Homotopy classification of 4-manifolds with

- finite abelian 2-generator fundamental group
- dihedral fundamental group

Based on joint work with Daniel Kasprowski, Johnny Nicholson, Mark Powell

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Plan: Classification of 4-manifolds

This talk will be in the topological category

•) Warmup: Homeomorphism classification of simply-connected 4-manifolds

•) Our ignorance for non-trivial fundamental groups

$\rightsquigarrow$ $\boxed{\text{Homotopy classification}}$

Poincaré 4-complexes

Whitehead's Γ-construction

[Hambleton-Kreck]

$\pi_1 = \text{finite}$

$\rightsquigarrow$ quadratic 2-type

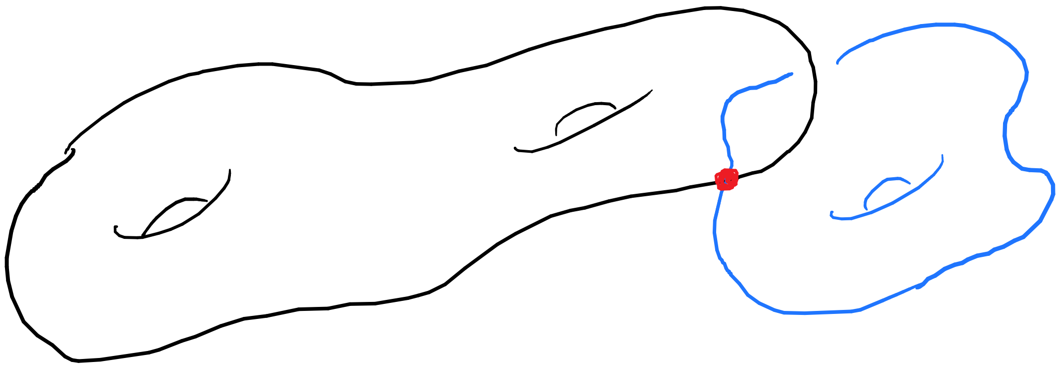
What we ([Kasprowski-Powell-R, 2020], [Kasprowski-Nicholson-R, 2020]) found for $\pi_1 \cong \mathbb{Z}/n \times \mathbb{Z}/m$ and $\pi_1 \cong \mathrm{Dih}_{2\cdot m}$

Simply-connected oriented 4-manifolds and intersection forms

closed, connected, oriented 4-mfld.

Intersection form $\quad H_2(M^4) \otimes_{\mathbb{Z}} H_2(M^4) \xrightarrow{\lambda_M} \mathbb{Z}$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [A] & & [B] \end{array}$$



[Milnor (1958)] Homotopy classification of simply-connected closed oriented 4-mflds.

$M \simeq_{\text{htpy eq.}} N \quad$ iff. $\quad \lambda_M \cong_{\text{isometric}} \lambda_N$

[Freedman (1984)] Homeomorphism ——— " ———

(Intersection form + Kirby-Siebenmann invariant)

Simply-connected oriented 4-manifolds and intersection forms

$$H_2(M^4) \otimes_{\mathbb{Z}} H_2(M^4) \longrightarrow \mathbb{Z}$$

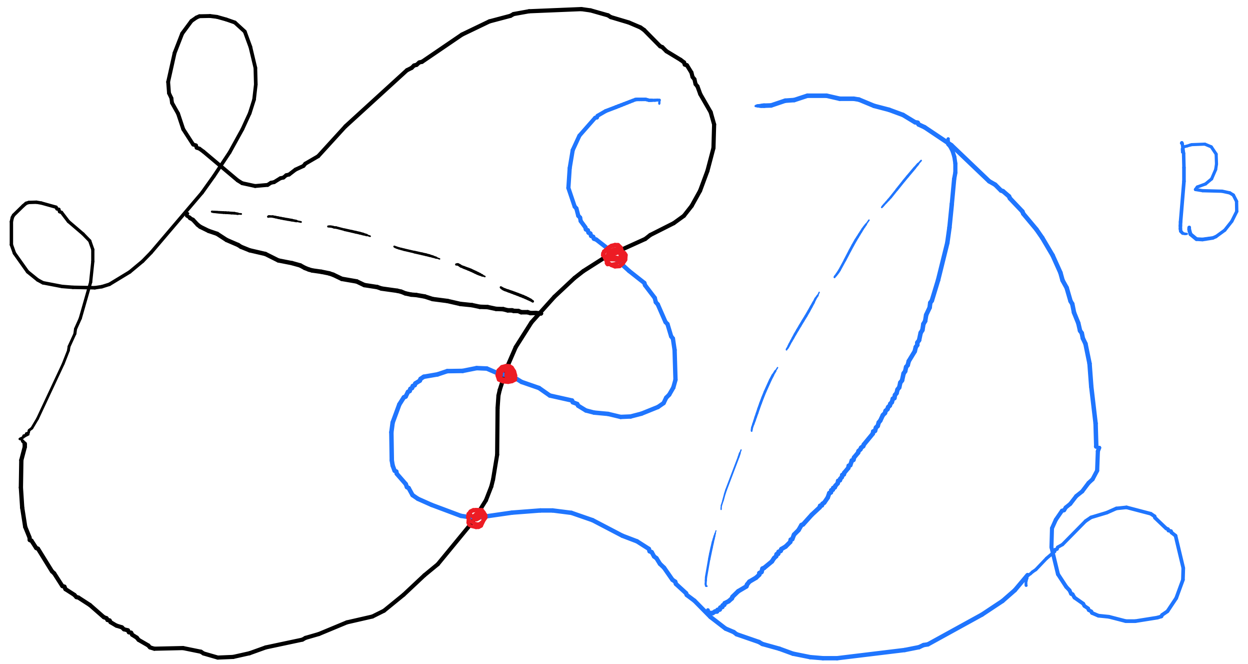
[Hurewicz] $\quad \cong \qquad\qquad \cong$

$$\pi_2(M) \otimes \pi_2(M) \longrightarrow \mathbb{Z}[\{1\}]$$

$$\begin{array}{cc} \Psi & \Psi \\ [A] & [B] \end{array}$$

group ring of the trivial group $\{1\} = \pi_1(M)$

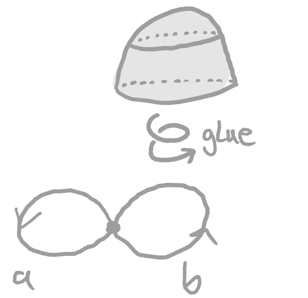
$A =$ immersed 2-sphere



Fun fact:

Any finitely presented group appears as $\pi_1\left(\begin{array}{c}\text{closed, smooth, oriented}\\ 4-\text{manifold}\end{array}\right)$

$\langle a, b \mid b^2 = e \rangle$



- Given presentation $\pi = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

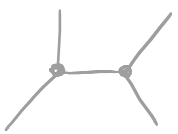
- build 2-complex $K(\pi) = \left(\bigvee_{\substack{\text{generators}\\ g_i}} S^1 \right) \cup_{\text{relations}} \bigcup^{m} D^2$

- $K(\pi) \hookrightarrow \mathbb{R}^5$

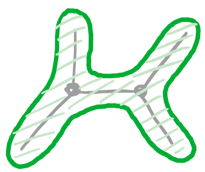
- take a closed tubular neighborhood $\nu K(\pi)$ ← 5-mfld.

  $\rightsquigarrow$ boundary $\partial \nu K(\pi)$ ← closed 4-mfld. with fundamental group π

$K(\pi) \subset \mathbb{R}^5$



$\partial \nu K(\pi)$



$\nu K(\pi)$

↗ <u>Markov's thm.:</u> Classification of 4-manifolds is undecidable in general

Some results for non-trivial fundamental groups:

For $\pi_1 \cong \mathbb{Z}$: [Freedman-Quinn, 1990]

- orientation character
- equivariant intersection form on π_2 } ↗ more on this soon
- Kirby-Siebenmann invariant

[Hambleton-Kreck, 1988]

Applied Freedman's results for manifolds with π_1 finite (finite groups are "good" in the sense of Freedman)

completed homeomorphism classification for finite cyclic groups $\mathbb{Z}/n$

$\rightsquigarrow$ $\boxed{\text{Homotopy classification}}$

Def.: oriented Poincaré 4-complex:

- finite CW-complex X
- oriented with a fundamental class $[X] \in H_4(X; \mathbb{Z})$

s.th. X "satisfies Poincaré duality", i.e.

$$- \cap [X] : C^{4-*}(X; \mathbb{Z}[\pi_1 X]) \longrightarrow C_*(X; \mathbb{Z}[\pi_1 X])$$

is a simple chain homotopy equivalence.

Ex.: every closed, oriented topological 4-manifold is homotopy equivalent to a Poincaré 4-complex

(but there are Poincaré 4-complexes which are not homotopy equivalent to any closed, topological 4-manifold [Hambleton-Milgram, 1978])

Def.: quadratic 2-type:

$$[\ \pi_1(X,*),\ \pi_2(X,*),\ k_X,\ \lambda_X\ ]$$

π_2 as a π_1-module

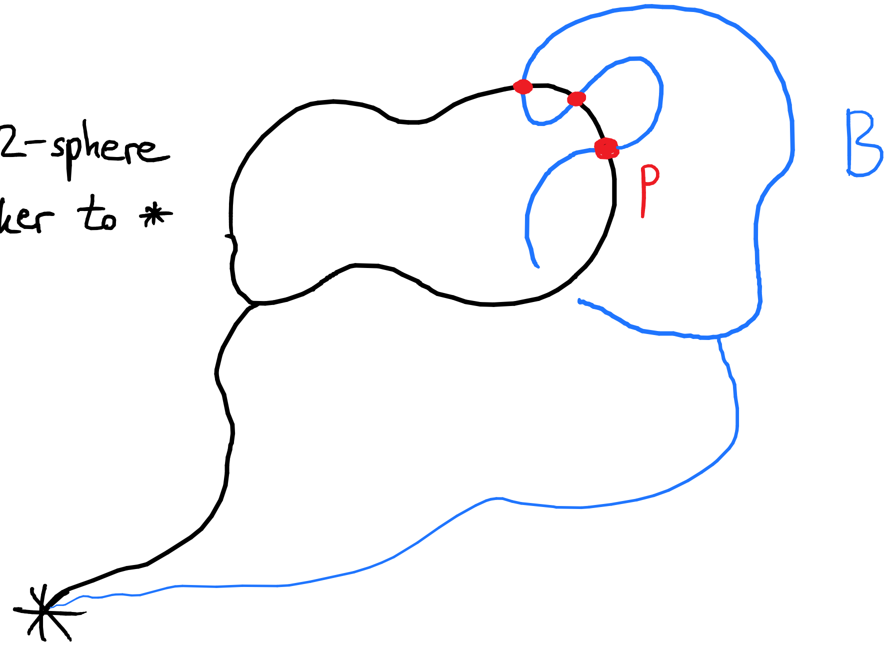
k-invariant

$k_X \in H^3(\pi_1(X);\pi_2(X))$

Equivariant intersection form:

$$\lambda_X : \underset{[A]}{\underset{\Psi}{\pi_2(X)}} \otimes \underset{[B]}{\underset{\Psi}{\pi_2(X)}} \longrightarrow \mathbb{Z}[\pi_1(X)]$$

$A =$ immersed 2-sphere with whisker to $*$

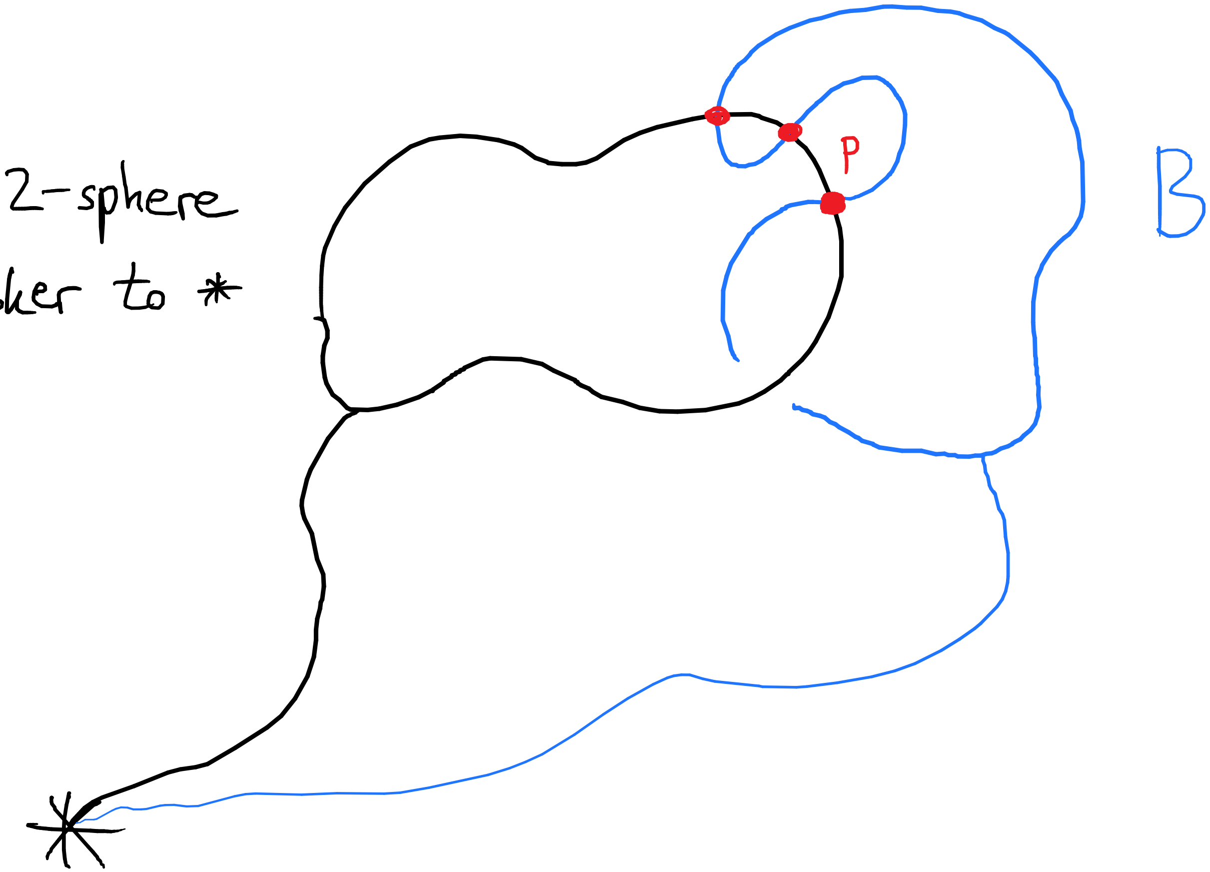


basepoint $*$

Equivariant intersection form:

$$\begin{array}{ccccc} \lambda_X : \pi_2(X,*) & \otimes & \pi_2(X,*) & \longrightarrow & \mathbb{Z}[\pi_1(X,*)] \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ [A] & & [B] & \longmapsto & \displaystyle\sum_{p \in A \pitchfork B} ? \end{array}$$

$A =$ immersed 2-sphere with whisker to $*$



basepoint $*$

Equivariant intersection form:

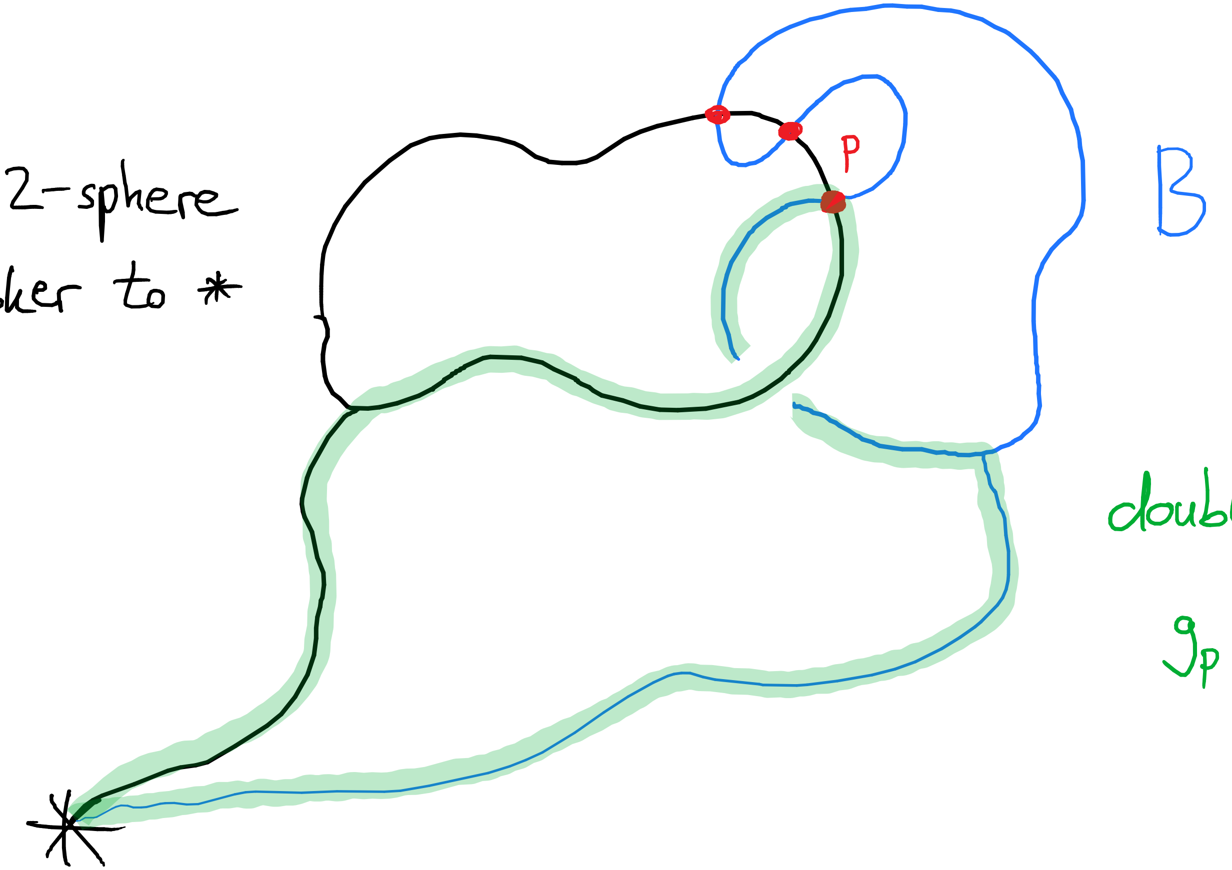
$$\lambda_X : \pi_2(X,*) \otimes \pi_2(X,*) \longrightarrow \mathbb{Z}[\pi_1(X,*)]$$

$$[A] \otimes [B] \longmapsto \sum_{p \in A \pitchfork B} \pm \, g_p$$

sign of intersection point p

double point loop

$A =$ immersed 2-sphere with whisker to $*$



double point loop

$g_p \in \pi_1(X)$

basepoint $*$

"History" of homotopy classification of 4-dim. Poincaré complexes:

Poincaré 4-complex $\rightsquigarrow$ quadratic 2-type

X $\qquad\qquad$ $[\pi_1(X,*), \pi_2(X,*), k_X, \lambda_X]$

•) [Hambleton-Kreck, 1988] 4-dim. oriented Poincaré complex with <u>fundamental group with 4-periodic cohomology</u> is classified up to homotopy by their quadratic 2-type

includes complexes with finite cyclic π_1

•) [Bauer, 1988] true if the 2-Sylow subgroup of π_1 has 4-periodic cohomology

M a module with an

action $\pi \circlearrowright M$ of the group π

Fixed points (invariants)

$$M^{\pi} := \{ m \in M \mid g \cdot m = m \text{ for } g \in \pi \}$$

$$\cong \mathrm{Hom}_{\mathbb{Z}[\pi]}(\mathbb{Z}, M)$$

right derived functor $\rightsquigarrow$

$$H^n(\pi; M) \cong \mathrm{Ext}^n_{\mathbb{Z}[\pi]}(\mathbb{Z}, M)$$

<u>Group cohomology</u>

Orbits (coinvariants)

$$M_{\pi} := M \big/ \langle m - g \cdot m \mid m \in M, g \in \pi \rangle$$

$$\cong \mathbb{Z} \otimes_{\mathbb{Z}[\pi]} M$$

$\rightsquigarrow$ Left derived functor

$$H_n(\pi; M) \cong \mathrm{Tor}_n^{\mathbb{Z}[\pi]}(\mathbb{Z}, M)$$

<u>Group homology</u>

π group, M a π-module

Tate (co)homology: Combine group homology & group cohomology for finite groups into a useful package

"norm map"

$$H_0(\pi; M) \xrightarrow{\cdot N} H^0(\pi; M)$$

$$\begin{array}{ccc} H_0(\pi;M) & \xrightarrow{\cdot N} & H^0(\pi;M) \\ \cong & & \cong \\ M_\pi & & M^\pi \\ \text{orbits} & & \text{fixed points} \end{array}$$

$$[m] \longmapsto \sum_{g \in \pi} g \cdot m$$

$$\hat{H}_n(\pi; M) := H_n(\pi; M) \quad \text{for } n \geq 1$$

$$\hat{H}_0(\pi; M) := \ker(\cdot N)$$

$$\hat{H}_{-1}(\pi; M) := \operatorname{coker}(\cdot N)$$

$$\hat{H}_n(\pi; M) := H^{-n-1}(\pi; M) \quad \text{for } n \leq -2$$

π is a **group with periodic cohomology** if

there exists $\quad u \in \hat{H}^d(\pi; \mathbb{Z})$

s.th. cup product $u \cup -$ is invertible

with a periodicity isomorphism $\quad u \cup - : \hat{H}^n(\pi; M) \xrightarrow{\cong} \hat{H}^{n+d}(\pi; M)$

$d = 4$

$$\begin{tikzcd}[column sep=small]
\cdots & \hat{H}^{-1}(\pi; M) \arrow[rrrr, dash, bend right, "\cong"'] & \hat{H}^{0}(\pi; M) \arrow[rrrr, dash, bend right, "\cong"'] & \hat{H}^{1}(\pi; M) \arrow[rrrr, dash, bend right, "\cong"'] & \hat{H}^{2}(\pi; M) & \hat{H}^{3}(\pi; M) & \hat{H}^{4}(\pi; M) & \hat{H}^{5}(\pi; M) & \cdots
\end{tikzcd}$$

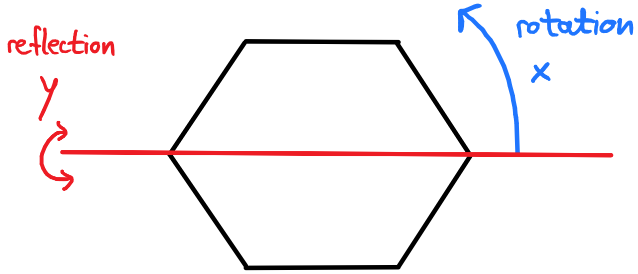
$\cdots \qquad \cdots$

Fact: A finite group π has d-periodic cohomology (for some $d \in \mathbb{N}$)

$\Leftrightarrow$ the abelian subgroups of π are cyclic

(i.e. π does not contain a subgroup of the form $\mathbb{Z}/p \times \mathbb{Z}/p$)

$\mathrm{Dih}_{2\cdot 6}$ = symmetries of a 6-gon



Dihedral group $\mathrm{Dih}_{2\cdot n} = \langle x, y \mid x^n = 1 = y^2,\ yxy^{-1} = x^{-1} \rangle$

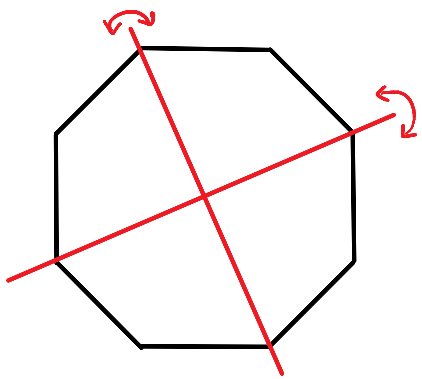
x: rotation of order n

y: reflection

•) If n is odd then $\mathrm{Dih}_{2\cdot n}$ has 4-periodic cohomology and so [Hambleton-Kreck] applies directly

•) If n is even not 4-periodic

subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2 < \mathrm{Dih}_{2\cdot 8}$



$$H_k(\mathrm{Dih}_{2\cdot n}; \mathbb{Z}) = \begin{cases} (\mathbb{Z}/2)^{\frac{k-1}{2}} & k \equiv 1 \mod 4 \\ (\mathbb{Z}/2)^{\frac{k+2}{2}} & k \equiv 2 \mod 4 \\ (\mathbb{Z}/2)^{\frac{k-1}{2}} & k \equiv 3 \mod 4 \\ \mathbb{Z}/n \oplus (\mathbb{Z}/2)^{\frac{k}{2}} & k \equiv 0 \mod 4 \end{cases}$$

Groups with periodic cohomology & free actions on spheres

no group element $\neq 1$ fixes a point of the space

> **Fact:** A finite group π which acts freely on a (finite-dim.) homology k-sphere has periodic cohomology with period $k+1$.

Ex.: •) Finite subgroups $\pi < \mathbb{S}^3 \cong SU(2)$

like finite cyclic groups $\mathbb{Z}/n$

(generalized) quaternion groups Q_{4n}

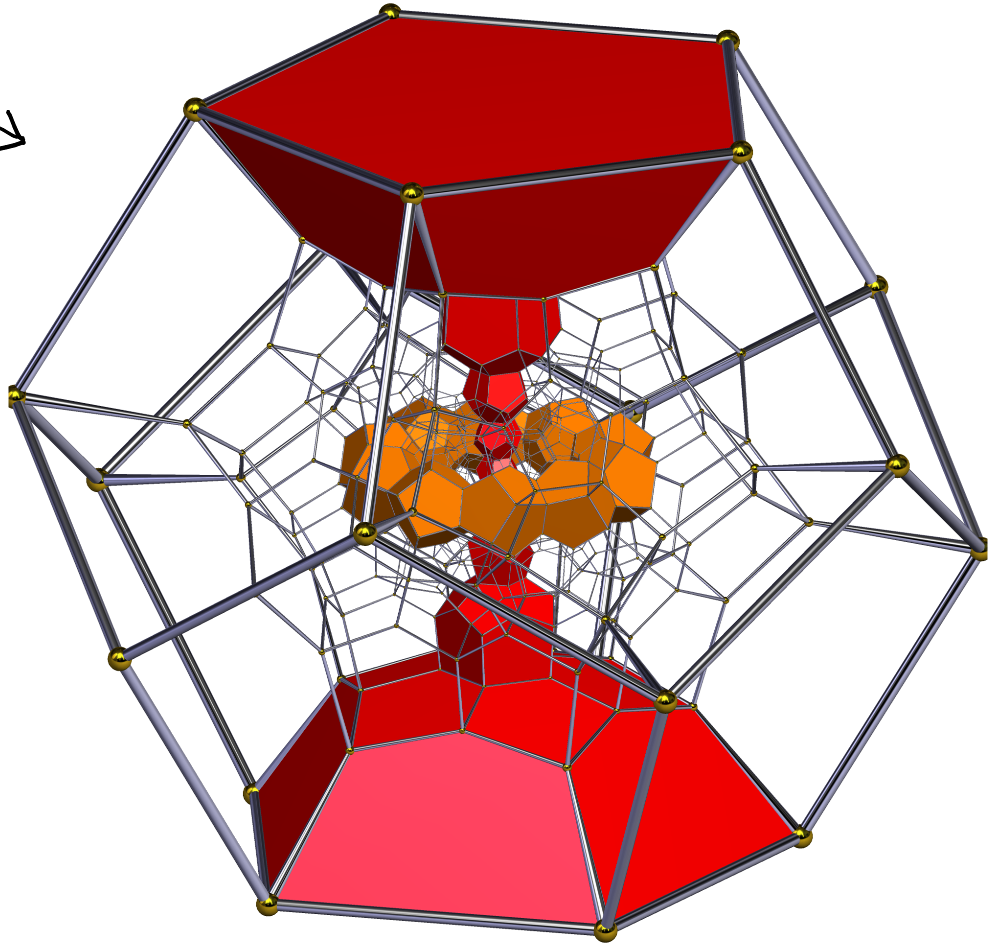
binary tetrahedral group A_4^* ; binary octahedral S_4^* ; binary dodecahedral A_5^*

•) Finite subgroups $\pi < SO(4) \circlearrowright \mathbb{S}^3$ with free action

by rotations

like central products $P_1^* \circ P_2^*$ of binary polyhedral groups ??

120-cell



free action of

binary dodecahedral group

$A_5^* \curvearrowright S^3$

Side remark: [Milnor]

If π acts freely on a sphere

$\Rightarrow$ π has at most one element of order 2

$\Rightarrow$ $Dih_{2\cdot n}$ cannot act freely on a sphere

Thm. Let π be a finite group s.th. its 2-Sylow subgroup is

- abelian with at most 2 generators or [Kasprowski-Powell-R, 2020]
- dihedral [Kasprowski-Nicholson-R, 2020]

Then oriented 4-dimensional Poincaré complexes X, X' with fundamental group π are homotopy equivalent

if and only if

their quadratic 2-types are isomorphic.

$$[\pi_1(X), \pi_2(X), k_X, \lambda_X : \pi_2(X) \otimes \pi_2(X) \to \mathbb{Z}[\pi_1(X)]] \sim [\pi_1(X'), \pi_2(X'), k_{X'}, \lambda_{X'} : \pi_2(X') \otimes \pi_2(X') \to \mathbb{Z}[\pi_1(X')]]$$

Rmk: Any isometry of the quadratic 2-types of X and X' is realized by a homotopy equivalence $X \to X'$.

[Hambleton-Kreck, Teichner]: If $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X))$ is $\mathbb{Z}$-torsion free,

4-dim. Poincaré complexes with finite fundamental group $\pi = \pi_1(X)$

are homotopy equivalent if and only if their quadratic 2-types are isomorphic.

Whitehead's Γ-groups: Let A be a $\mathbb{Z}[\pi]$-module.

For A with free abelian underlying $\mathbb{Z}$-module

$$\Gamma(A) = \langle\, b\otimes b,\ b\otimes b' + b'\otimes b \,\rangle_{\substack{b \neq b' \in \mathbb{Z}\text{-basis } \mathcal{B} \\ \text{of } A}} \subset A \otimes A$$

$\mathbb{Z}[\pi]$-module via the action $\pi \curvearrowright \Gamma(A) \longrightarrow \Gamma(A)$

$$g,\ \sum a_i \otimes b_i \longmapsto \sum (g\cdot a_i) \otimes (g \cdot b_i)$$

Whitehead observed that for a CW-complex L,

$\Gamma(\pi_2(L))$ fits into an exact sequence

$$H_4(\tilde{L};\mathbb{Z}) \longrightarrow \Gamma(\pi_2(L)) \xrightarrow{\text{"precomposing with Hopf map } \eta\text{"}} \pi_3(L) \xrightarrow{\text{Hurewicz}} H_3(\tilde{L};\mathbb{Z}) \longrightarrow 0$$

Useful fact: For A, A' free $\mathbb{Z}$-modules

$$\Gamma(A \oplus A') \cong \Gamma(A) \oplus (A \otimes_{\mathbb{Z}} A') \oplus \Gamma(A')$$

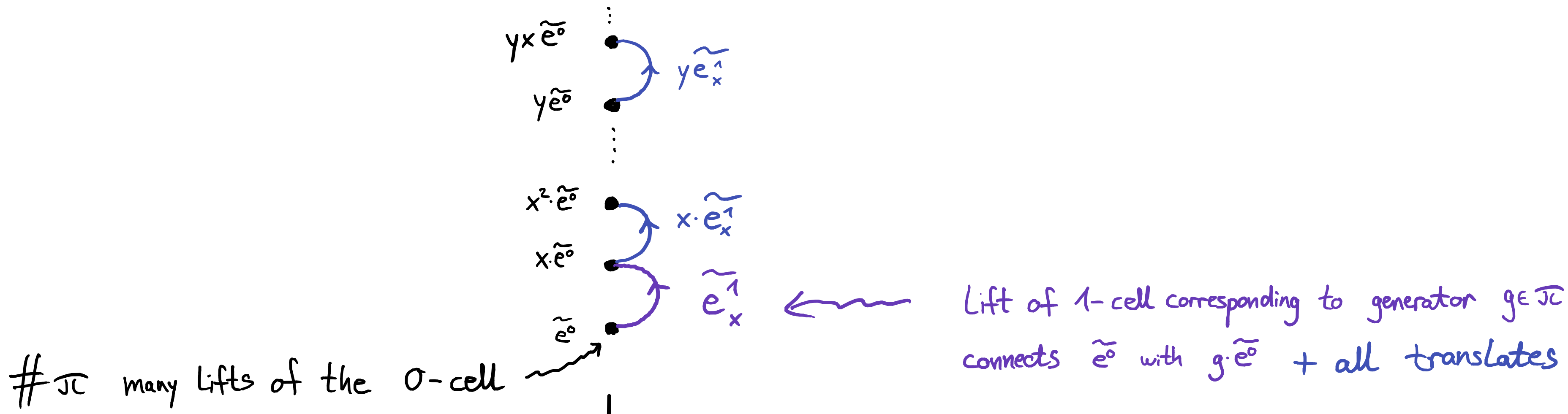
$$\pi_2(K(\pi)) \cong \pi_2(\widetilde{K(\pi)}) \cong H_2(\widetilde{K(\pi)};\mathbb{Z}) \cong H_2(K(\pi);\mathbb{Z}[\pi])$$

universal cover of $K(\pi)$

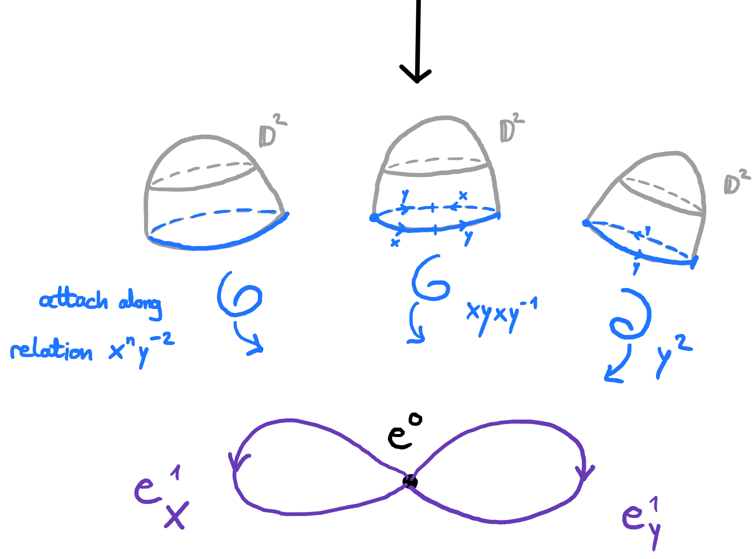
Hurewicz theorem: For simply connected space, the first nonzero homotopy group agrees with homology

this notation means you take local coefficients in $\mathbb{Z}[\pi]$

$\boxed{\widetilde{K(\pi)}}$



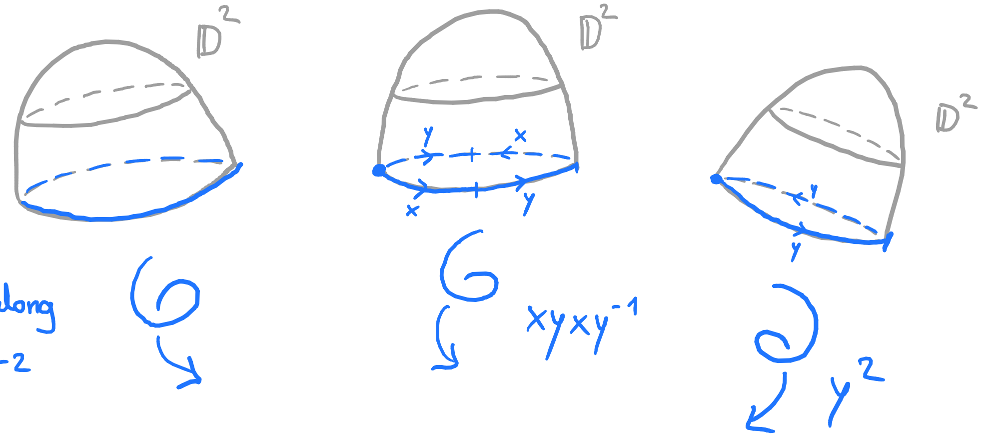
$K(\pi) =$



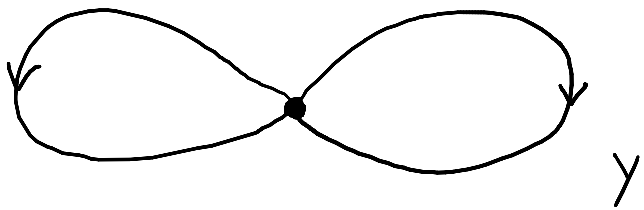
The differential d_2 in the universal cover is given by **Fox – derivatives**

running example: $\pi := \mathrm{Dih}_{2\cdot n} = \langle x, y \mid x^n y^{-2}, xyxy^{-1}, y^2 \rangle$

attach 2-cell along relation $x^n y^{-2}$

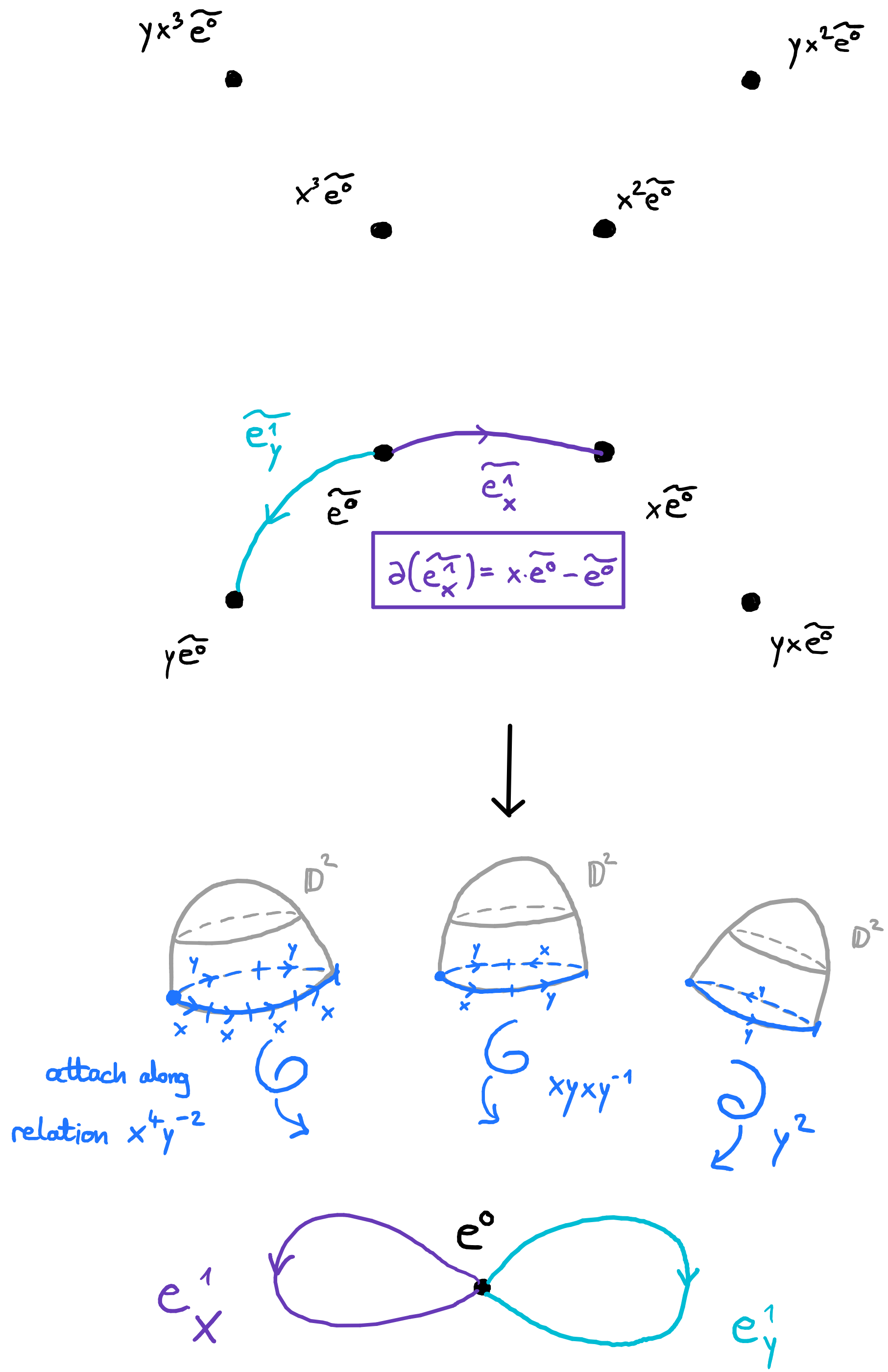


presentation 2-complex $K(\pi) =$



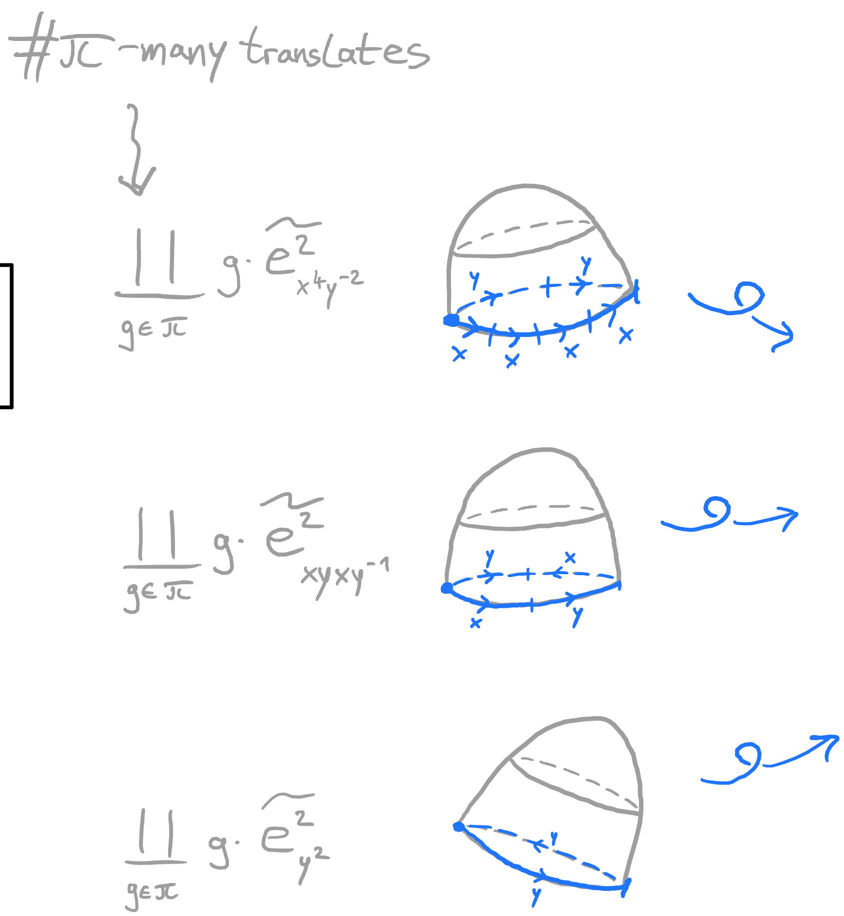
1-cells for generators

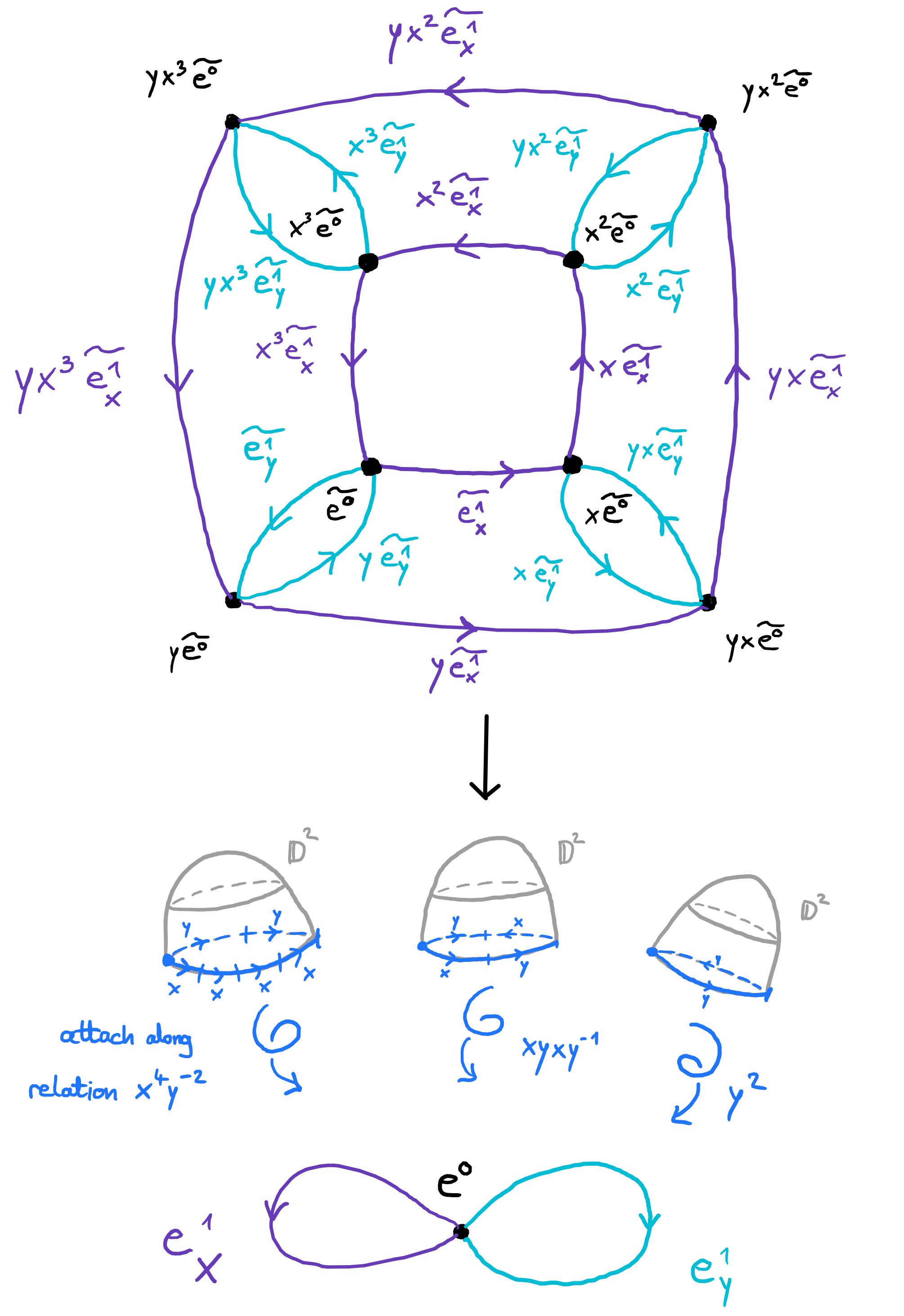
$\boxed{\widetilde{K}(\pi)}$



$K(\pi = \langle x, y \mid x^4y^{-2}, xyxy^{-1}, y^2 \rangle) =$

$\boxed{\widetilde{K(\pi)}}$





$$K(\pi = \langle x,y \mid x^4y^{-2}, xyxy^{-1}, y^2\rangle) =$$

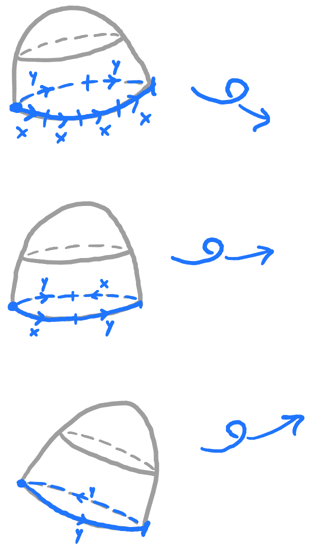
Cellular chain complex of $\widetilde{K(\pi)}$:

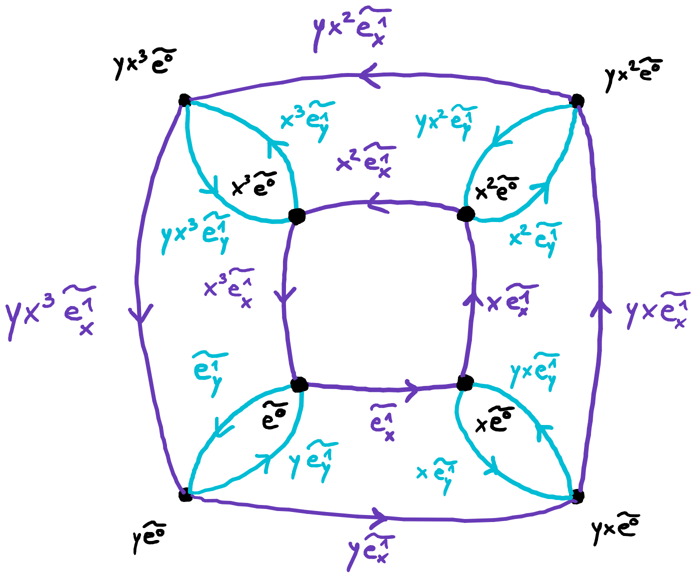
$$\begin{array}{ccccc}
C_2^{cell} & \xrightarrow{\quad d_2 \quad} & C_1^{cell} & \xrightarrow{\quad d_1 \quad} & C_0^{cell} \\
\wr\| & & \wr\| & & \wr\| \\
\mathbb{Z}[\pi]^{\oplus \text{relations}} & \longrightarrow & \mathbb{Z}[\pi]^{\oplus \text{generators}} & \longrightarrow & \mathbb{Z}[\pi] \\
\wr\| & & \wr\| & & \wr\| \\
\mathbb{Z}[\pi]\langle \widetilde{e}^2_{x^4y^{-2}} \rangle \oplus \mathbb{Z}[\pi]\langle \widetilde{e}^2_{xyxy^{-1}} \rangle \oplus \mathbb{Z}[\pi]\langle \widetilde{e}^2_{y^2} \rangle & \longrightarrow & \mathbb{Z}[\pi]\langle \widetilde{e}^1_x \rangle \oplus \mathbb{Z}[\pi]\langle \widetilde{e}^1_y \rangle & \longrightarrow & \mathbb{Z}[\pi]\langle \widetilde{e}^0 \rangle
\end{array}$$

$\coprod_{g\in\pi} g\cdot \widetilde{e}^2_{x^4y^{-2}}$

$\coprod_{g\in\pi} g\cdot \widetilde{e}^2_{xyxy^{-1}}$

$\coprod_{g\in\pi} g\cdot \widetilde{e}^2_{y^2}$



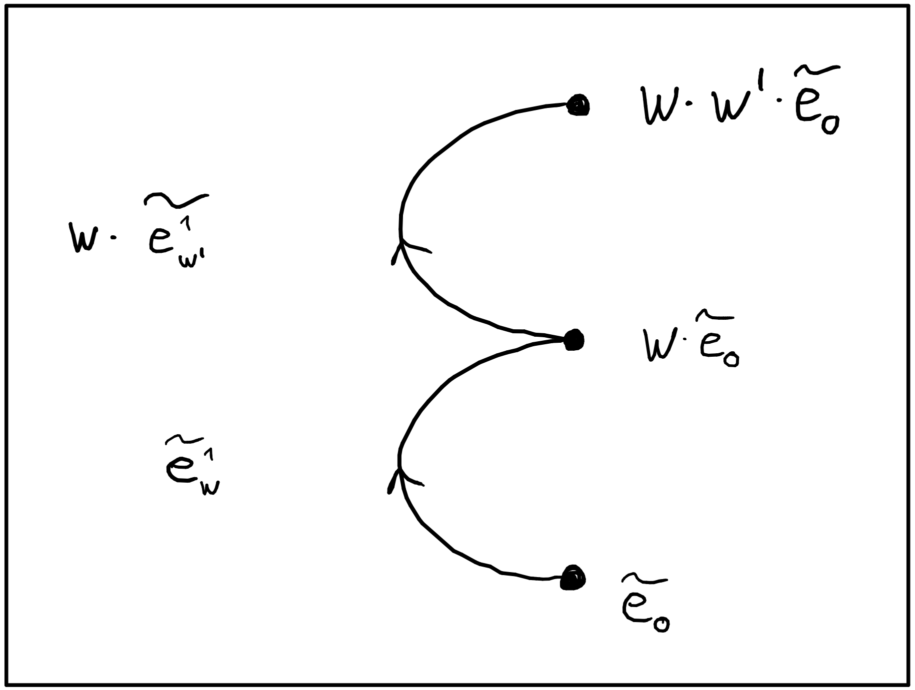


The differential d_2 in the universal cover is given by <u>Fox-derivatives</u>

$$\partial_a : \mathbb{Z}[F_r] \longrightarrow \mathbb{Z}[F_r]$$

$$\partial_a(b) = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{else} \end{cases} \qquad \partial_a(1) = 0$$

$$\partial_a(w \cdot w') = \partial_a(w) + w \cdot \partial_a(w')$$ "weird Leibnitz rule"



$$\mathrm{Dih}_{2\cdot n} = \langle x, y \mid x^n y^{-2},\ xyxy^{-1},\ y^2 \rangle$$

$$\begin{aligned}
\partial_x(x^n) = \partial_x(x \cdot x^{n-1}) &= \partial_x(x) + x \cdot \partial_x(x^{n-1}) \\
&= 1 + x \cdot \partial_x(x \cdot x^{n-1}) \\
&\ \ \vdots \\
&= 1 + x + x^2 + \cdots + x^{n-1}
\end{aligned}$$

$$\begin{aligned}
\partial_x(xyxy^{-1}) &= \partial_x(x) + x \cdot \partial_x(yxy^{-1}) \\
&= 1 + x \cdot y \cdot \partial_x(xy^{-1}) \\
&= 1 + x \cdot y \cdot \Big(\underbrace{\partial_x(x)}_{=1} + x \cdot \underbrace{\partial_x(y^{-1})}_{=0} \Big) \\
&= 1 + x \cdot y
\end{aligned}$$

$$\mathbb{Z}[\pi]^{\overset{\#\text{relations}}{\oplus 3}} \xrightarrow{\cdot \begin{pmatrix} 1+x+x^2+\ldots+x^{n-1} & \partial_y(x^n) \\ 1+xy & \partial_y(xyxy^{-1}) \\ \partial_x(y^2) & \partial_y(y^2) \end{pmatrix}} \mathbb{Z}[\pi]^{\overset{\#\text{generators}}{\oplus 2}} \xrightarrow{\cdot \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}[\pi]$$

[Hambleton–Kreck] For X finite 4-dim. oriented Poincaré complex with $\pi_1(X)$ finite:

Have short exact sequence of <u>stable</u> isomorphism classes of $\mathbb{Z}\pi$-modules

$$\boxed{0 \to \ker d_2 \to \pi_2(X) \oplus \mathbb{Z}\pi_1^{\oplus r} \to \operatorname{coker} d^2 \to 0}$$

d_2 from a free $\mathbb{Z}\pi_1$-module resolution

(C_*, d_*) of the trivial $\mathbb{Z}[\pi_1]$-module $\mathbb{Z}$

Example of such a differential d_2 for the presentation

$\langle x, y \mid x^n y^{-2}, xyxy^{-1}, y^2 \rangle$ of the dihedral group $D_{2\cdot n}$

$$C_*(\mathcal{P}): \quad 0 \to \ker(d_2) \to \mathbb{Z}\pi^3 \xrightarrow[d_2]{\cdot\begin{pmatrix} N_x & -(1+y) \\ 1+xy & x-1 \\ 0 & y+1 \end{pmatrix}} \mathbb{Z}\pi^2 \xrightarrow[d_1]{\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

X finite 4-dim. oriented Poincaré complex with $\pi_1(X)$ finite

Have short exact sequence of stable isomorphism classes of $\mathbb{Z}\pi$-modules

$$\boxed{0 \longrightarrow \ker d_2 \longrightarrow \pi_2(X) \oplus \mathbb{Z}\pi_1^{\oplus r} \longrightarrow \operatorname{coker} d^2 \longrightarrow 0}$$

d_2 from a free $\mathbb{Z}\pi_1$ -module resolution

(C_*, d_*) of the trivial $\mathbb{Z}[\pi_1]$-module $\mathbb{Z}$

Strategy for showing that $\operatorname{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X))\right) = 0$:

1. Show that $\operatorname{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\ker d_2)\right) = 0$

2. Show that $\operatorname{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\operatorname{coker} d^2)\right) = 0$

X finite 4-dim. oriented Poincaré complex with $\pi_1(X)$ finite

$$\boxed{0 \longrightarrow \ker d_2 \longrightarrow \pi_2(X) \oplus \mathbb{Z}\pi_1^{\oplus r} \longrightarrow \operatorname{coker} d^2 \longrightarrow 0}$$

<u>The choice of resolution (C_*, d_*) does not matter for computing $\operatorname{Tors}(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X)))$:</u>

- for any two choices of resolution $d_*, \tilde{d}_*$ the $\mathbb{Z}\pi$-modules

$$\operatorname{Ker} d_2 \cong_{\text{stably}} \operatorname{Ker} \tilde{d}_2 \quad ; \quad \operatorname{Coker} d^2 \cong_{\text{stably}} \operatorname{Coker} \tilde{d}_2$$

are stably isomorphic

- $\operatorname{Tors}(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(D))$ does not change if we stabilize

$$D \rightsquigarrow D \oplus \mathbb{Z}[\pi_1(X)]^{\oplus r}$$

Summary: Classifying 4-manifolds is hard !

Fix a fundamental group — we looked at finite groups $\mathbb{Z}/n \times \mathbb{Z}/m$ and $Dih_{2\cdot m}$

Try to find invariants that pin down the homotopy type of an oriented 4-dimensional Poincaré complex X with finite fundamental group π

$\rightsquigarrow$ quadratic 2-type $[\pi_1(X), \pi_2(X), k_X, \lambda_X]$

Our result:
[Kasprowski-Powell-R, 2020]
[Kasprowski-Nicholson-R, 2020]

If the 2-Sylow subgroup of π is $\mathbb{Z}/2^k \times \mathbb{Z}/2^l$ or $Dih_{2\cdot k}$, the isometry class of the 2-type is enough!

How? Using results of Hambleton-Kreck, Teichner:

Enough to show that $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X))$ is torsion free.

Excerpt from the proof for $\mathbb{Z}/_n \times \mathbb{Z}/_m$:

Proof. For the group $\pi = \langle a, b \mid a^n, b^m, [a,b]\rangle$ let $N_a := \sum_{i=0}^{n-1} a^i$ and $N_b := \sum_{i=0}^{m-1} b^i$. Let $C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$ be the chain complex corresponding to the presentation $\langle a, b \mid a^n, b^m, [a,b]\rangle$. Extend this to the standard free resolution of $\mathbb{Z}$ as a $\mathbb{Z}\pi$-module:

$$
\begin{tikzcd}
C_4 \arrow[d, "d_4"] & \mathbb{Z}\pi \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "b-1"] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"] \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "b-1"] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"] \\
C_3 \arrow[d, "d_3"] & & \mathbb{Z}\pi \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "-N_b"] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] & \\
C_2 \arrow[d, "d_2"] & & & \mathbb{Z}\pi \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "b-1"] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"] & & \\
C_1 \arrow[d, "d_1"] & & & & \mathbb{Z}\pi \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"] & & & \\
C_0 & & & & & \mathbb{Z}\pi & & & &
\end{tikzcd}
$$

By exactness, $\ker d_2 \cong \operatorname{im} d_3 \cong C_3/\ker d_3 \cong \operatorname{coker} d_4$. From this it follows that

$$\ker d_2 \cong (\mathbb{Z}\pi)^4/\langle (N_a, 0, 0, 0), (b-1, 1-a, 0, 0), (0, N_b, N_a, 0), (0, 0, b-1, 1-a), (0, 0, 0, N_b)\rangle.$$

$$\mathrm{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\ker d_2)\right)$$

$$\cong$$

Example 5.1. The following is a complete list of all groups of order at most 16 such that $\widehat{H}_0(\pi; \Gamma(\ker d_2))$ is non-trivial. The group $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$ is the quaternion group.

zeroth Tate-homology

π	$\widehat{H}_0(\pi; \Gamma(\ker d_2))$
$\mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^4$
$Q_8 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^4$

I.e. for these groups, the [Hambleton-Kreck] strategy does not apply, and we don't know whether the homotopy type is determined by the quadratic 2-type.

Realization of quadratic 2-types

In other words: Which quadratic 2-types $[\pi_1(X), \pi_2(X), k_X, \lambda_X]$

actually occur for a manifold $X = M^4$?

Stabilization of 4-manifolds: $M \rightsquigarrow M \#^r S^2 \times S^2$

$$[\pi_1, \pi_2, k, \lambda] \rightsquigarrow [\pi_1, \pi_2 \oplus \mathbb{Z}[\pi]^{\oplus 2r}, i_*(k), \lambda \oplus H(\mathbb{Z}[\pi]^{\oplus r})]$$

hyperbolic form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus r}$$

[Hambleton-Kreck] For π_1 finite, a quadratic 2-type is realizable by a topological 4-manifold $\Leftrightarrow$ it is stably realizable.

Strategy to study the homeomorphism classification of top. 4-mflds with π_1 finite:

Solve "cancellation" for quadratic 2-types; then apply surgery theory

Thanks !