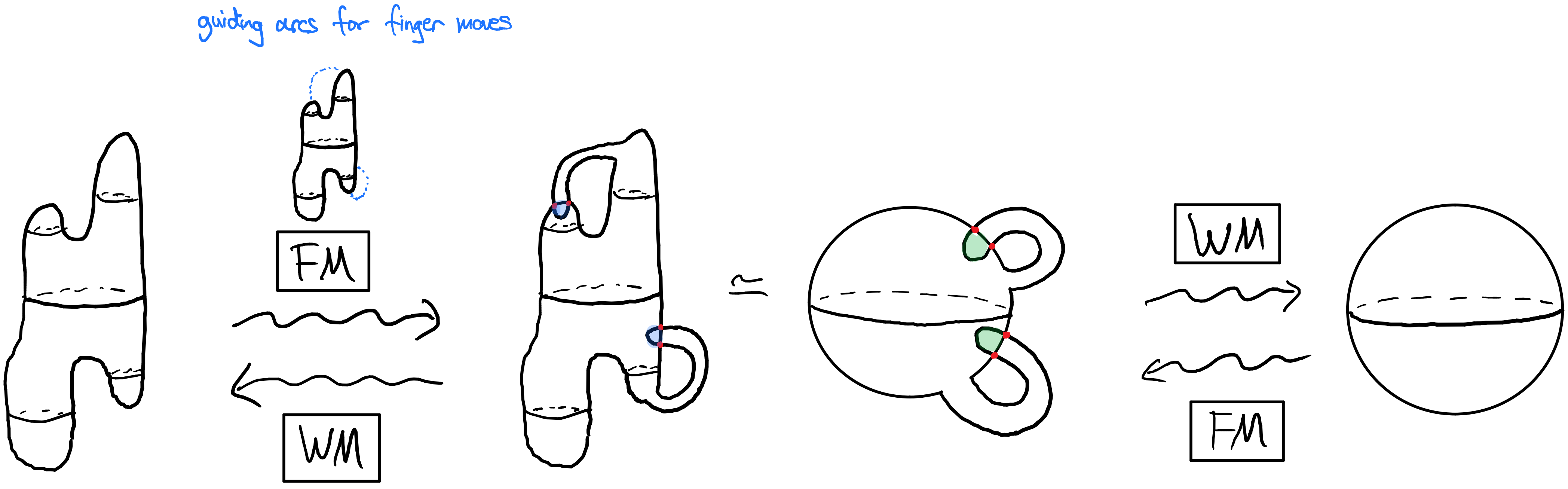
Similarly, any 2-knot $S^2 \overset{S}{\hookrightarrow} S^4$ is <u>(regularly) homotopic</u> to unknot

$\pi_2(S^4) = \{0\}$

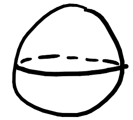

Past
Present
Future
Finger move

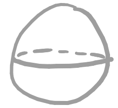

Past
Present
Future
Whitney move

Schematic of a regular homotopy



immersed middle level

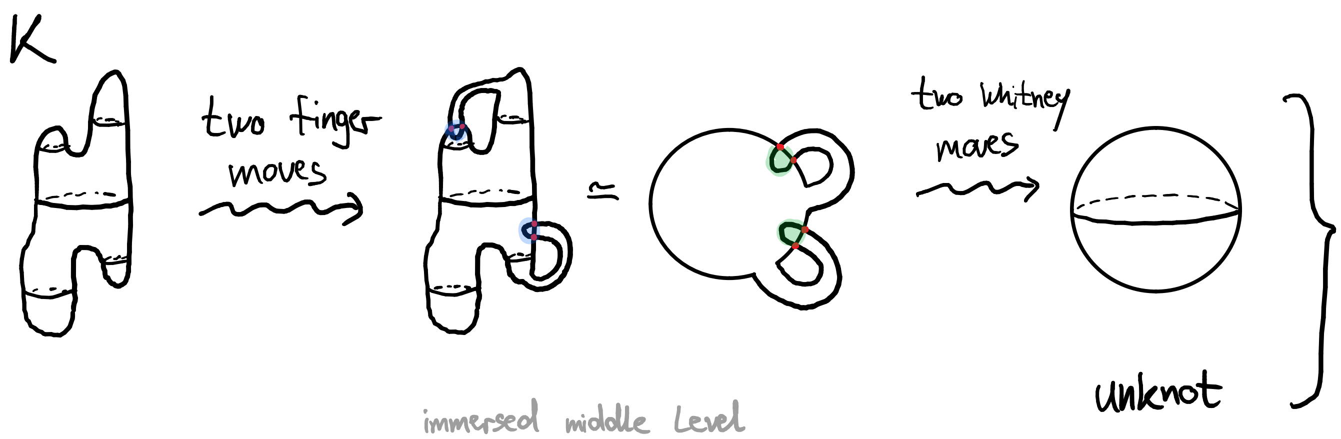
$\pi_2(S^4) = \{0\} \rightsquigarrow$ any knotted 2-sphere $K: S^2 \hookrightarrow S^4$ is (regularly) homotopic to the unknot 

$\pi_2(S^4) = \{0\}$ $\rightsquigarrow$ any knotted 2-sphere $K: S^2 \hookrightarrow S^4$ is (regularly) homotopic to the unknot 

We define the **Casson–Whitney unknotting number**

$$u_{CW}(K)$$

as the minimal number of Finger moves in a regular homotopy from K to the unknot



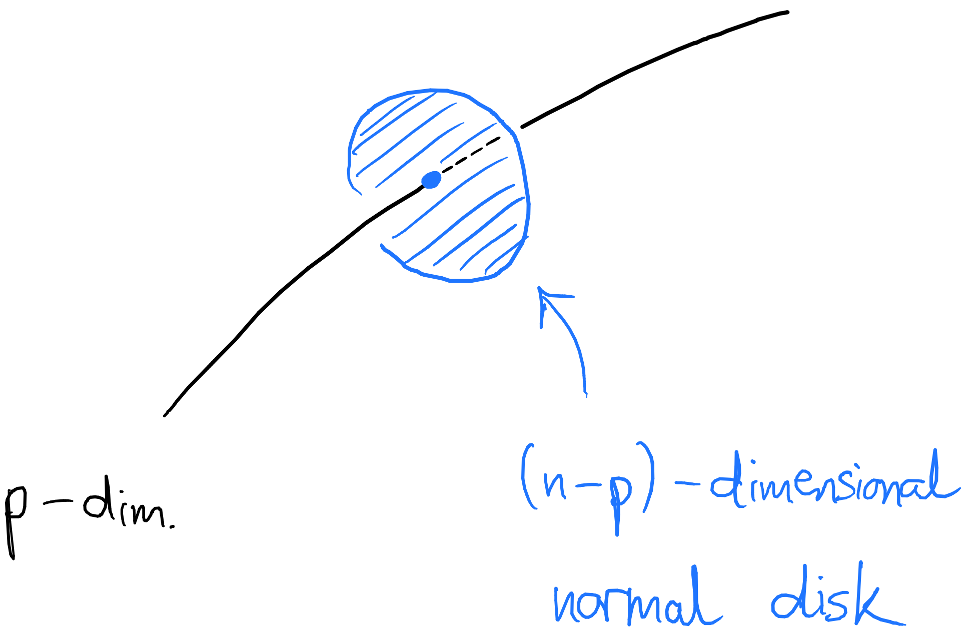
If this is the shortest seq. then here: $u_{CW}(K) = 2$

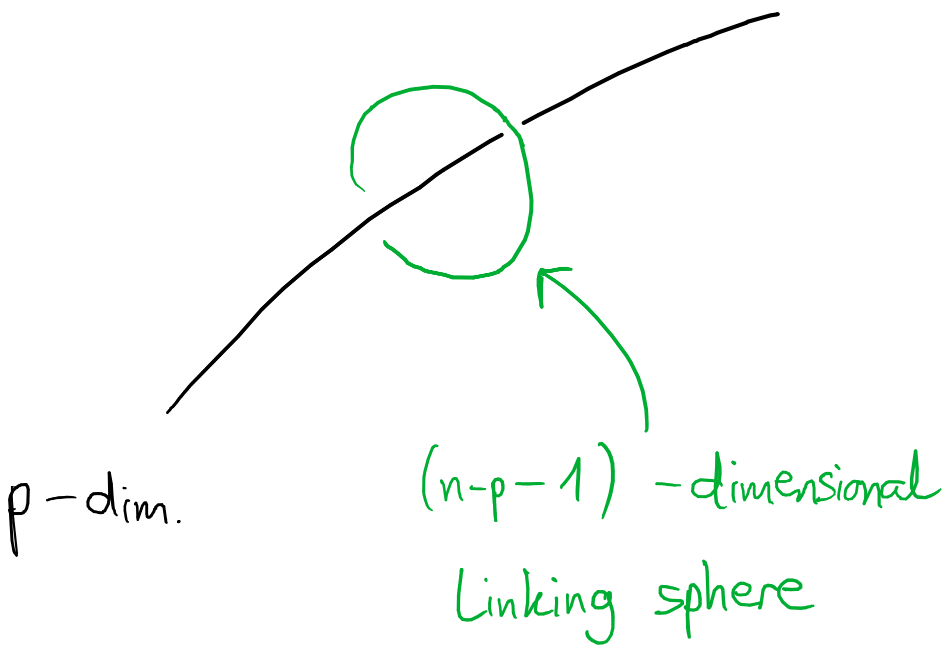
Idea: Study knotted surfaces $S \subset S^4$

via the fundamental group of their complement

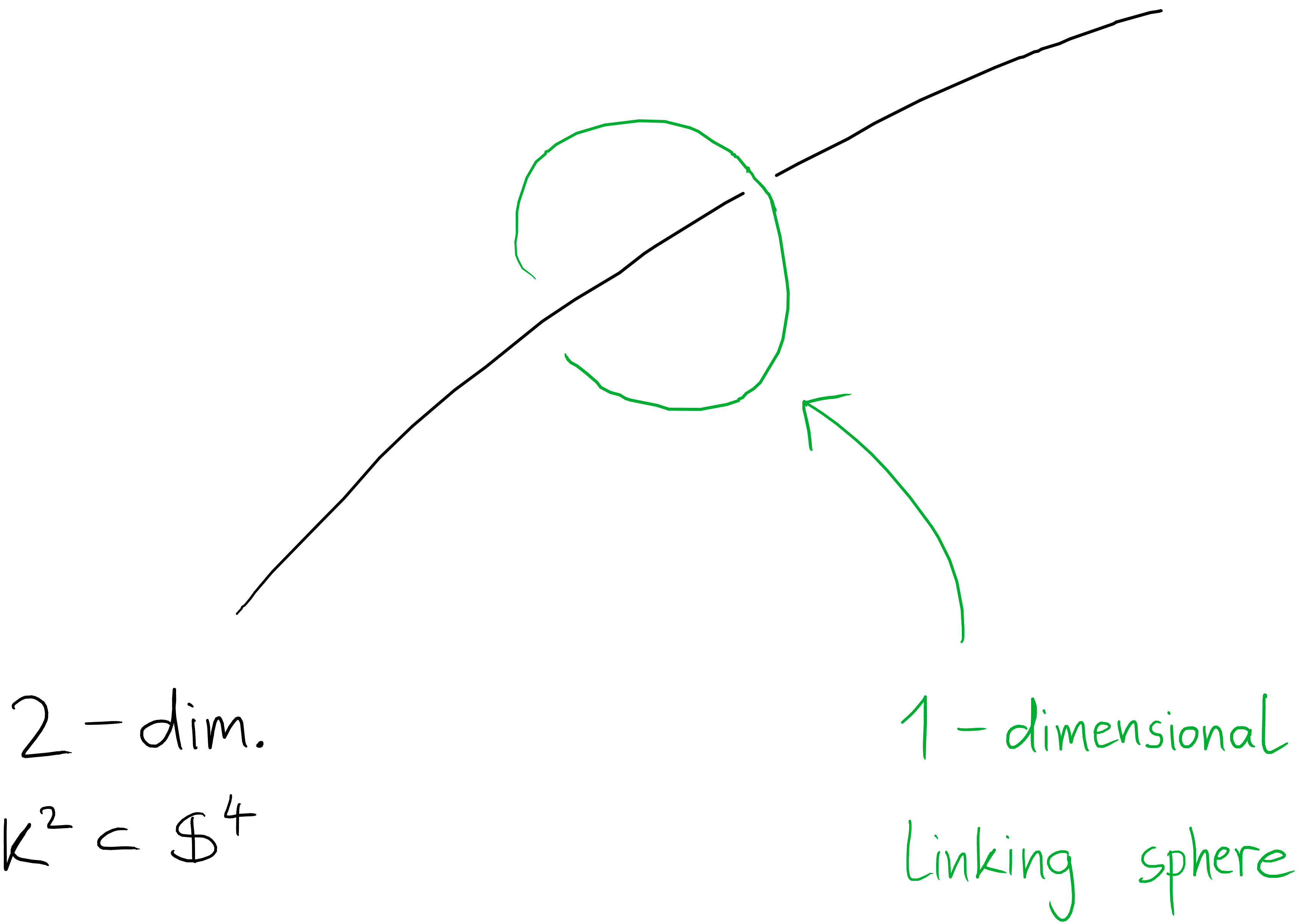
$\pi_1(S^4 - S, *)$

ambient space $\mathbb{R}^n$

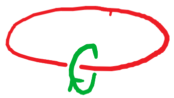




If ambient dimension is 4-dimensional:



unknot

$\pi_1\Big(S^3 \setminus$  $\Big) \cong \mathbb{Z}$

generated by a meridian

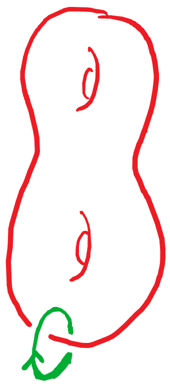
Corollary of Dehn's lemma:

$\pi_1(S^3 \setminus K) \cong \mathbb{Z}$

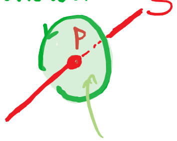
$\Rightarrow K$ is unknotted

---

unknotted surface S

$\pi_1\Big(S^4 \setminus$  $\Big) \cong \mathbb{Z}$

meridian: boundary of a normal 2-disk of S at point p



fiber of the normal disk bundle

BIG open question:

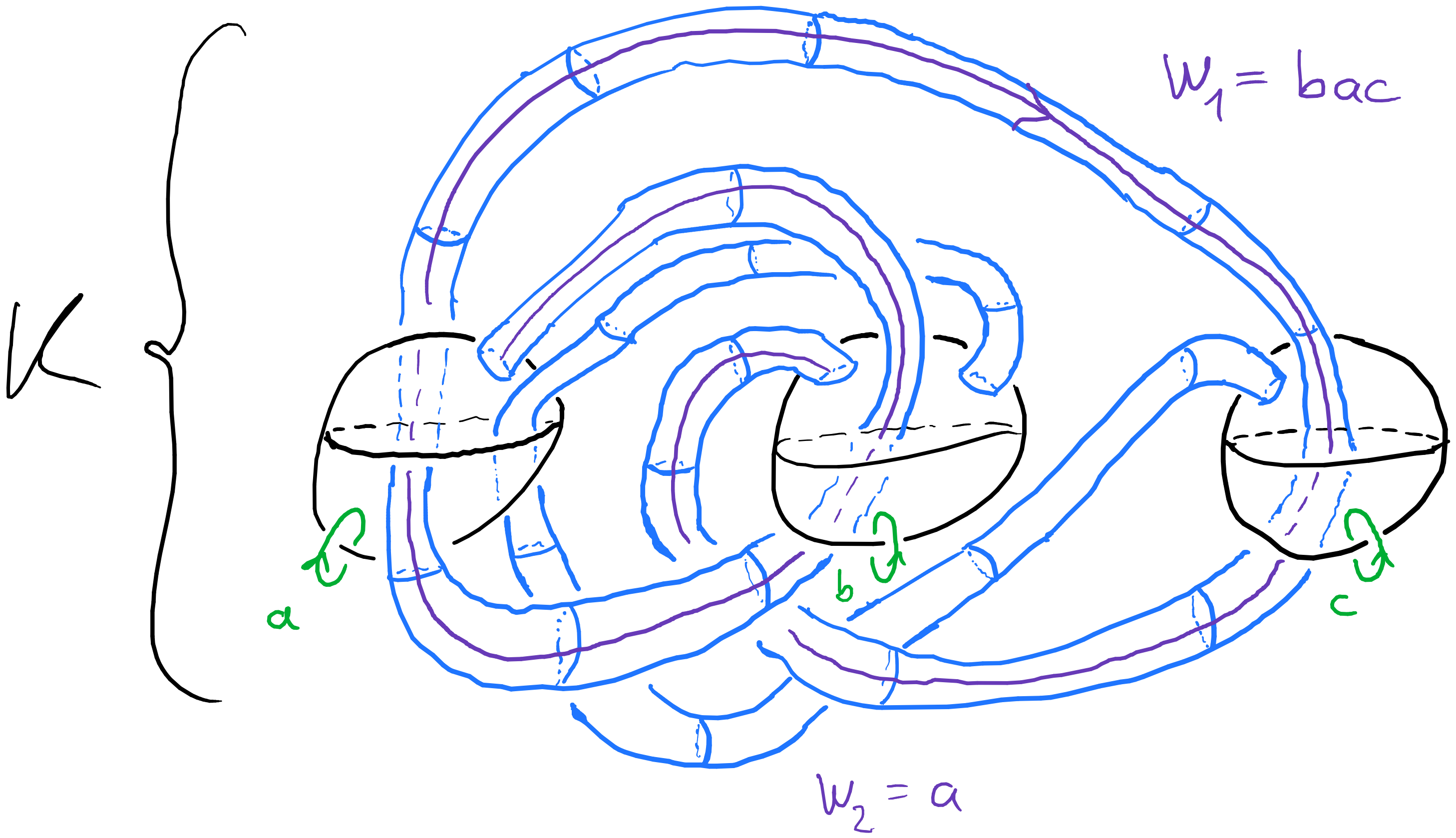
Does π_1 characterize smoothly unknotted surfaces in 4-dim. space?

Algebraic effect of stabilization:

$$\boxed{\pi_1\left(S^4 - (S + h^1)\right) \cong \pi_1(S^4 - S) \big/ \langle\langle w^{-1} a w = b \rangle\rangle}$$

So a stabilization can make two meridians equal

Example: $\pi_1(\mathbb{S}^4 - \text{ribbon 2-knot})$



$$\langle a, b, c \mid b = w_1^{-1} a w_1 , \quad c = w_2^{-1} b w_2 \rangle$$

$$\Leftrightarrow b = (bac)^{-1} a (bac) \qquad \Leftrightarrow c = a^{-1} b a$$

Algebraic effect of finger move:

$$\pi_1(S^4 - S^{\text{fing.}}) \cong \pi_1(S^4 - S) \big/ \langle\langle [w^{-1} a w, b] \rangle\rangle$$

Immersion after finger move on S

A finger move can make a pair of meridians commute.

Algebraic versions of the unknotting #s:

Finger move: $\pi_1(S^4 - S^{\text{fing}}) \cong \pi_1(S^4 - S) / \langle\langle [w^{-1}aw, a] \rangle\rangle$

Stabilization: $\pi_1(S^4 - S^{\text{stab}}) \cong \pi_1(S^4 - S) / \langle\langle w^{-1}aw = a \rangle\rangle$

$a_{cw}(K) :=$ min. # of Finger move relations $[w_i^{-1} a_i w_i, a_i]$ (a_i = meridians)

such that $\pi_1(S^4 - K) / \langle\langle [w_1^{-1}a_1w_1, a_1], [w_2^{-1}a_2w_2, a_2], \dots, [w_k^{-1}a_kw_k, a_k] \rangle\rangle$

is abelian ($\Leftrightarrow \cong \mathbb{Z}$)

$a_{stab}(K) :=$ min. # of 1-handle relations $a_i = w_i^{-1} \cdot a_i \cdot w_i$

such that $\pi_1(S^4 - K) / \langle\langle a_1 = w_1^{-1}a_1w_1, a_2 = w_2^{-1}a_2w_2, \dots, a_k = w_k^{-1}a_kw_k \rangle\rangle$

is abelian

this is the best algebraic lower bound for the Casson-Whitney number we know of

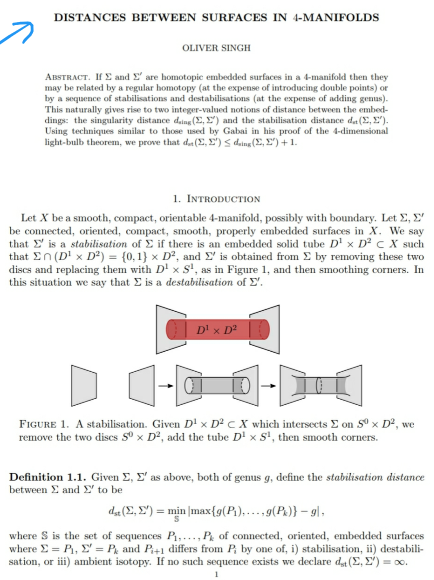
$$
\begin{array}{ccc}
a_{cw}(K) & \leq & u_{cw}(K) \\
\rotatebox{90}{$\geq$} & & \\
a_{stab}(K) & \leq & u_{stab}(K) \\
\rotatebox{90}{$\geq$} & &
\end{array}
$$

minimal size of generating set of Alexander module of K

(Nakanishi index)

$$
\begin{array}{ccc}
a_{cw}(K) & \leq & u_{cw}(K) \\
\rotatebox{90}{$\leq$} & & ?? \\
a_{stab}(K) & \leq & u_{stab}(K)
\end{array}
$$

Oliver Singh's paper was very inspirational



DISTANCES BETWEEN SURFACES IN 4-MANIFOLDS

OLIVER SINGH

ABSTRACT. If Σ and Σ' are homotopic embedded surfaces in a 4-manifold then they may be related by a regular homotopy (at the expense of introducing double points) or by a sequence of stabilisations and destabilisations (at the expense of adding genus). This naturally gives rise to two integer-valued notions of distance between the embeddings: the singularity distance $d_{\mathrm{sing}}(\Sigma, \Sigma')$ and the stabilisation distance $d_{\mathrm{st}}(\Sigma, \Sigma')$. Using techniques similar to those used by Gabai in his proof of the 4-dimensional light-bulb theorem, we prove that $d_{\mathrm{st}}(\Sigma, \Sigma') \leq d_{\mathrm{sing}}(\Sigma, \Sigma') + 1$.

1. INTRODUCTION

Let X be a smooth, compact, orientable 4-manifold, possibly with boundary. Let Σ, Σ' be connected, oriented, compact, smooth, properly embedded surfaces in X. We say that Σ' is a *stabilisation* of Σ if there is an embedded solid tube $D^1 \times D^2 \subset X$ such that $\Sigma \cap (D^1 \times D^2) = \{0, 1\} \times D^2$, and Σ' is obtained from Σ by removing these two discs and replacing them with $D^1 \times S^1$, as in Figure 1, and then smoothing corners. In this situation we say that Σ is a *destabilisation* of Σ'.

FIGURE 1. A stabilisation. Given $D^1 \times D^2 \subset X$ which intersects Σ on $S^0 \times D^2$, we remove the two discs $S^0 \times D^2$, add the tube $D^1 \times S^1$, then smooth corners.

Definition 1.1. Given Σ, Σ' as above, both of genus g, define the *stabilisation distance* between Σ and Σ' to be

$$d_{\mathrm{st}}(\Sigma, \Sigma') = \min_{\mathbb{S}} |\max\{g(P_1), \ldots, g(P_k)\} - g|,$$

where $\mathbb{S}$ is the set of sequences $P_1, \ldots, P_k$ of connected, oriented, embedded surfaces where $\Sigma = P_1$, $\Sigma' = P_k$ and P_{i+1} differs from P_i by one of, i) stabilisation, ii) destabilisation, or iii) ambient isotopy. If no such sequence exists we declare $d_{\mathrm{st}}(\Sigma, \Sigma') = \infty$.

1

By carefully manipulating the regular homotopies to the unknot, we can show

$$\boxed{u_{stab}(K) \leq u_{cw}(K) + 1}$$

the smooth unknotting conjecture would imply that the +1 is not necessary

and

$$\boxed{u_{cw}(K) = 1 \Rightarrow u_{stab}(K) = 1}$$

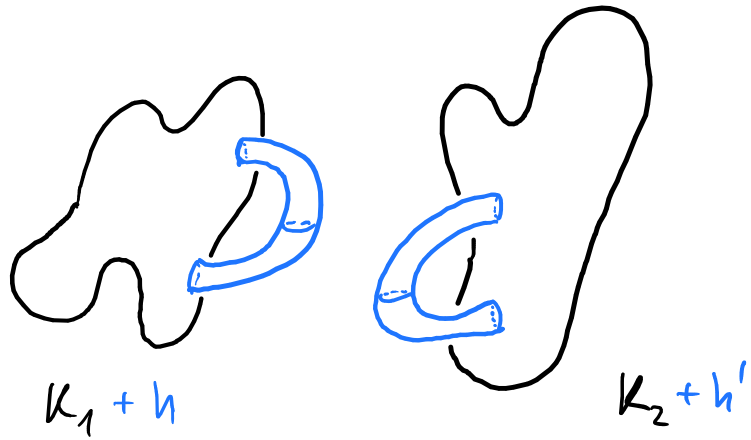
$$\boxed{\text{Of course } \quad u_{stab}(K_1 \# K_2) \le u_{stab}(K_1) + u_{stab}(K_2)}$$

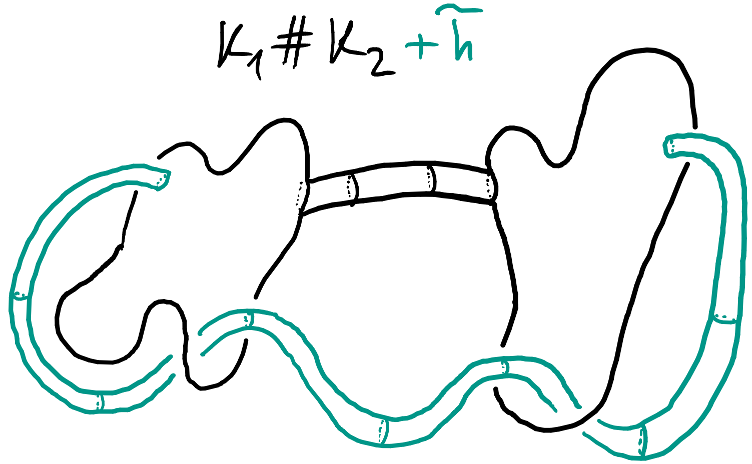
But u_{stab} can fail to be additive:

[Miyazaki] has an example of ribbon 2-knots K_1, K_2 with

$$u_{stab}(K_i) = 1,$$

but there is a new 1-handle which transforms $K_1 \# K_2$ into an unknotted torus.





Miyazaki's example:

$\tau\left(T(2,p) \# T(2,q)\right)$ for $q = p+2$
$q = p+4$
$q = p+6$ if $\gcd(p,p+6)=1$

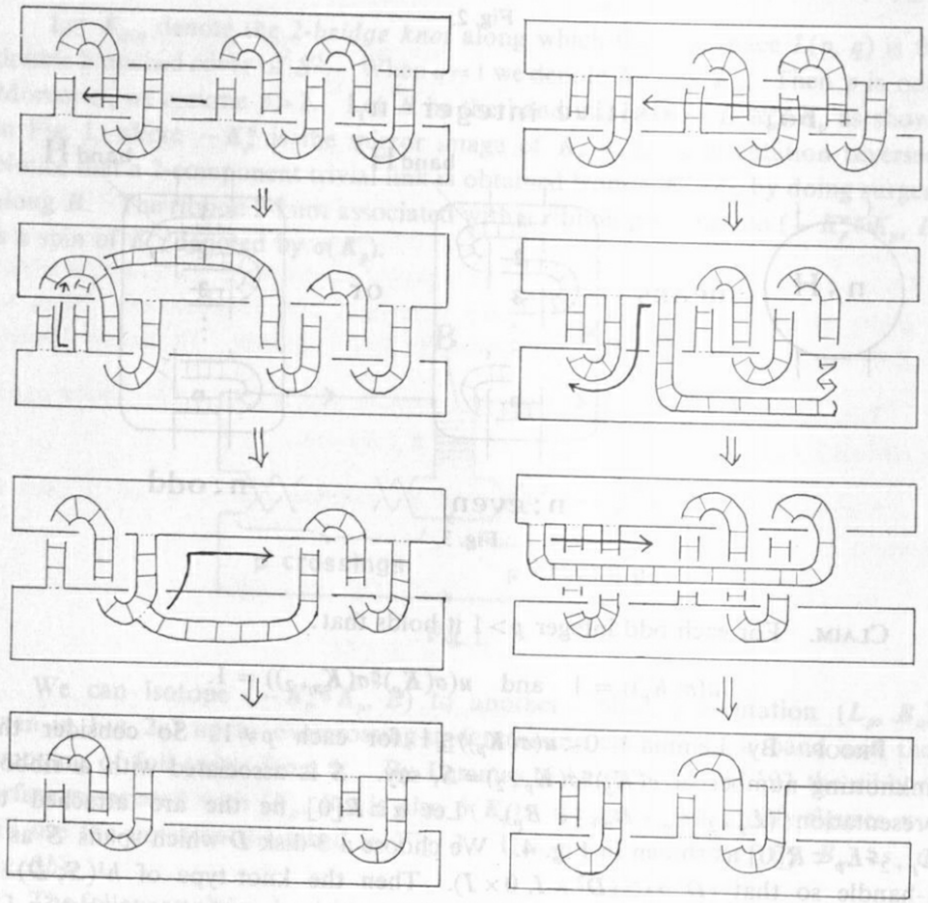
$$u_{stab.}\left(\tau\left(T(2,p) \# T(2,q)\right)\right) = 1$$

MIYAZAKI, K.
KOBE J. MATH.,
3 (1986), 77–85

ON THE RELATIONSHIP AMONG UNKNOTTING NUMBER, KNOTTING GENUS AND ALEXANDER INVARIANT FOR 2-KNOTS

By Katura MIYAZAKI

(Received May 24, 1985)



a) b)

Fig. 6.

Miyazaki explicitly draws the isotopy of the fusion bands which shows that the torus you get after one stabilization is unknotted.

Have examples with $\underbrace{u_{stab}(K)}_{=1} \neq \underbrace{u_{cw}(K)}_{=2}$

Used $a_{cw}(K)$ to find the lower bound by showing that a single finger move relation is not enough to abelianize the group:

positive generator of the evaluation of the Alexander ideal at $t=-1$

Thm.: For K_1, K_2 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$ have $u_{cw}(K_1 \# K_2) \geq 2$

Pf. sketch that $u_{cw}(K_1 \# K_2) \geq 2$:

K_1, K_2 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$

Will show that a relation of the form $[\text{mer.}, w^{-1} \cdot \text{mer.} \cdot w]$ does not abelianize $\pi(K_1 \# K_2)$

- Determinant condition $\rightsquigarrow$ $\pi K_i \twoheadrightarrow \mathrm{Dih}_{p_i} \cong \mathbb{Z}/p_i \rtimes \mathbb{Z}/2$

- Group of connected sum admits surjection $\pi(K_1 \# K_2) \xrightarrow{\phi} \underbrace{(\mathbb{Z}/p_1 * \mathbb{Z}/p_2) \rtimes \mathbb{Z}/2}_{=:G}$

- Enough: Induced image $G/\langle\langle \phi([\text{mer.}, w^{-1}\,\text{mer.}\,w]) \rangle\rangle$ not abelian

- Look at commutator subgroup: Want to show $\mathbb{Z}/p_1 * \mathbb{Z}/p_2 / \langle\langle [z, v^{-1} z v] \rangle\rangle$ is not trivial

  $z = \phi(\text{meridian})$ $\quad v = \phi(w)$

- Rewrite $[z, v^{-1} z v] = [z,v]^2$, show this normally generates

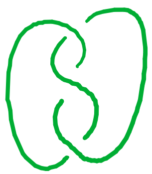
$\rightsquigarrow$ then use a Freiheitssatz of [Fine, Howie, Rosenberger (1988)] to conclude that $\mathbb{Z}/p_1 * \mathbb{Z}/p_2 / \langle\langle g^2 \rangle\rangle$ is nontrivial for any $g \in \mathbb{Z}/p_1 * \mathbb{Z}/p_2$

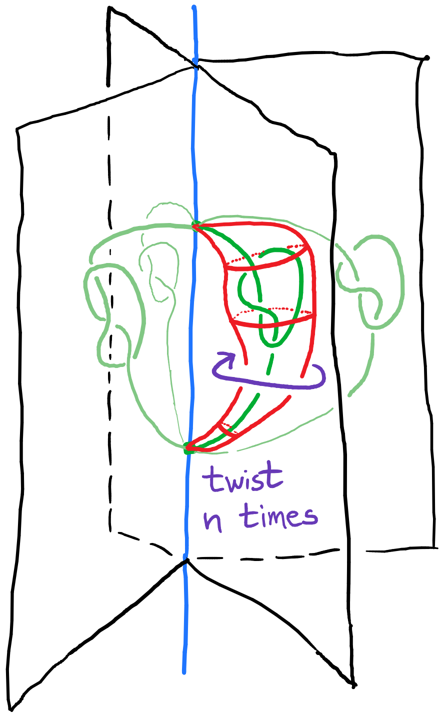
$\square$

Prop.: $u_{cw}(\tau^n k) \leq u(k)$

n-twist spin of $k: S^1 \hookrightarrow S^3$

classical unknotting number of the 1-knot k

$\tau^n($ $k =$  $)$ $=$



Corollary: $a_{cw}(\tau^n k)$ is a lower bound for the classical unknotting number.

Thanks!