2021 Graduate Student Topology and Geometry Conference, 30 min talk

Homotopy classification of 4-manifolds with

- finite abelian 2-generator fundamental group
- dihedral fundamental group

Based on joint work with Daniel Kasprowski,
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Plan: Classification of 4-manifolds

This talk will be in the topological category

- Warmup: Homeomorphism classification of simply-connected 4-manifolds
- Our ignorance for non-trivial fundamental groups

$\rightsquigarrow$ $\boxed{\text{Homotopy classification}}$

Poincaré 4-complexes

[Hambleton-Kreck]

π_1 = finite

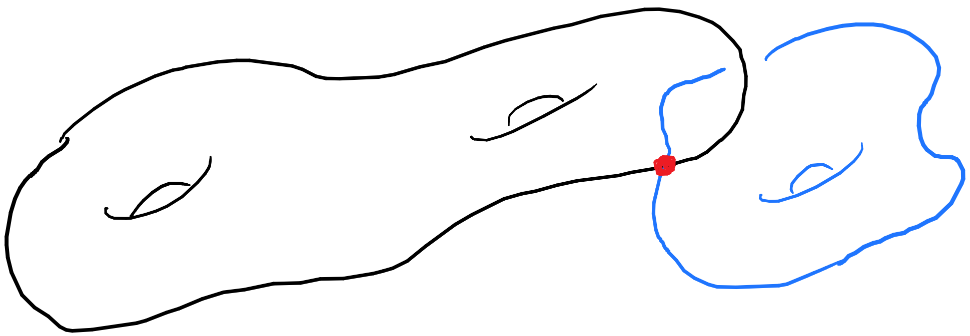
$\rightsquigarrow$ quadratic 2-type

Whitehead's Γ-construction

What we ([Kasprowski-Powell-R, 2020], [Kasprowski-Nicholson-R, 2020]) found for $\pi_1 \cong \mathbb{Z}/n \times \mathbb{Z}/m$ and $\pi_1 \cong \mathrm{Dih}_{2\cdot m}$

Simply-connected oriented 4-manifolds and intersection forms

Intersection form $$\underset{\underset{[A]}{\rotatebox{90}{$\in$}}}{H_2(M^4)} \otimes_{\mathbb{Z}} \underset{\underset{[B]}{\rotatebox{90}{$\in$}}}{H_2(M^4)} \xrightarrow{\lambda_M} \mathbb{Z}$$



[Milnor (1958)] Homotopy classification of simply-connected closed oriented 4-mflds.

$M \simeq_{\text{htpy eq.}} N$ iff. $\lambda_M \cong_{\text{isometric}} \lambda_N$

[Freedman (1984)] Homeomorphism ———— " ————

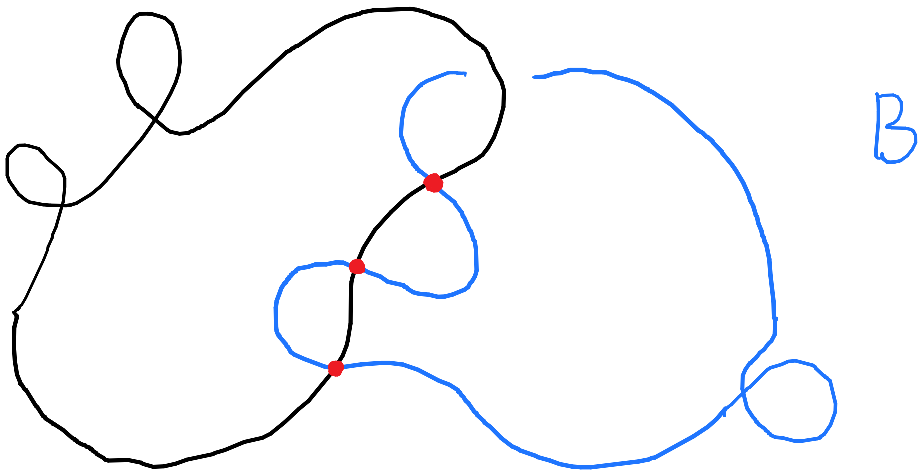
(Intersection form + Kirby-Siebenmann invariant)

Simply-connected oriented 4-manifolds and intersection forms

$$\begin{array}{ccccc} H_2(M^4) & \otimes_{\mathbb{Z}} & H_2(M^4) & \longrightarrow & \mathbb{Z} \\ \text{Hurewicz} & \cong & & & \cong \\ \pi_2(M) & \otimes & \pi_2(M) & \longrightarrow & \mathbb{Z}[\{1\}] \\ \cup\!\! \mid & & \cup\!\! \mid & & \\ [A] & & [B] & & \end{array}$$

group ring of the trivial group $\{1\} = \pi_1(M)$

$A =$ immersed 2-sphere

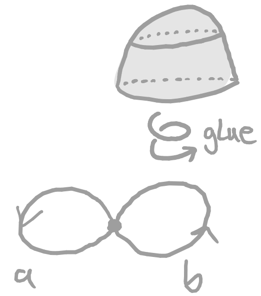


Fun fact:

Any finitely presented group appears as $\pi_1\left(\begin{array}{c}\text{closed, smooth, oriented} \\ \text{4-manifold}\end{array}\right)$

Given presentation $\quad \pi = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$ $\qquad \langle a, b \mid b^2 = e \rangle$

build 2-complex $\quad K(\pi) = \left(\bigvee_{\substack{\text{generators} \\ g_i}} S^1 \right) \cup_{\text{relations}} \bigcup^m D^2$



$K(\pi) \hookrightarrow \mathbb{R}^5$

take a closed tubular neighborhood $\quad \nu K(\pi) \leftarrow$ 5-mfld.

boundary $\quad \partial \nu K(\pi) \leftarrow$ closed 4-mfld. with fundamental group π

($\nearrow$ Markov's thm.: Classification of 4-manifolds is undecidable in general)

Some results for non-trivial fundamental groups:

[Freedman–Quinn, 1990]

For $\pi_1 \cong \mathbb{Z}$:

- orientation character
- equivariant intersection form on π_2
- Kirby–Siebenmann invariant

} ↗ more on this soon

[Hambleton–Kreck, 1988]

Applied Freedman's results for manifolds with π_1 finite

(finite groups are "good" in the sense of Freedman)

completed homeomorphism classification for finite cyclic groups $\mathbb{Z}/n$

$\rightsquigarrow$ Homotopy classification

Def.: oriented Poincaré 4-complex:

- finite CW-complex X
- oriented with a fundamental class $[X] \in H_4(X;\mathbb{Z})$

s.th. X "satisfies Poincaré duality", i.e.

$$- \cap [X] : C^{4-*}(X;\mathbb{Z}[\pi_1 X]) \longrightarrow C_*(X;\mathbb{Z}[\pi_1 X])$$

is a simple chain homotopy equivalence.

Ex.: every closed, oriented topological 4-manifold is homotopy equivalent to a Poincaré 4-complex

(but there are Poincaré 4-complexes which are not homotopy equivalent to any closed, topological 4-manifold [Hambleton-Milgram, 1978])

Def.: quadratic 2-type:

$$[\,\pi_1(X,*),\ \pi_2(X,*),\ k_X,\ \lambda_X\,]$$

π_2 as a π_1-module

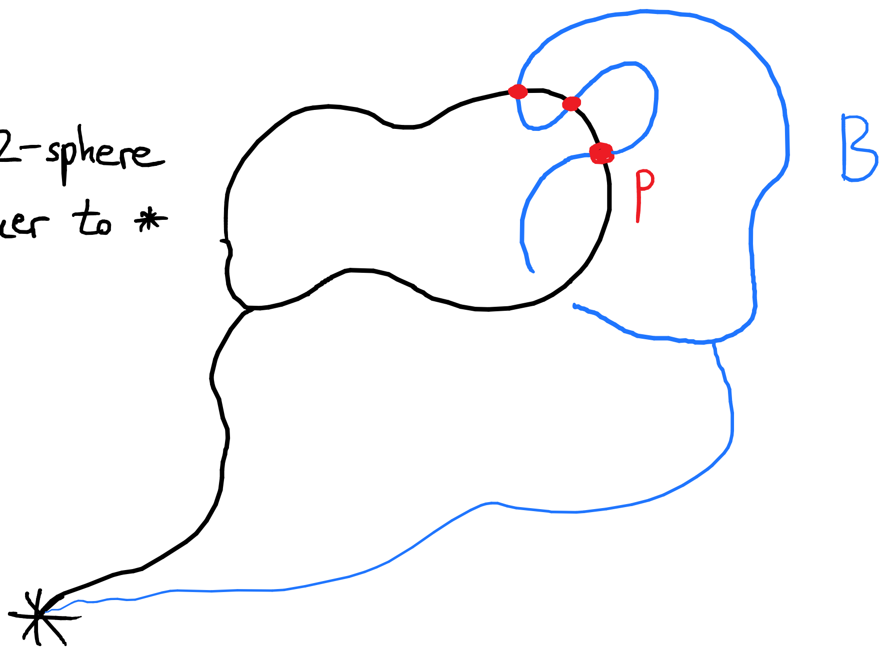
k-invariant

$k_X \in H^3(\pi_1(X);\pi_2(X))$

Equivariant intersection form:

$$\lambda_X : \underset{\substack{\cup \\ [A]}}{\pi_2(X)} \otimes \underset{\substack{\cup \\ [B]}}{\pi_2(X)} \longrightarrow \mathbb{Z}[\pi_1(X)]$$

$A =$ immersed 2-sphere with whisker to $*$



basepoint $*$

Equivariant intersection form:

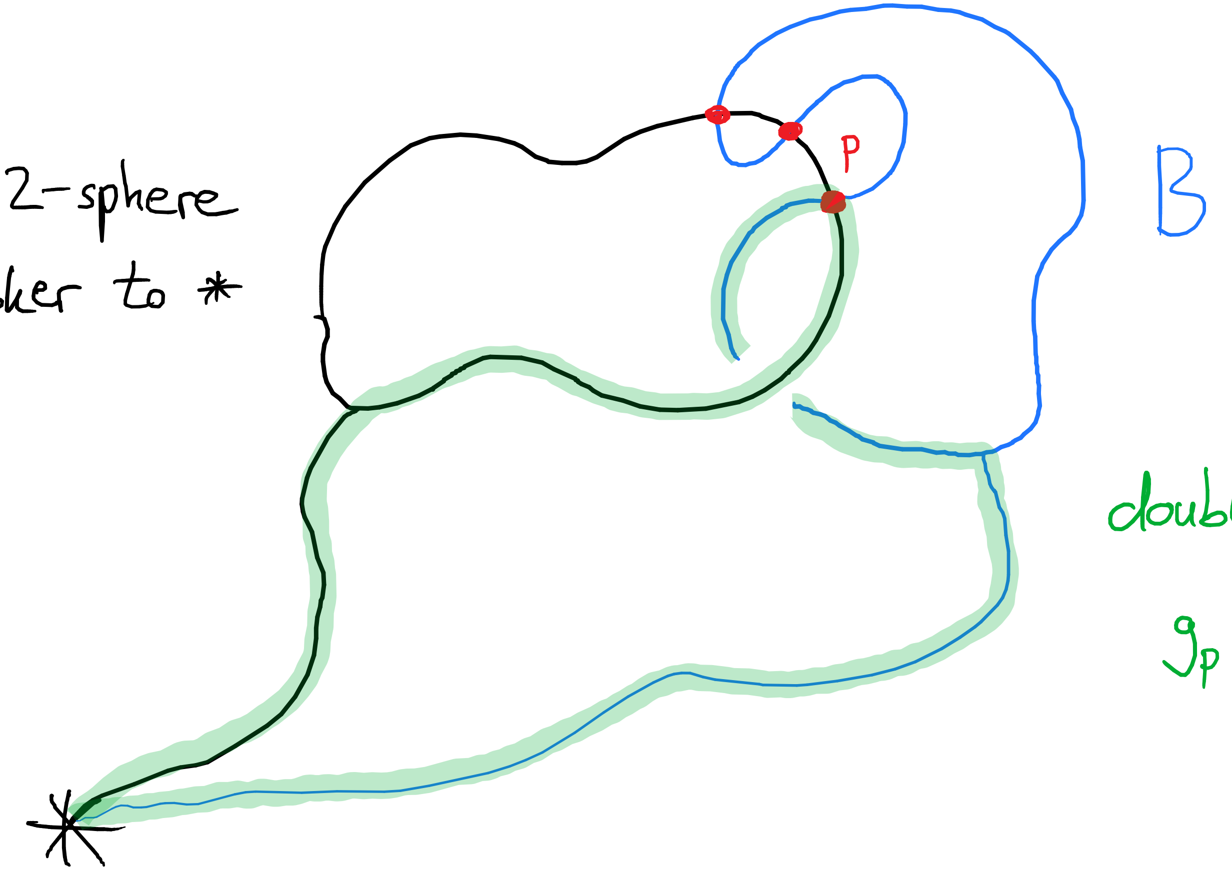
$$\lambda_X : \pi_2(X) \otimes \pi_2(X) \longrightarrow \mathbb{Z}[\pi_1(X)]$$

$$\begin{array}{ccccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ [A] & & [B] & \longmapsto & \displaystyle\sum_{p \in A \pitchfork B} \pm\, g_p \end{array}$$

sign of intersection point p

double point loop

A = immersed 2-sphere with whisker to *



basepoint *

double point loop

$g_p \in \pi_1(X)$

"History" of homotopy classification of 4-dim. Poincaré complexes:

Poincaré 4-complex $\rightsquigarrow$ quadratic 2-type

$$[\,\pi_1(X,*),\ \pi_2(X,*),\ k_X,\ \lambda_X\,]$$

[Hambleton-Kreck, 1988] 4-dim. oriented Poincaré complex with fundamental group with 4-periodic cohomology is classified up to homotopy by their quadratic 2-type

includes complexes with finite cyclic π_1

[Bauer, 1988] true if the 2-Sylow subgroup of π_1 has 4-periodic cohomology

Thm. Let π be a finite group s.th. its 2-Sylow subgroup is

- abelian with at most 2 generators or [Kasprowski-Powell-R, 2020]
- dihedral [Kasprowski-Nicholson-R, 2020]

Then oriented 4-dimensional Poincaré complexes X, X' with fundamental group π are homotopy equivalent if and only if their quadratic 2-types are isomorphic.

$$[\pi_1(X), \pi_2(X), k_X, \lambda_X : \pi_2(X) \otimes \pi_2(X) \to \mathbb{Z}[\pi_1(X)]] \sim [\pi_1(X'), \pi_2(X'), k_{X'}, \lambda_{X'} : \pi_2(X') \otimes \pi_2(X') \to \mathbb{Z}[\pi_1(X')]]$$

[Hambleton-Kreck, Teichner]: If $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X))$ is $\mathbb{Z}$-torsion free,

4-dim. Poincaré complexes with finite fundamental group $\pi = \pi_1(X)$

are homotopy equivalent if and only if their quadratic 2-types are isomorphic.

Whitehead's Γ-groups: Let A be a $\mathbb{Z}[\pi]$-module.

For A with free abelian underlying $\mathbb{Z}$-module

$$\Gamma(A) = \langle b \otimes b,\ b \otimes b' + b' \otimes b \rangle_{\substack{b \neq b' \in \mathbb{Z}\text{-basis } \mathcal{B} \\ \text{of } A}} \subset A \otimes A$$

$\mathbb{Z}[\pi]$-module via the action $\pi \curvearrowright \Gamma(A) \longrightarrow \Gamma(A)$

$$g,\ \sum a_i \otimes b_i \longmapsto \sum (g \cdot a_i) \otimes (g \cdot b_i)$$

Whitehead observed that for a CW-complex L,

$\Gamma(\pi_2(L))$ fits into an exact sequence

$$H_4(\tilde{L};\mathbb{Z}) \longrightarrow \Gamma(\pi_2(L)) \xrightarrow{\text{"precomposing with Hopf map } \eta\text{"}} \pi_3(L) \xrightarrow{\text{Hurewicz}} H_3(\tilde{L};\mathbb{Z}) \longrightarrow 0$$

<u>Useful fact:</u> For A, A' free $\mathbb{Z}$-modules

$$\Gamma(A \oplus A') \cong \Gamma(A) \oplus (A \otimes_{\mathbb{Z}} A') \oplus \Gamma(A')$$

Have short exact sequence of stable isomorphism classes of $\mathbb{Z}\pi$-modules

$$\boxed{0 \longrightarrow \ker d_2 \longrightarrow \pi_2(X) \oplus \mathbb{Z}\pi_1^{\oplus r} \longrightarrow \operatorname{coker} d^2 \longrightarrow 0}$$

d_2 from a free $\mathbb{Z}\pi_1$-module resolution

(C_*, d_*) of the trivial $\mathbb{Z}[\pi_1]$-module $\mathbb{Z}$

Example of such a differential d_2 for the presentation

$\langle x, y \mid x^n \cdot y^{-2}, xyxy^{-1}, y^2 \rangle$ of the dihedral group $D_{2\cdot n}$

$$C_*(\mathcal{P}): \quad 0 \to \ker(d_2) \to \mathbb{Z}\pi^3 \xrightarrow[d_2]{\cdot\begin{pmatrix} N_x & -(1+y) \\ 1+xy & x-1 \\ 0 & y+1 \end{pmatrix}} \mathbb{Z}\pi^2 \xrightarrow[d_1]{\cdot\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}} \mathbb{Z}\pi \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

<u>Strategy for showing that $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X)) = 0$:</u>

- Show that $\operatorname{Tors}(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\ker d_2)) = 0$
- Show that $\operatorname{Tors}(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\operatorname{coker} d^2)) = 0$

$$\boxed{0 \longrightarrow \ker d_2 \longrightarrow \pi_2(X) \oplus \mathbb{Z}\pi_1^{\oplus r} \longrightarrow \operatorname{coker} d^2 \longrightarrow 0}$$

The choice of resolution (C_*, d_*) does not matter for computing $\operatorname{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X))\right)$:

•) for any two choices of resolution $d_*, \widetilde{d_*}$ the $\mathbb{Z}\pi$-modules

$$\operatorname{Ker} d_2 \cong_{\text{stably}} \operatorname{Ker} \widetilde{d_2}$$

are stably isomorphic

$$\operatorname{Coker} d^2 \cong_{\text{stably}} \operatorname{Coker} \widetilde{d_2}$$

•) $\operatorname{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(D)\right)$ does not change if we stabilize

$$D \rightsquigarrow D \oplus \mathbb{Z}[\pi_1(X)]^{\oplus r}$$

Summary: Classifying 4-manifolds is hard

Fix a fundamental group — we looked at finite groups $\mathbb{Z}/n \times \mathbb{Z}/m$ and $Dih_{2\cdot m}$

Try to find invariants that pin down the homotopy type of an oriented 4-dimensional Poincaré complex X with finite fundamental group π

$\rightsquigarrow$ quadratic 2-type $[\pi_1(X), \pi_2(X), k_X, \lambda_X]$

Our result: If the 2-Sylow subgroup of π is $\mathbb{Z}/2^k \times \mathbb{Z}/2^l$ or $Dih_{2\cdot k}$,

[Kasprowski–Powell–R, 2020]

[Kasprowski–Nicholson–R, 2020]

the isometry class of the 2-type is enough!

How? Using results of Hambleton–Kreck, Teichner:

Enough to show that $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\pi_2(X))$ is torsion free.

Excerpt from the proof for $\mathbb{Z}/_n \times \mathbb{Z}/_m$:

Proof. For the group $\pi = \langle a, b \mid a^n, b^m, [a,b] \rangle$ let $N_a := \sum_{i=0}^{n-1} a^i$ and $N_b := \sum_{i=0}^{m-1} b^i$. Let $C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$ be the chain complex corresponding to the presentation $\langle a, b \mid a^n, b^m, [a,b] \rangle$. Extend this to the standard free resolution of $\mathbb{Z}$ as a $\mathbb{Z}\pi$-module:

$$
\begin{tikzcd}
C_4 \arrow[d, "d_4"] & \mathbb{Z}\pi \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "b-1"'] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"'] \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "b-1"'] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"'] \\
C_3 \arrow[d, "d_3"] & & \mathbb{Z}\pi \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"'] \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "-N_b"'] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"'] & \\
C_2 \arrow[d, "d_2"] & & & \mathbb{Z}\pi \arrow[dr, "N_a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "b-1"'] \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "N_b"'] & & \\
C_1 \arrow[d, "d_1"] & & & & \mathbb{Z}\pi \arrow[dr, "1-a"] & \oplus & \mathbb{Z}\pi \arrow[dl, "1-b"'] & & & \\
C_0 & & & & & \mathbb{Z}\pi & & & &
\end{tikzcd}
$$

By exactness, $\ker d_2 \cong \operatorname{im} d_3 \cong C_3 / \ker d_3 \cong \operatorname{coker} d_4$. From this it follows that

$$\ker d_2 \cong (\mathbb{Z}\pi)^4 / \langle (N_a, 0, 0, 0), (b-1, 1-a, 0, 0), (0, N_b, N_a, 0), (0, 0, b-1, 1-a), (0, 0, 0, N_b) \rangle.$$

$\mathrm{Tors}\left(\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} \Gamma(\ker d_2)\right)$

$\cong$

Example 5.1. The following is a complete list of all groups of order at most 16 such that $\widehat{H}_0(\pi; \Gamma(\ker d_2))$ is non-trivial. The group $Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = ijk \rangle$ is the quaternion group.

zeroth Tate-homology

π	$\widehat{H}_0(\pi; \Gamma(\ker d_2))$
$\mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^4$
$Q_8 \times \mathbb{Z}/2$	$(\mathbb{Z}/2)^4$

Thanks!