Baby Geometri Seminar

2021-03-10 , 14:15 CET

60 min talk

Zeeman's fibration theorem & trisecting knotted surface groups

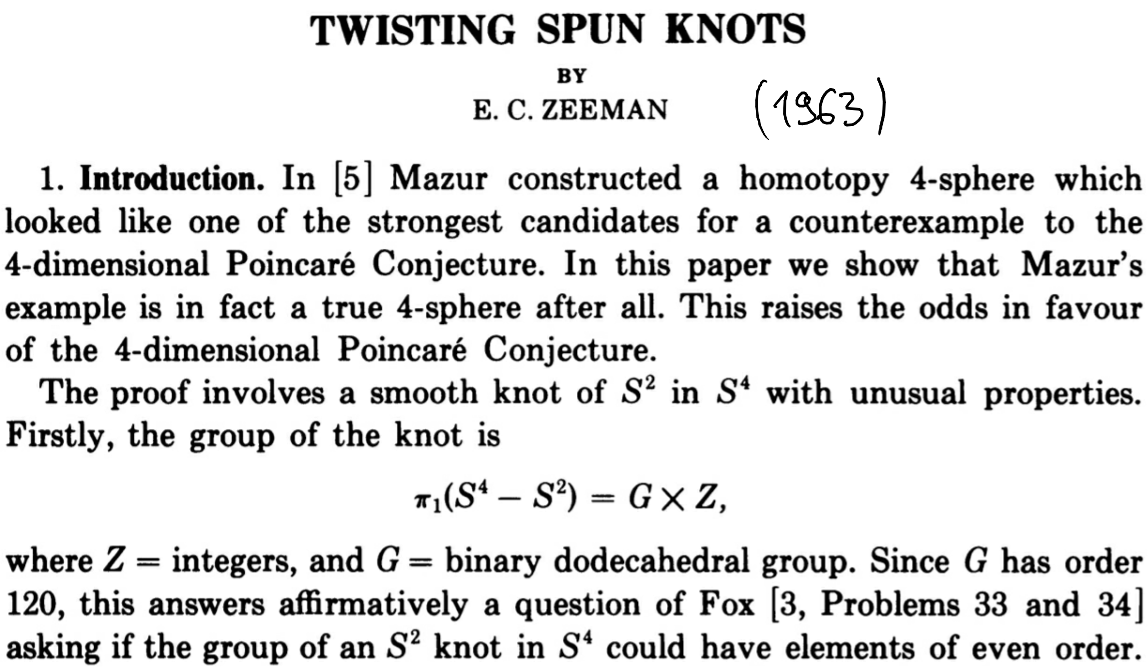
Benjamin Matthias Ruppik,

3rd year PhD student at the Max Planck Institute for Mathematics, Bonn

Zeeman's fibration theorem:

$m \neq 0$

The m-twist spin of a classical knot k is fibred by the punctured m-fold cyclic branched cover of k.



TWISTING SPUN KNOTS

BY

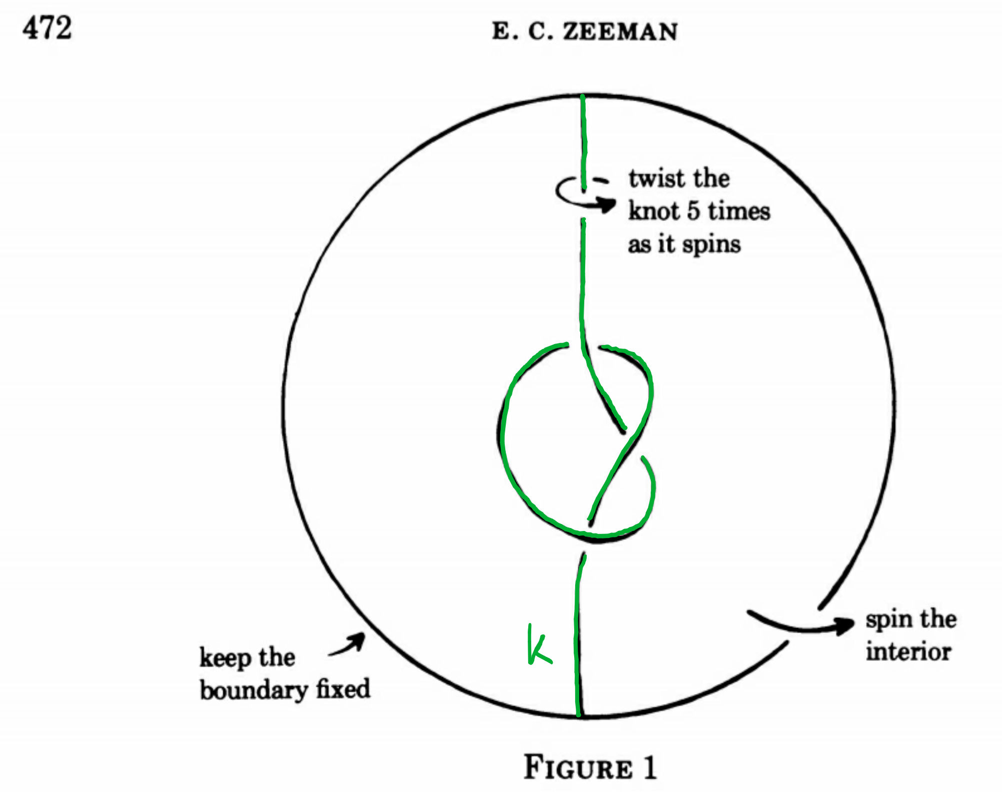
E. C. ZEEMAN (1963)

1. **Introduction.** In [5] Mazur constructed a homotopy 4-sphere which looked like one of the strongest candidates for a counterexample to the 4-dimensional Poincaré Conjecture. In this paper we show that Mazur's example is in fact a true 4-sphere after all. This raises the odds in favour of the 4-dimensional Poincaré Conjecture.

The proof involves a smooth knot of S^2 in S^4 with unusual properties. Firstly, the group of the knot is

$$\pi_1(S^4 - S^2) = G \times Z,$$

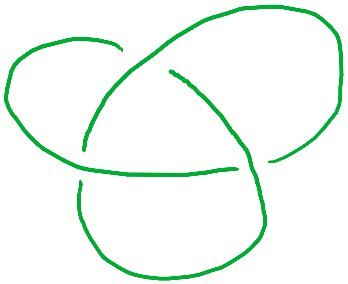
where Z = integers, and G = binary dodecahedral group. Since G has order 120, this answers affirmatively a question of Fox [3, Problems 33 and 34] asking if the group of an S^2 knot in S^4 could have elements of even order.



472 E. C. ZEEMAN

FIGURE 1

$k : S^1 \hookrightarrow S^3$ classical knot



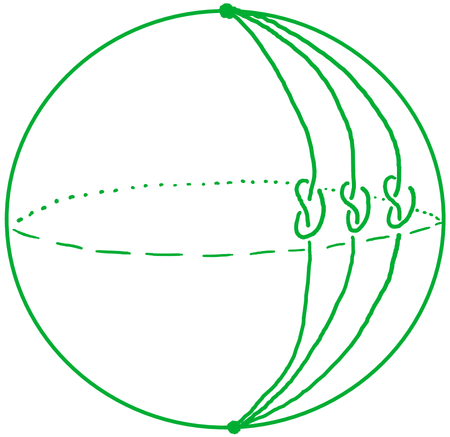
Zeeman's fibration theorem: $m \neq 0$

The m-twist spin of a classical knot k is fibred by the punctured m-fold cyclic branched cover of k.

$\tau^m(k) : S^2 \hookrightarrow S^4$

n-twist spin of k is a knotted 2-sphere embedded in the 4-sphere

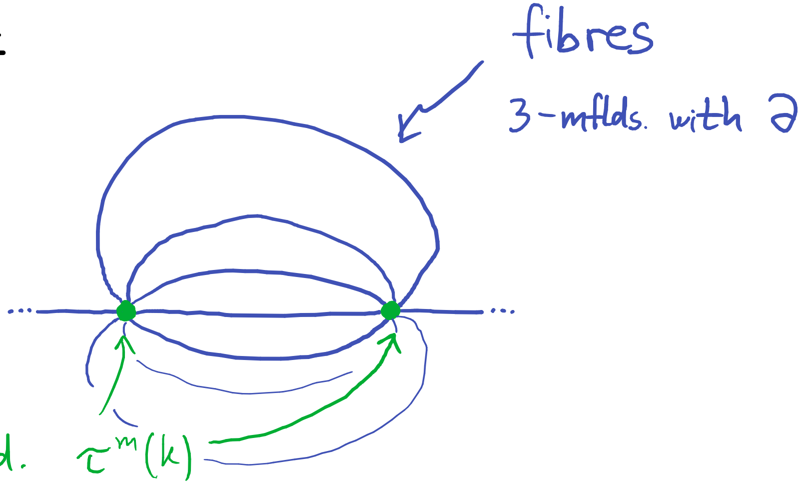
"2-knot in S^4"



$$\begin{tikzcd}
\Sigma_m(k)^\circ = \Sigma_n(k) - \text{int}\, D^3 \arrow[d] \\
S^4 - \nu(\tau^m(k)) \arrow[d] \\
S^1
\end{tikzcd}$$

} fibre bundle over S^1

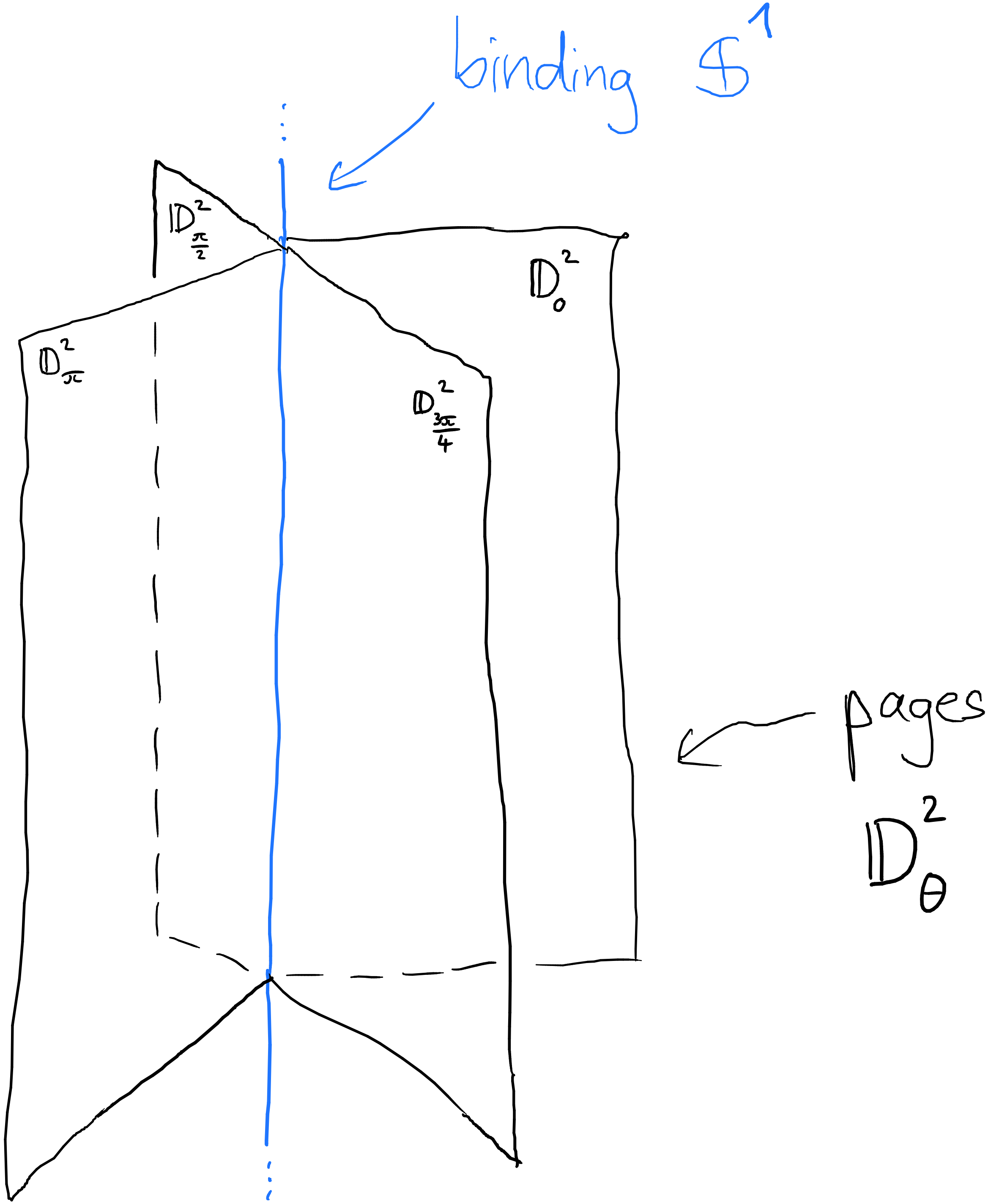
Schematic:



Spinning Warmup

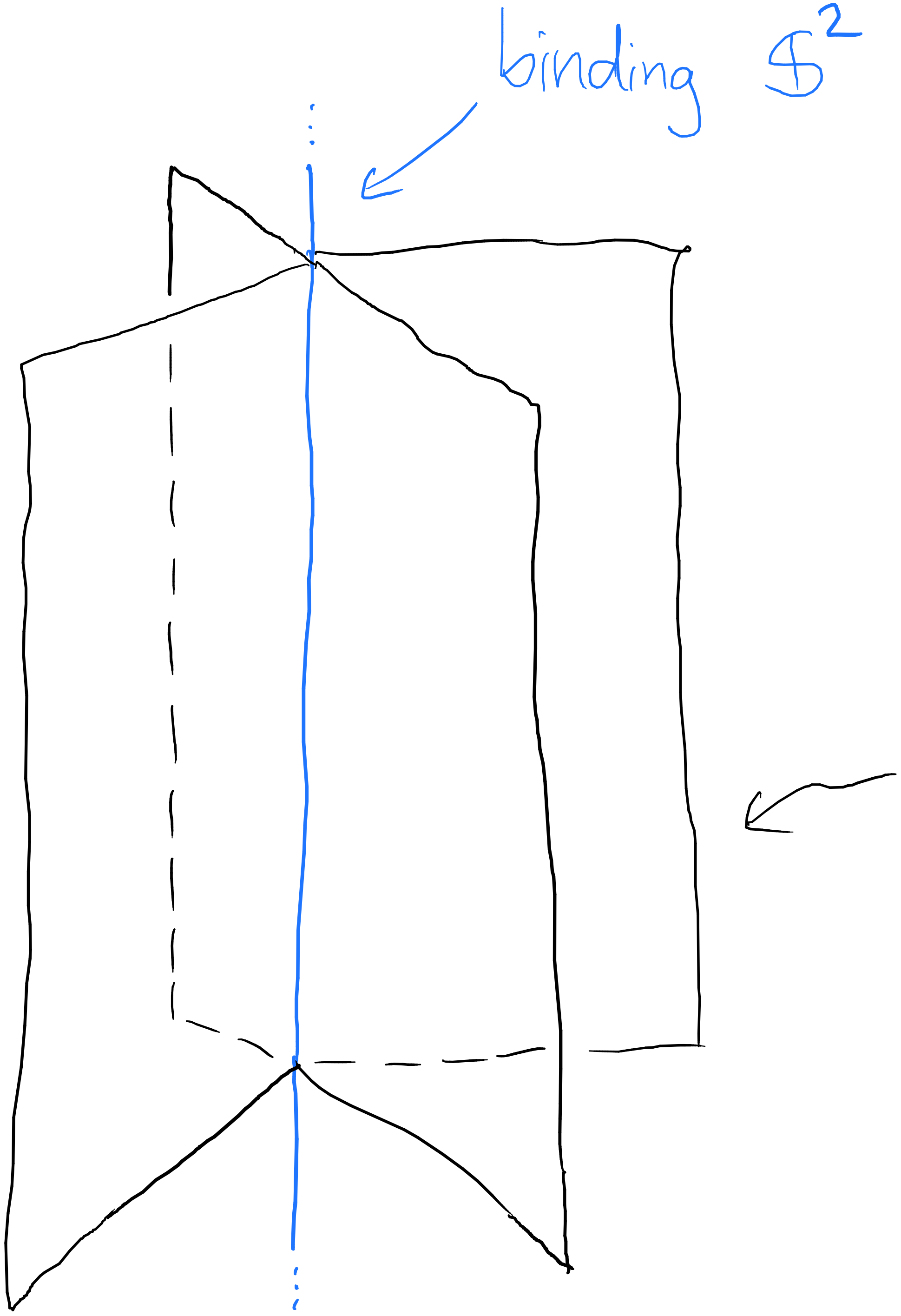
open book decomposition

of $\mathbb{S}^3$ =



Spinning

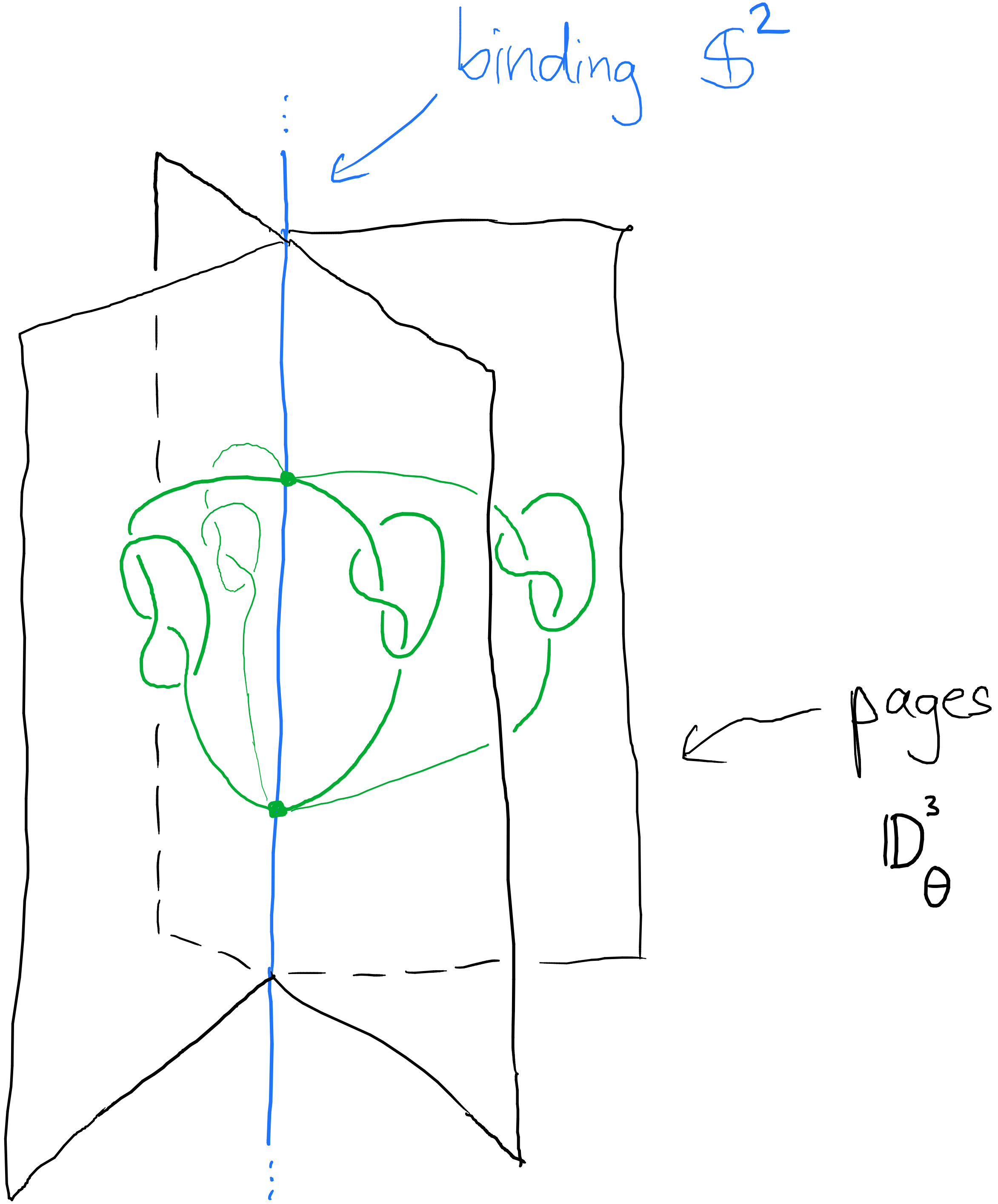
open book decomposition of S^4 =



Spinning

first described by [Emil Artin, 1925]

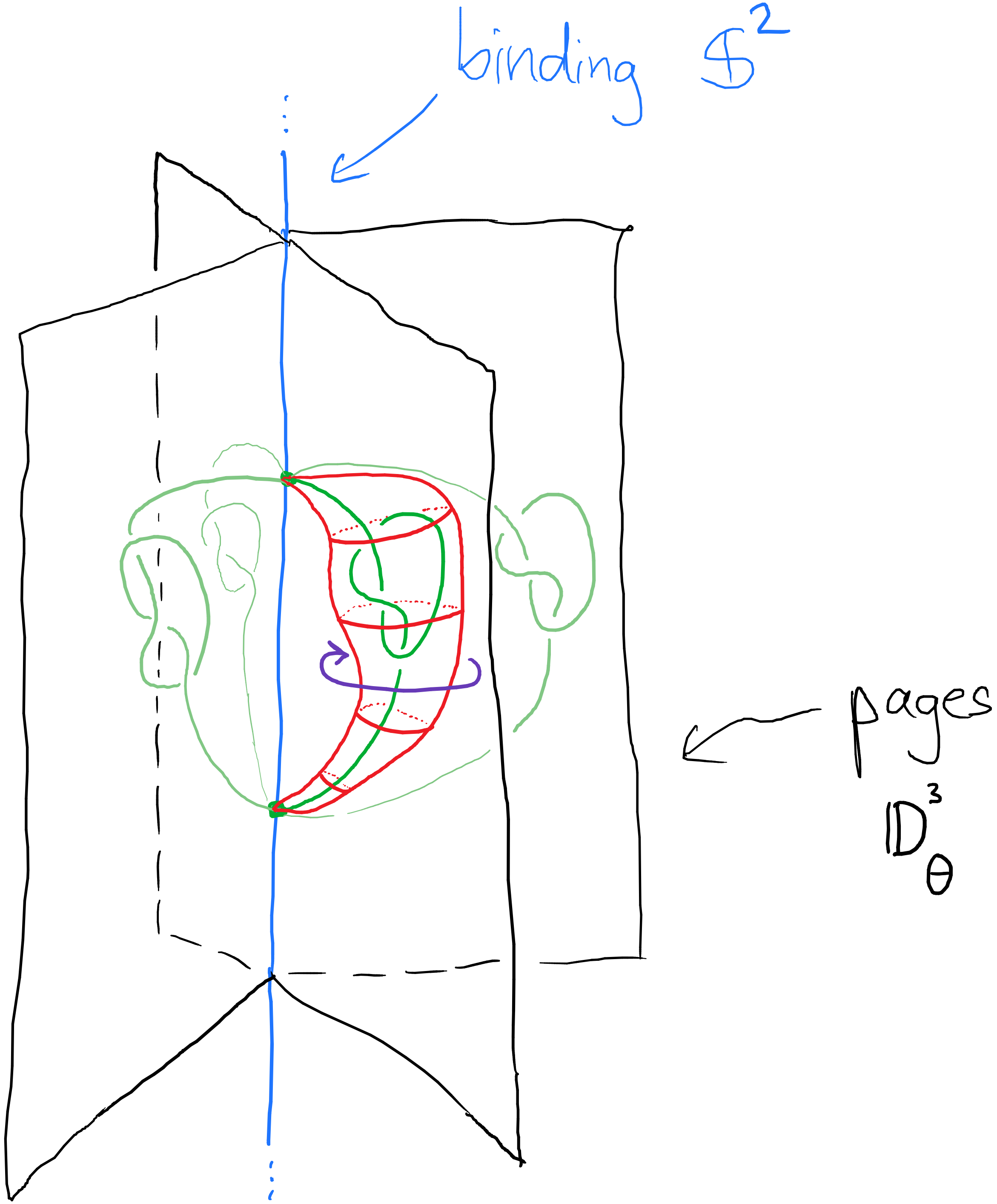
open book decomposition of $\mathbb{S}^4$ =



Twist-Spinning

[Zeeman, 1965]

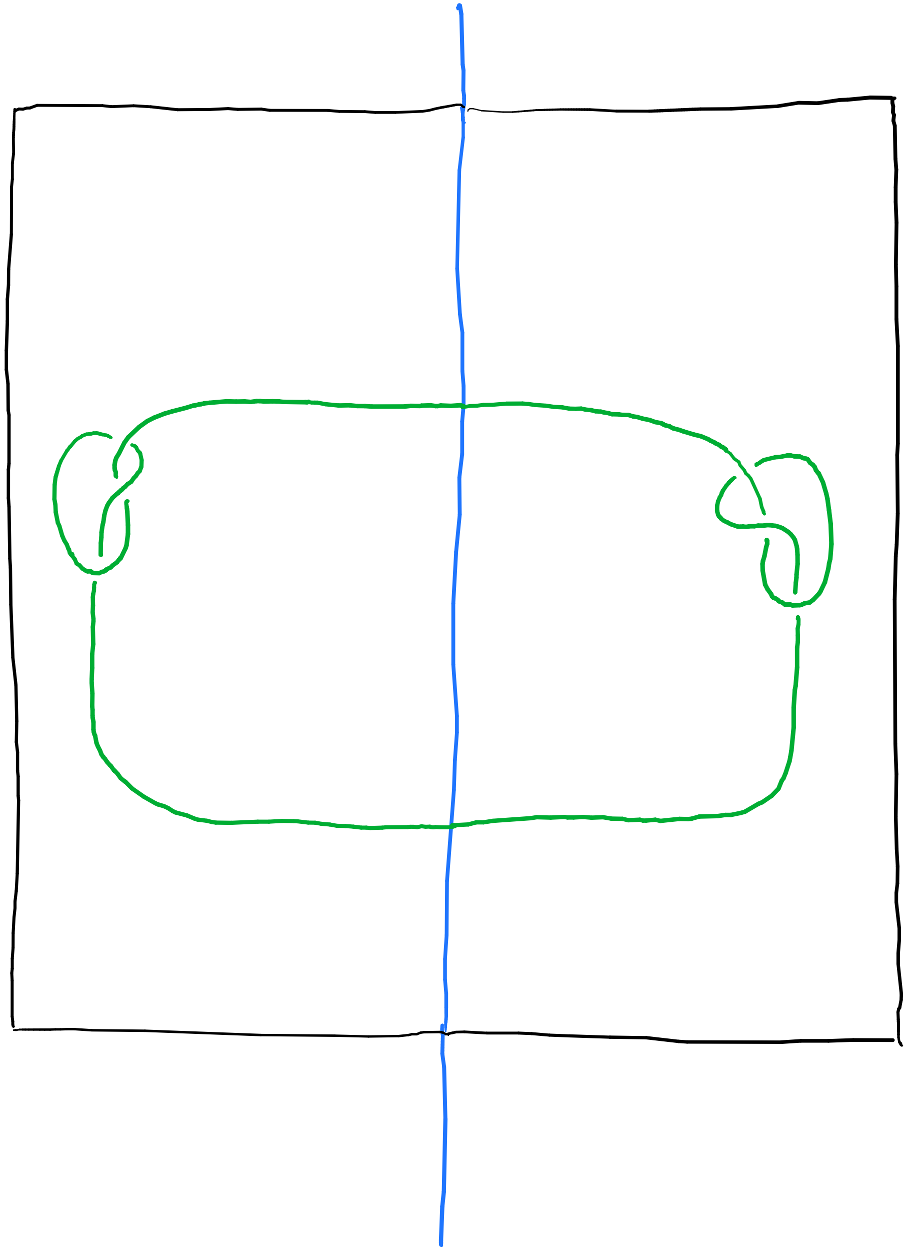
open book decomposition of S^4 =



Equatorial cross-section:

$$\tau^m(k) \cap S^3 = k \# r\overline{k}$$

the mirror image of the reverse of k

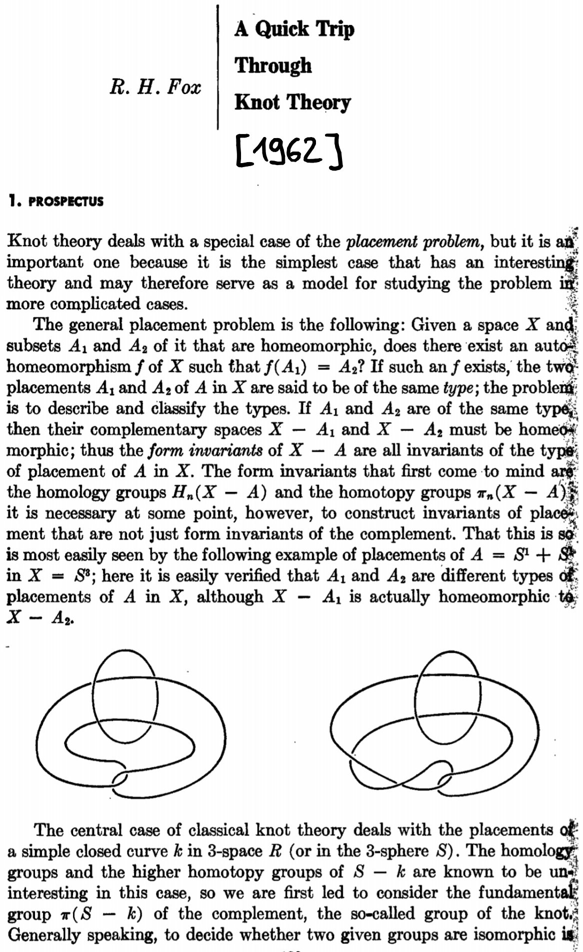


Corollary: The (± 1)-twist spin of k is fibered by $\Sigma_{\pm 1}(k)^\circ \cong \mathbb{D}^3$

$\Rightarrow \tau^{\pm 1}(k)$ is an unknotted 2-sphere

Corollary: $k \# (-k)$ is doubly slice.

this means that $k\#(-k)$ appears as the equatorial slice of an unknotted 2-sphere in $\mathbb{S}^4$



R. H. Fox

A Quick Trip
Through
Knot Theory

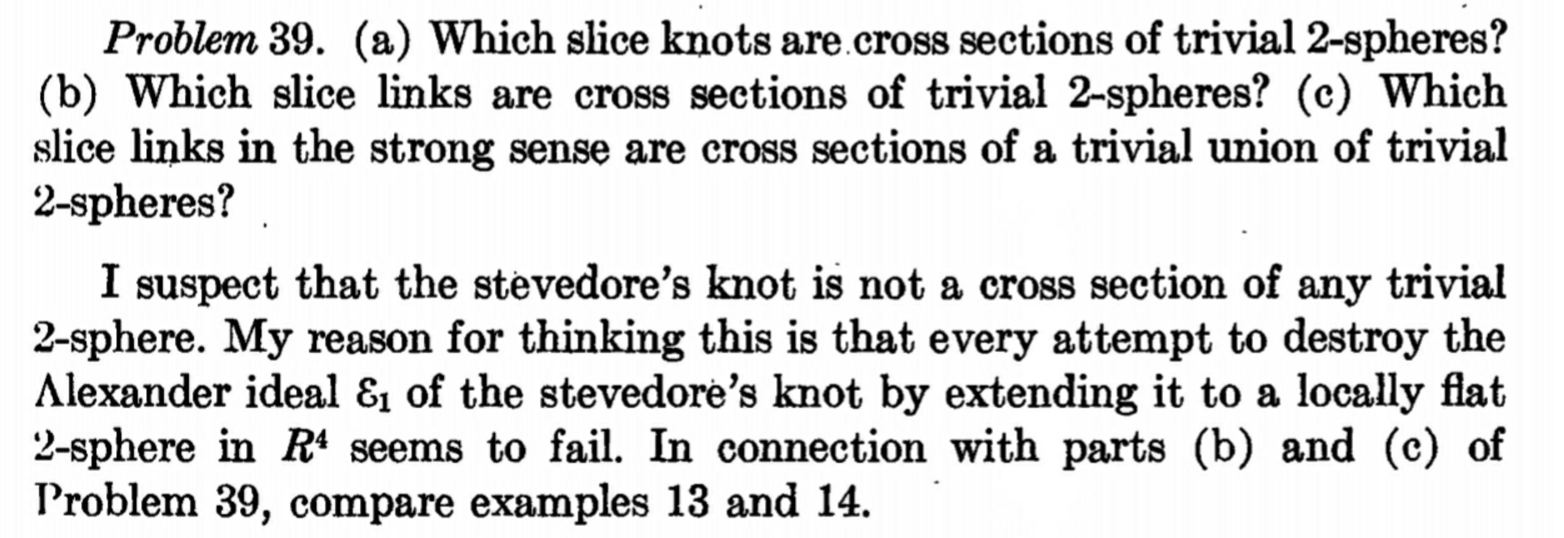
[1962]

1. PROSPECTUS

Knot theory deals with a special case of the *placement problem*, but it is an important one because it is the simplest case that has an interesting theory and may therefore serve as a model for studying the problem in more complicated cases.

The general placement problem is the following: Given a space X and subsets A_1 and A_2 of it that are homeomorphic, does there exist an autohomeomorphism f of X such that $f(A_1) = A_2$? If such an f exists, the two placements A_1 and A_2 of A in X are said to be of the same *type*; the problem is to describe and classify the types. If A_1 and A_2 are of the same type, then their complementary spaces $X - A_1$ and $X - A_2$ must be homeomorphic; thus the *form invariants* of $X - A$ are all invariants of the type of placement of A in X. The form invariants that first come to mind are the homology groups $H_n(X - A)$ and the homotopy groups $\pi_n(X - A)$; it is necessary at some point, however, to construct invariants of placement that are not just form invariants of the complement. That this is so is most easily seen by the following example of placements of $A = S^1 + S^1$ in $X = S^3$; here it is easily verified that A_1 and A_2 are different types of placements of A in X, although $X - A_1$ is actually homeomorphic to $X - A_2$.

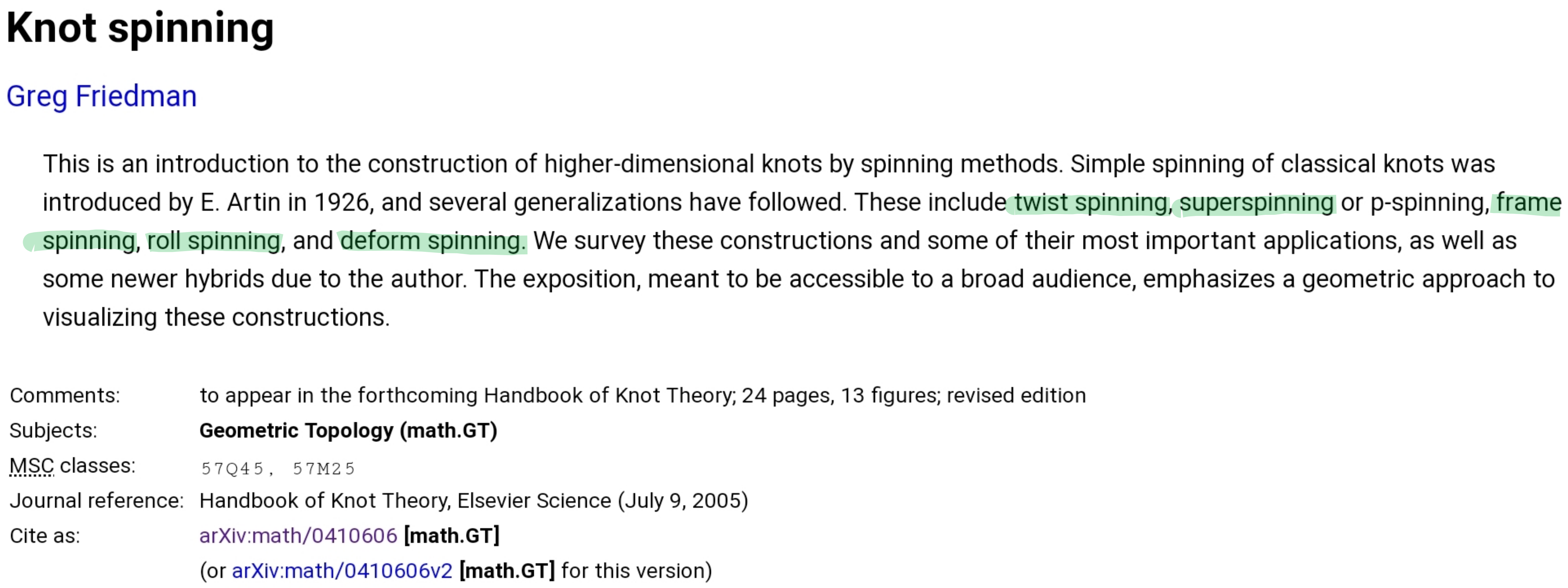
The central case of classical knot theory deals with the placements of a simple closed curve k in 3-space R (or in the 3-sphere S). The homology groups and the higher homotopy groups of $S - k$ are known to be uninteresting in this case, so we are first led to consider the fundamental group $\pi(S - k)$ of the complement, the so-called group of the knot. Generally speaking, to decide whether two given groups are isomorphic is



Problem 39. (a) Which slice knots are cross sections of trivial 2-spheres? (b) Which slice links are cross sections of trivial 2-spheres? (c) Which slice links in the strong sense are cross sections of a trivial union of trivial 2-spheres?

I suspect that the stevedore's knot is not a cross section of any trivial 2-sphere. My reason for thinking this is that every attempt to destroy the Alexander ideal $\mathcal{E}_1$ of the stevedore's knot by extending it to a locally flat 2-sphere in R^4 seems to fail. In connection with parts (b) and (c) of Problem 39, compare examples 13 and 14.

More spinning constructions in



Knot spinning

Greg Friedman

This is an introduction to the construction of higher-dimensional knots by spinning methods. Simple spinning of classical knots was introduced by E. Artin in 1926, and several generalizations have followed. These include twist spinning, superspinning or p-spinning, frame spinning, roll spinning, and deform spinning. We survey these constructions and some of their most important applications, as well as some newer hybrids due to the author. The exposition, meant to be accessible to a broad audience, emphasizes a geometric approach to visualizing these constructions.

Comments:	to appear in the forthcoming Handbook of Knot Theory; 24 pages, 13 figures; revised edition
Subjects:	**Geometric Topology (math.GT)**
MSC classes:	57Q45, 57M25
Journal reference:	Handbook of Knot Theory, Elsevier Science (July 9, 2005)
Cite as:	arXiv:math/0410606 **[math.GT]**
	(or arXiv:math/0410606v2 **[math.GT]** for this version)

Effect of (twist) spinning on the knot group

$\pi_1(\mathbb{S}^4 - \tau^0(k)) \cong \pi_1(\mathbb{S}^3 - k)$, i.e. 0-twist spinning preserves the knot group

μ is a meridian of k

$$\pi_1(\mathbb{S}^4 - \tau^m(k)) \cong \pi_1(\mathbb{S}^3 - k) \Big/ \begin{array}{c} \mu^m \cdot a = a \cdot \mu^m \\ \text{for all } a \in \pi_1(\mathbb{S}^3 - k) \end{array}$$

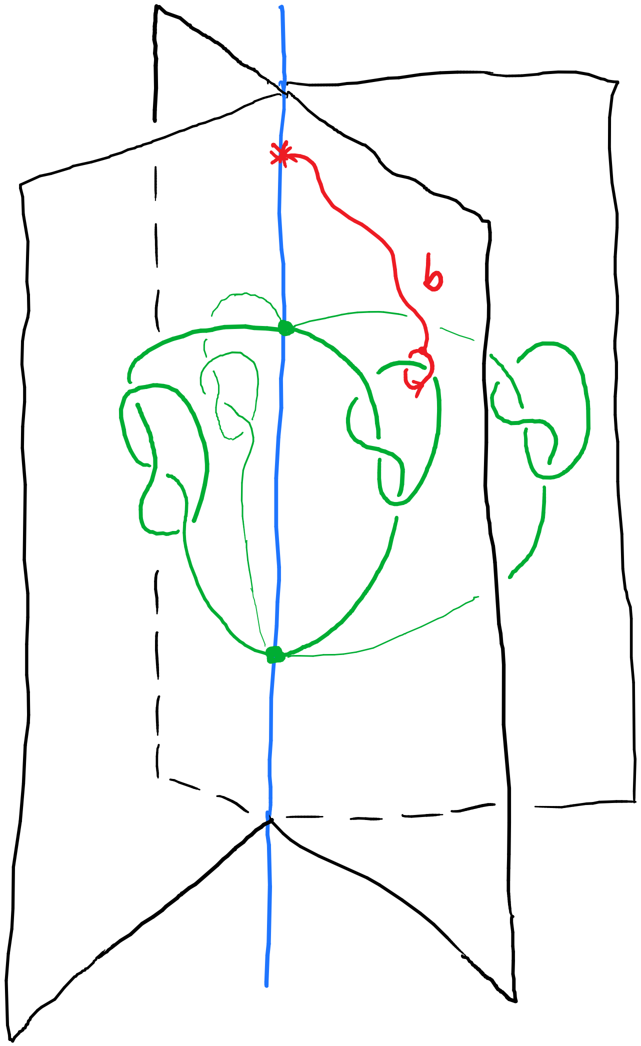
i.e. the knot group of k, but we force the m^{th} power of a meridian of K to be central in $\pi_1(\mathbb{S}^4 - \tau^m k)$

Explicit example: 5-twist spin of the trefoil

$\pi_1(S^3 - \text{trefoil}) \cong \langle a,b \mid aba = bab \rangle$

binary dodecahedral group

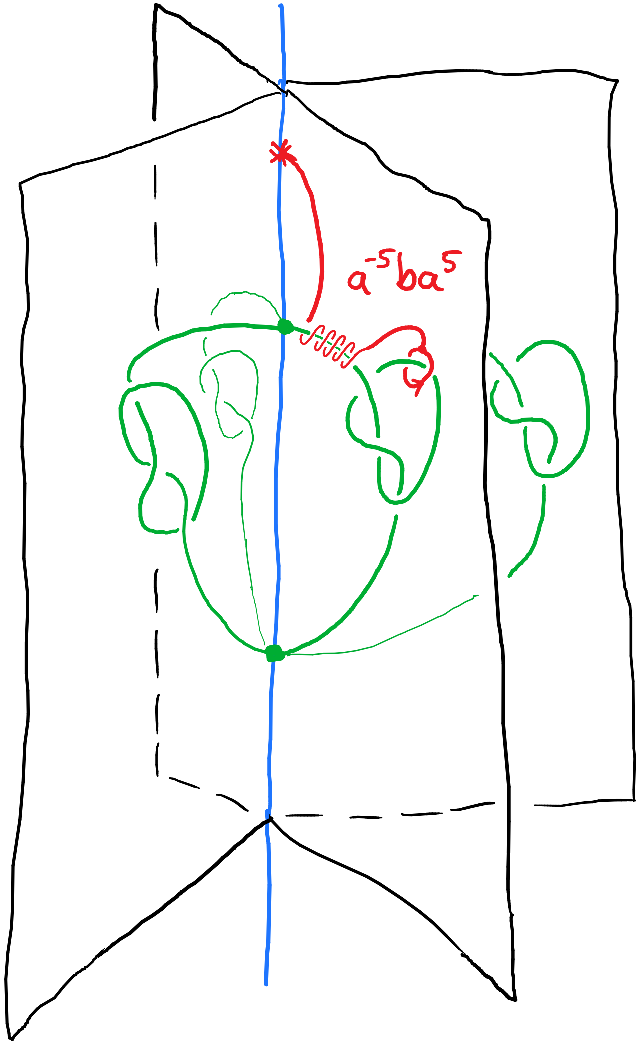
$\pi_1(S^4 - \tau^5(\text{trefoil})) \cong \langle a,b \mid aba = bab,\ b = a^{-5}ba^5 \rangle \cong \mathrm{Dod}^* \times \mathbb{Z}$



move loop around

⟶

the open book



Observation: There can be non-trivial torsion in a 2-knot group.

Compare with: Classical knot groups $\pi_1(\mathbb{S}^3 - k)$ are torsion free.

$\pi_2(\mathbb{S}^3 - k) = 0$

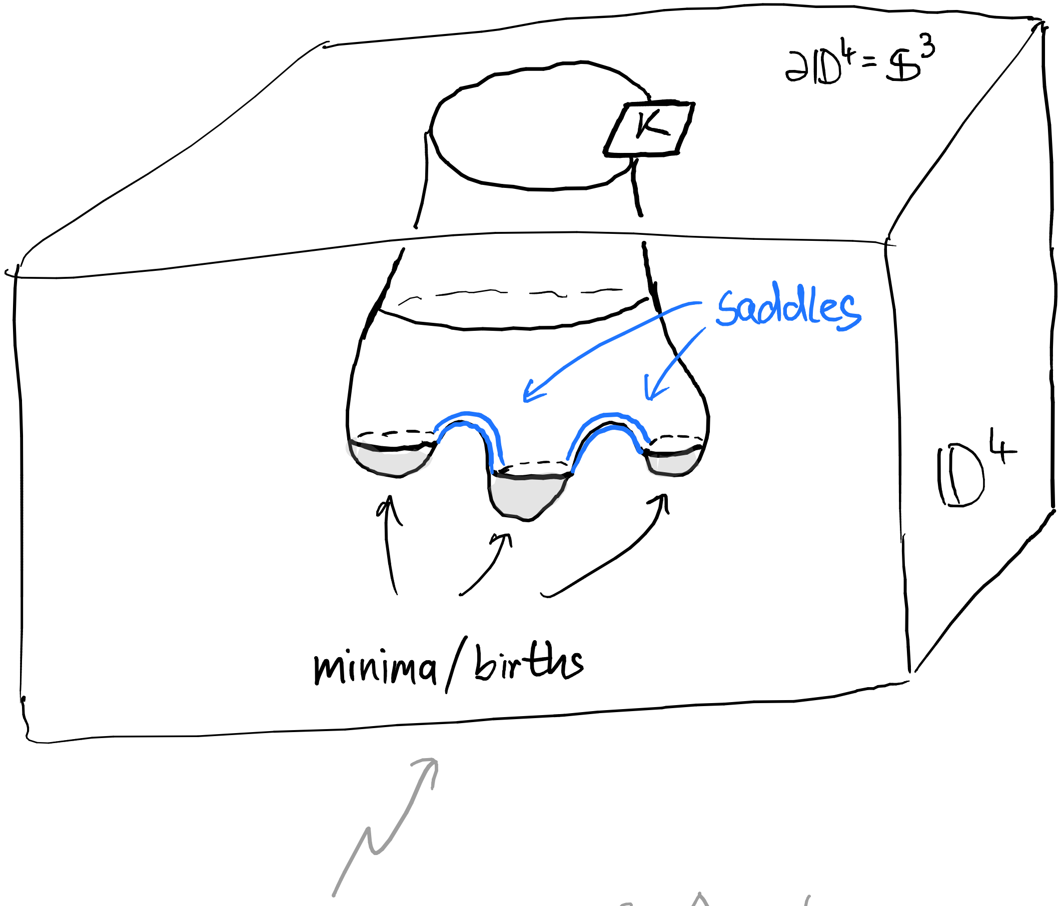
Consequence of the asphericity of classical knot complements,

$$K(\pi_1(\mathbb{S}^3 - k), 1) \simeq \mathbb{S}^3 - k.$$

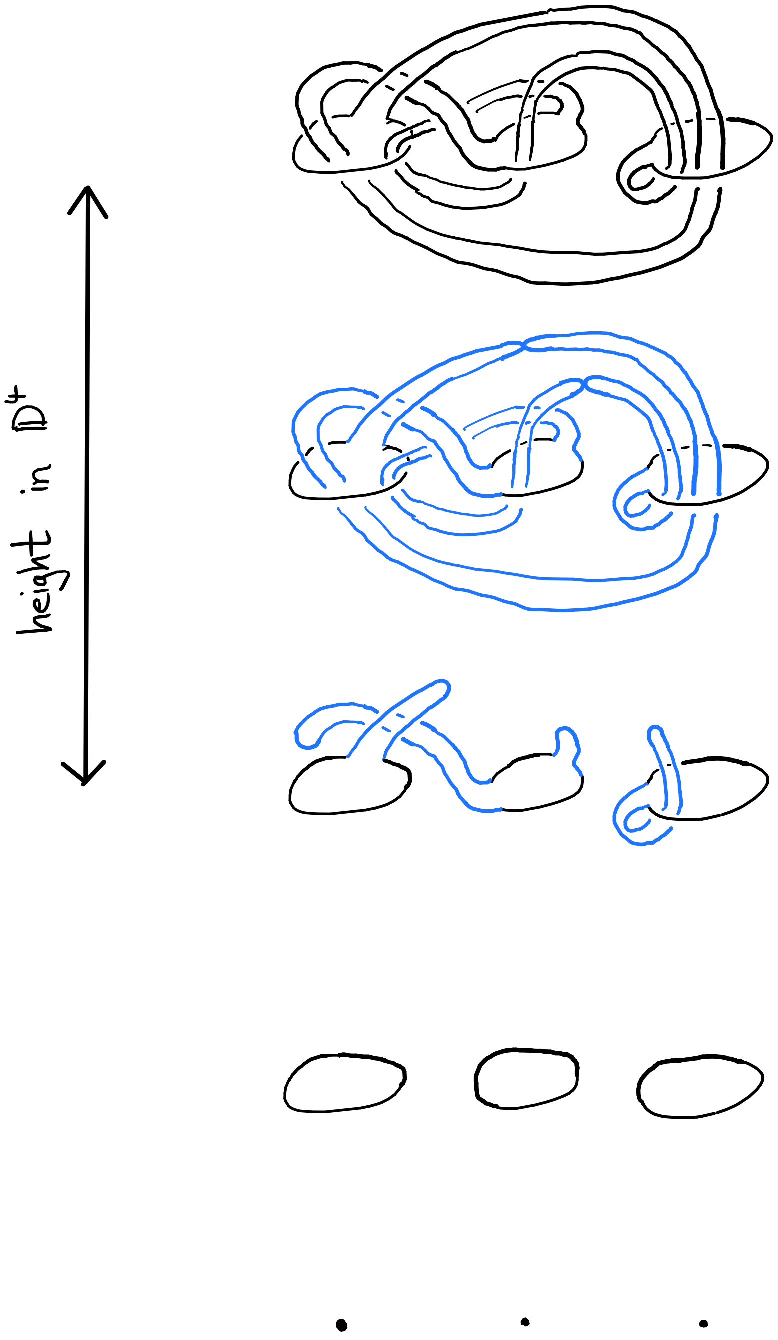
Eilenberg-MacLane space for the knot group

(The asphericity of classical knot complements follows from the sphere theorem [Papakyriakopoulos (1957)])

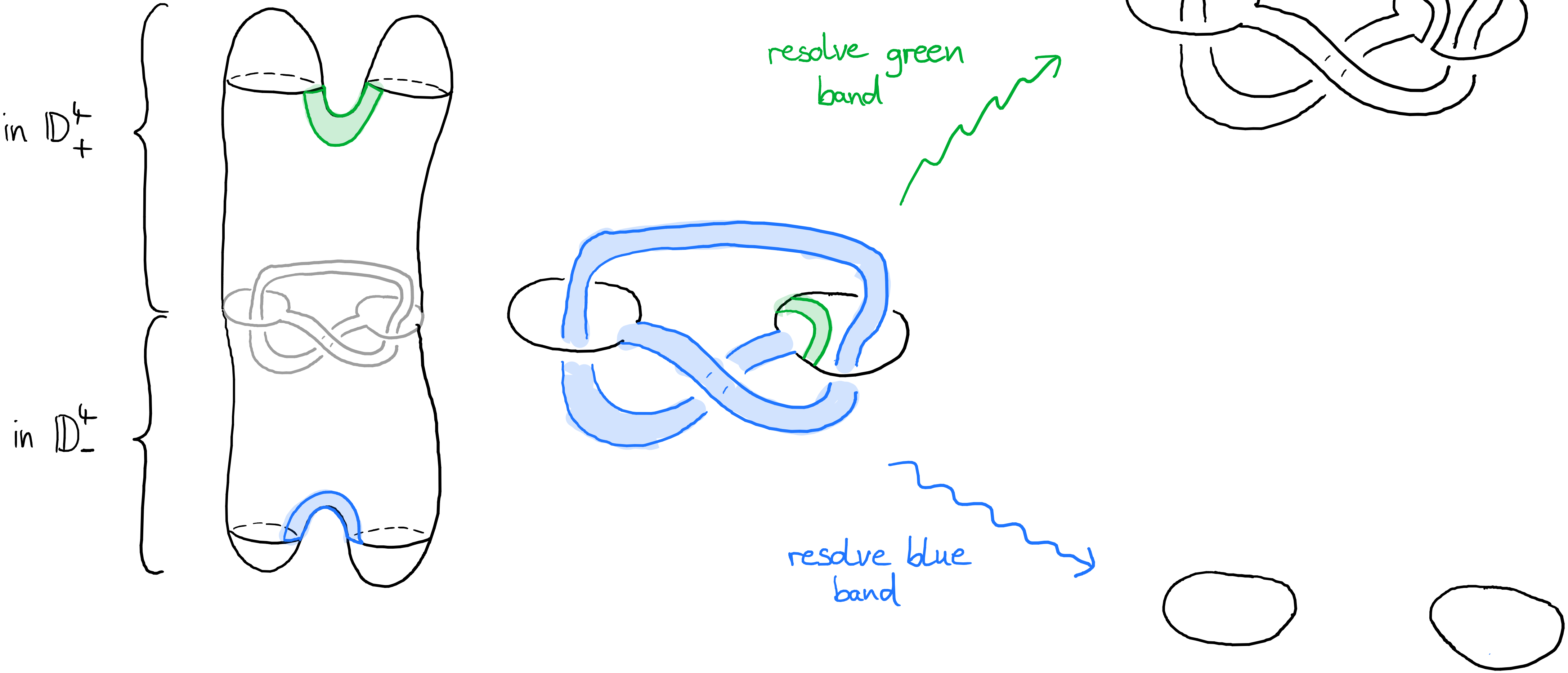
Describing knotted surfaces via movies



Ribbon disk $D^2 \overset{\Delta}{\hookrightarrow} D^4$



Banded unlink diagrams



Open question: Are ribbon disk complements aspherical?

I.e. is $\pi_2(\mathbb{D}^4 \setminus \Delta)$ always zero?

↑

$\Delta : \mathbb{D}^2 \hookrightarrow \mathbb{D}^4$ ribbon disk

↗ Whitehead conjecture:

Are subcomplexes of aspherical 2-dim. complexes aspherical as well?

"Can you create π_2 by removing something from a 2-complex?"

Open question: Are ribbon 2-knot groups torsion free?

(⇐ asphericity of ribbon disk complements would imply this)

⚠ Beware that there are many claimed proofs of this in the literature, but they have gaps!

[Yanagawa: On ribbon 2-knots II. The second homotopy group of the complementary domain (1969)], ...

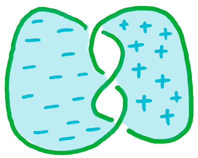
As a consequence, some of the literature on π_2(2-knot complement) is problematic

[Lomonaco: The homotopy groups of knots I. How to compute the algebraic 2-type (1981)], ...

Cyclic branched covers: $\Sigma_m(k)$

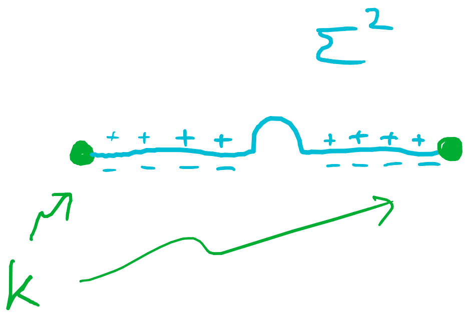
[Rolfsen: Knots and Links, 10.C]

S^3 branched over a classical knot $k: S^1 \hookrightarrow S^3$



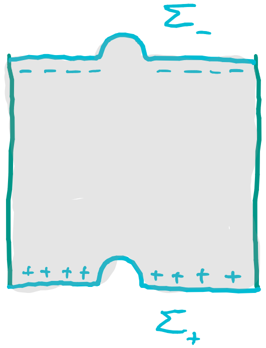
First: Cyclic unbranched cover corresponds to the epimorphism

$$\pi_1(S^3 - k) \twoheadrightarrow H_1(S^3 - k) \cong \mathbb{Z} \twoheadrightarrow \mathbb{Z}/m$$



cut S^3 open along Seifert surface Σ^2

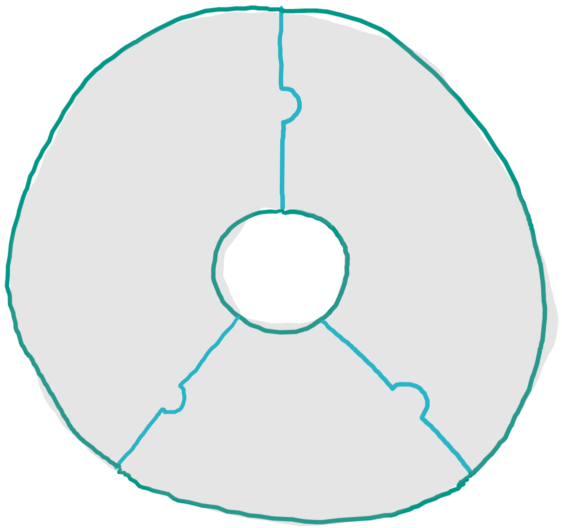
⇝

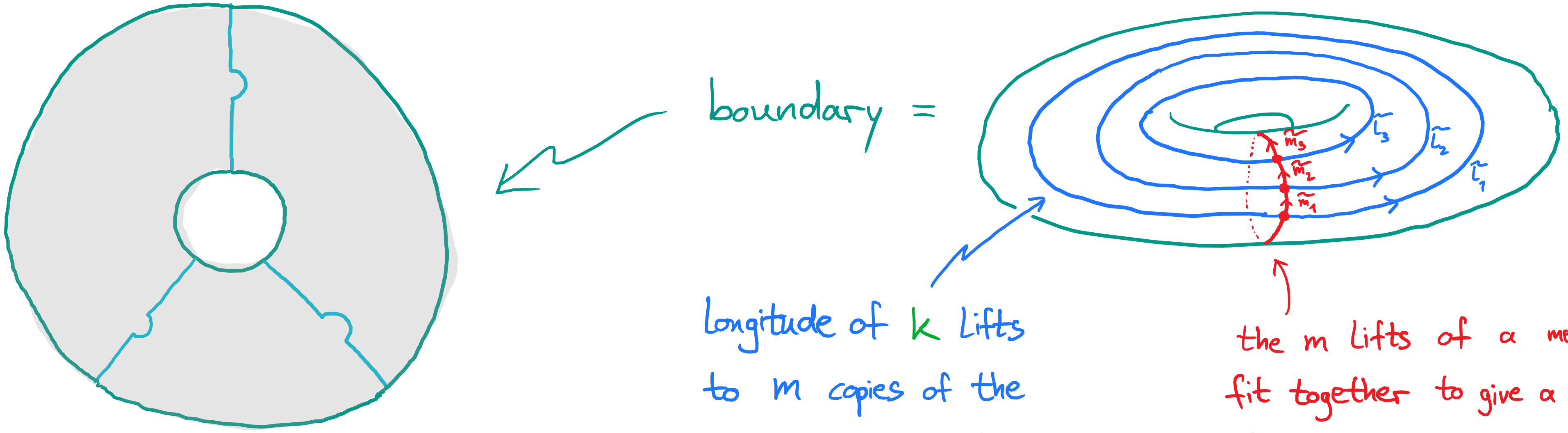


assemble m "puzzle pieces"

⇝

in cyclic pattern





longitude of k lifts to m copies of the longitude of boundary torus

the m lifts of a meridian of k fit together to give a meridian of boundary torus upstairs

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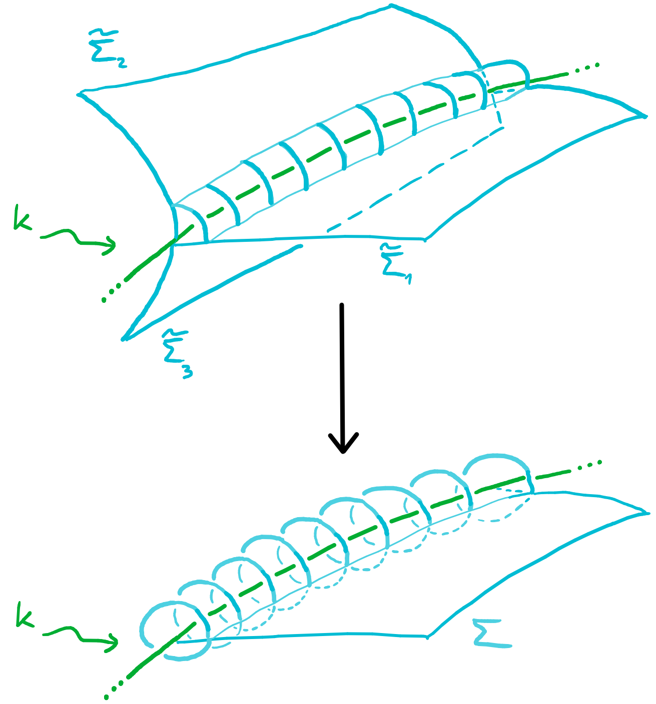
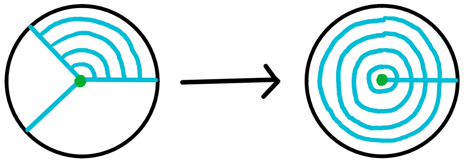
Finally: Plug in solid tori upstairs and downstairs, extend map by using $\mathbb{D}^2 \longrightarrow \mathbb{D}^2$ on each meridional disk $z \longmapsto \frac{z^m}{|z|^{m-1}}$



$$\begin{tikzcd}
\Sigma_m(k) \arrow[d] \\
(\mathbb{S}^3 - \nu(k)) \cup (\mathbb{S}^1 \times \mathbb{D}^2) = \mathbb{S}^3
\end{tikzcd}$$

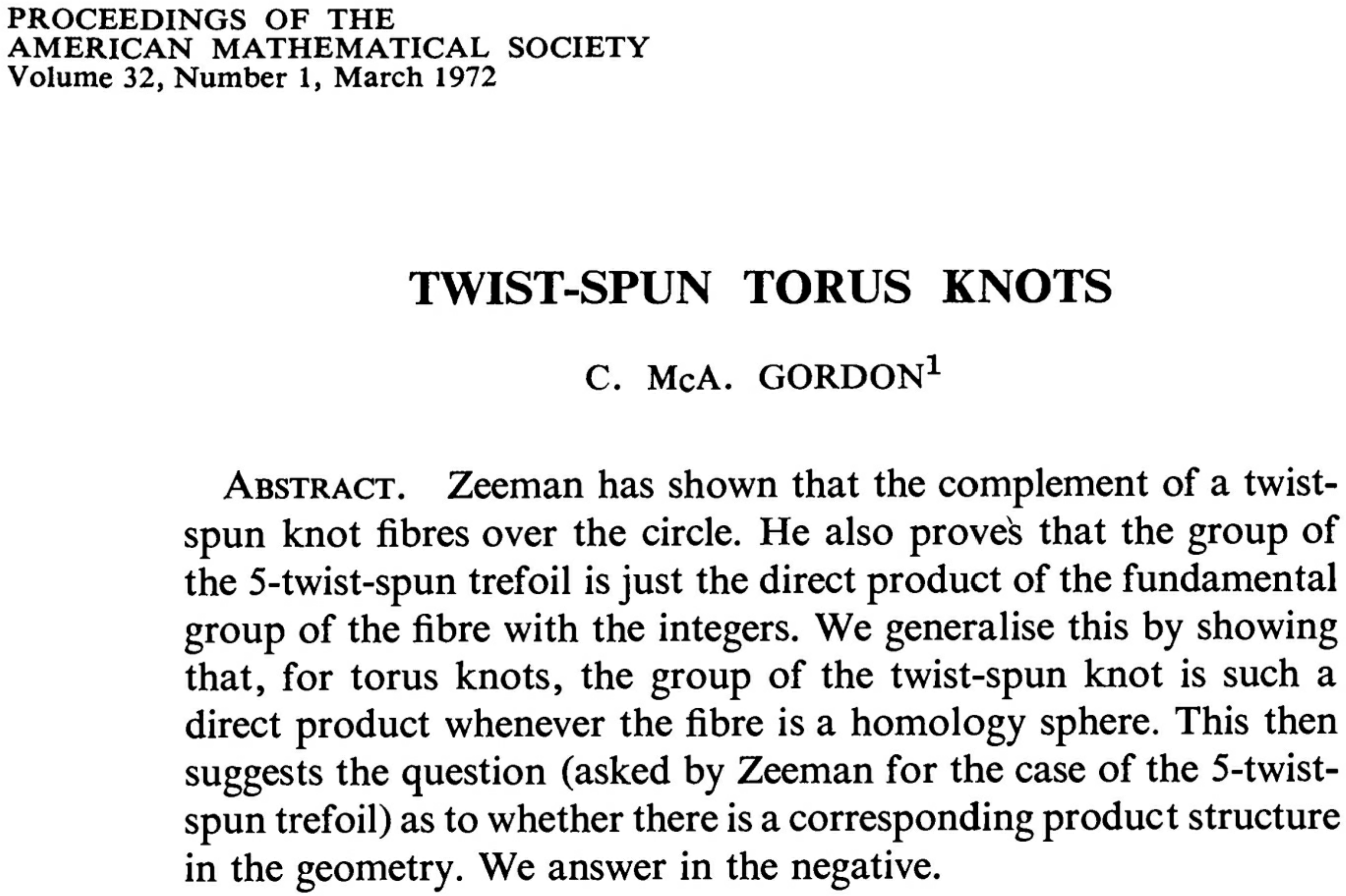


Interesting family of 2-knots: Pick p, q, r coprime

p-twist spin of (q, r)-torus knot $\tau^p T(q,r)$

q-twist spin of (p, r)-torus knot $\tau^q T(p,r)$

r-twist spin of (p, q)-torus knot $\tau^r T(p,q)$



PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 1, March 1972

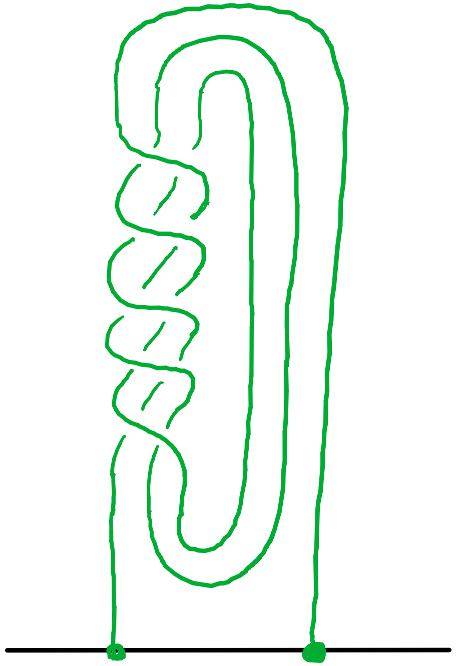
TWIST-SPUN TORUS KNOTS

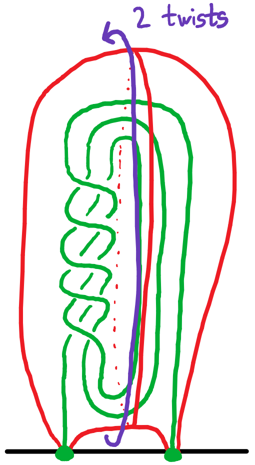
C. McA. GORDON[1]

ABSTRACT. Zeeman has shown that the complement of a twist-spun knot fibres over the circle. He also proves that the group of the 5-twist-spun trefoil is just the direct product of the fundamental group of the fibre with the integers. We generalise this by showing that, for torus knots, the group of the twist-spun knot is such a direct product whenever the fibre is a homology sphere. This then suggests the question (asked by Zeeman for the case of the 5-twist-spun trefoil) as to whether there is a corresponding product structure in the geometry. We answer in the negative.

2-twist spin of
(3,5)-torus knot

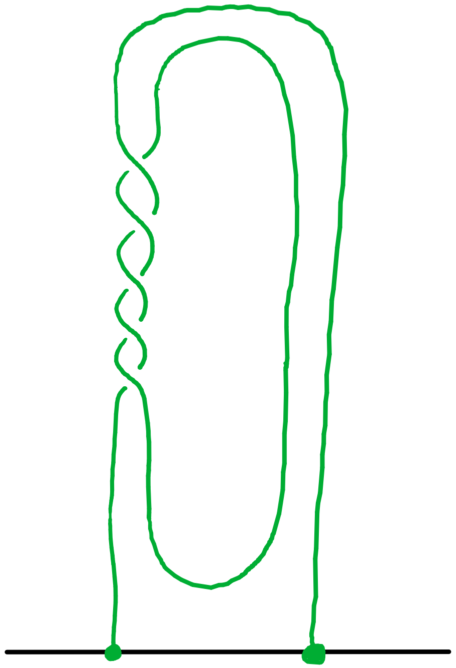
$\tau^2 T(3,5)$

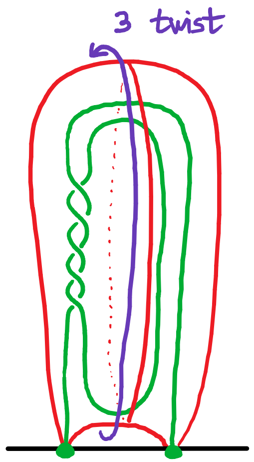




3-twist spin of
(2,5)-torus knot

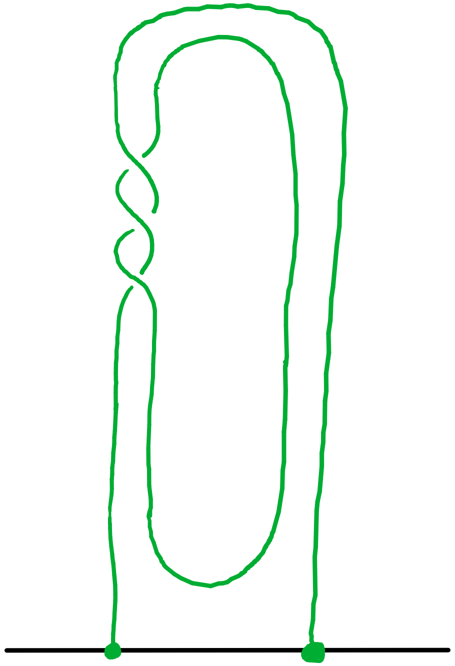
$\tau^3 T(2,5)$

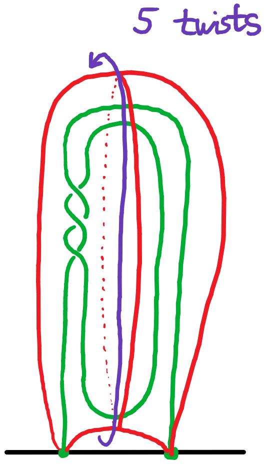




5-twist spin of
(2,3)-torus knot

$\tau^5 T(2,3)$





Take $\{p, q, r\} = \{2, 3, 5\}$

$\tau^2 T(3,5)$, $\tau^3 T(2,5)$ and $\tau^5 T(2,3)$

are all fibred by a punctured Brieskorn sphere

$\Sigma(2,3,5)$

Poincaré homology sphere

binary dodecahedral / icosahedral group

$$\pi_1(S^4 - \tau^p T(q,r)) \cong \mathbb{Z} \times Dod^*$$

but this isomorphism does not come

from a product structure on the complement [Gordon]

$S^4 - \nu(\tau^2 T(3,5))$ fibered by $\Sigma(2,3,5)^\circ$ ← bounded punctured Poincaré homology sphere

In particular: $\Sigma(2,3,5)^\circ \xhookrightarrow[\text{embedds}]{\text{smoothly}} S^4$

Surprising, because $\Sigma(2,3,5) \cancel{\xhookrightarrow{\text{smoothly}}} S^4$

If it did, would have

$S^4 =$

smooth homology 4-ball with $\partial =$

← $\Sigma(2,3,5)$

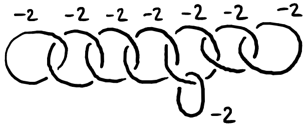
$U \cup_{\partial = \Sigma(2,3,5)} P_{E_8}$

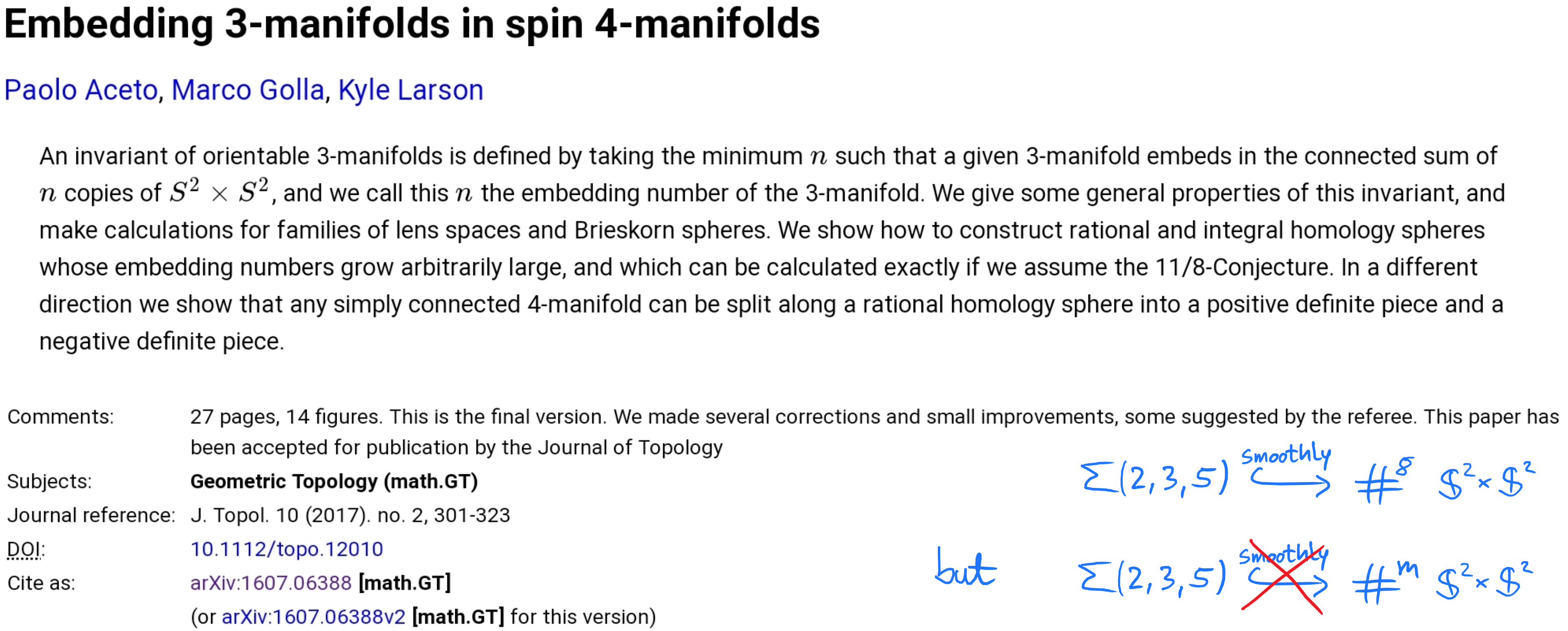
closed, smooth, spin 4-manifold with signature $= 8$

↯ to Rohlin's theorem

But we also know

$\Sigma(2,3,5) = \partial(E_8\text{-plumbing } P_{E_8})$





Embedding 3-manifolds in spin 4-manifolds

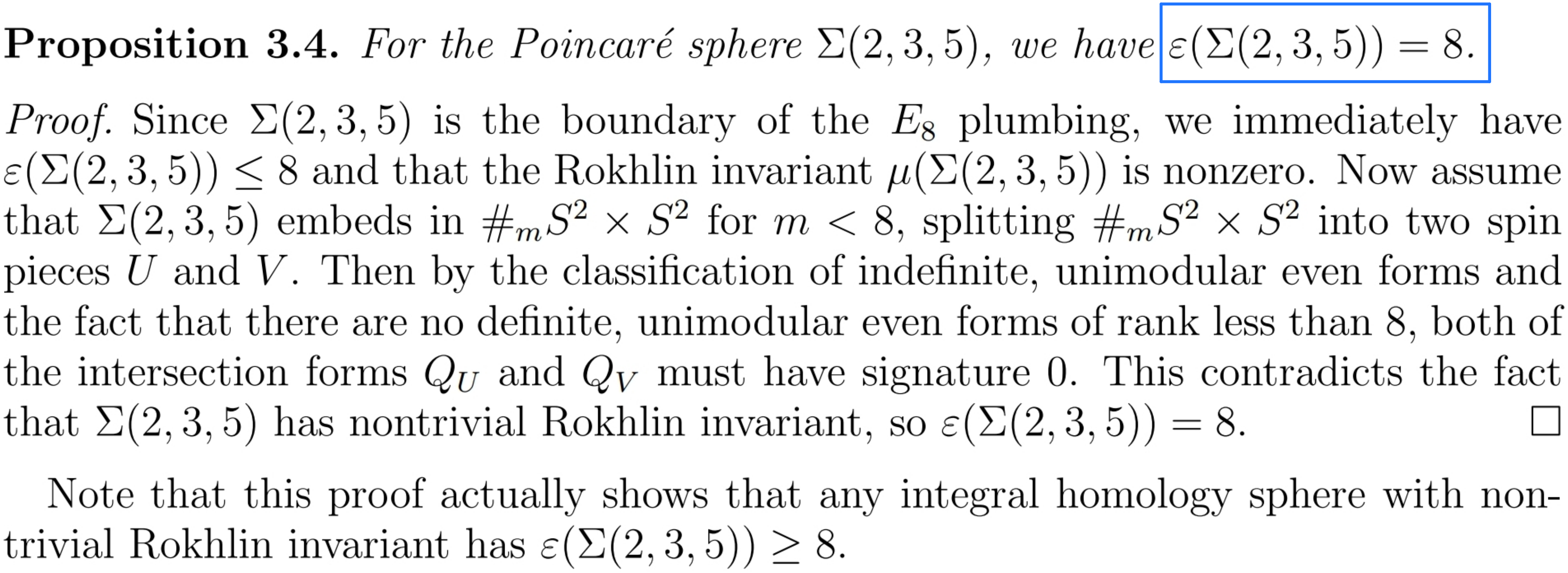
Paolo Aceto, Marco Golla, Kyle Larson

An invariant of orientable 3-manifolds is defined by taking the minimum n such that a given 3-manifold embeds in the connected sum of n copies of $S^2 \times S^2$, and we call this n the embedding number of the 3-manifold. We give some general properties of this invariant, and make calculations for families of lens spaces and Brieskorn spheres. We show how to construct rational and integral homology spheres whose embedding numbers grow arbitrarily large, and which can be calculated exactly if we assume the 11/8-Conjecture. In a different direction we show that any simply connected 4-manifold can be split along a rational homology sphere into a positive definite piece and a negative definite piece.

Comments: 27 pages, 14 figures. This is the final version. We made several corrections and small improvements, some suggested by the referee. This paper has been accepted for publication by the Journal of Topology
Subjects: **Geometric Topology (math.GT)**
Journal reference: J. Topol. 10 (2017). no. 2, 301-323
DOI: 10.1112/topo.12010
Cite as: arXiv:1607.06388 **[math.GT]**
(or arXiv:1607.06388v2 **[math.GT]** for this version)

$$\Sigma(2,3,5) \xhookrightarrow{\text{smoothly}} \#^8 S^2\times S^2$$

but $\Sigma(2,3,5) \cancel{\xhookrightarrow{\text{smoothly}}} \#^m S^2\times S^2$ for $m \le 7$



Proposition 3.4. *For the Poincaré sphere $\Sigma(2,3,5)$, we have* $\boxed{\varepsilon(\Sigma(2,3,5)) = 8.}$

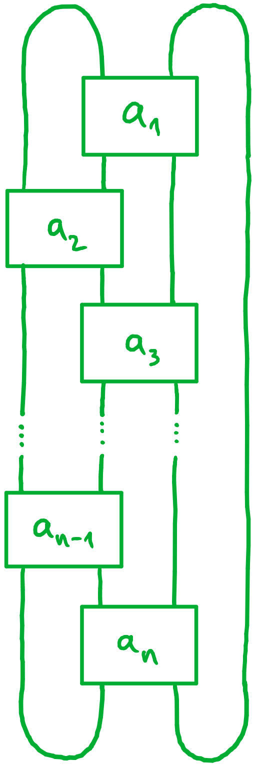
Proof. Since $\Sigma(2,3,5)$ is the boundary of the E_8 plumbing, we immediately have $\varepsilon(\Sigma(2,3,5)) \leq 8$ and that the Rokhlin invariant $\mu(\Sigma(2,3,5))$ is nonzero. Now assume that $\Sigma(2,3,5)$ embeds in $\#_m S^2 \times S^2$ for $m < 8$, splitting $\#_m S^2 \times S^2$ into two spin pieces U and V. Then by the classification of indefinite, unimodular even forms and the fact that there are no definite, unimodular even forms of rank less than 8, both of the intersection forms Q_U and Q_V must have signature 0. This contradicts the fact that $\Sigma(2,3,5)$ has nontrivial Rokhlin invariant, so $\varepsilon(\Sigma(2,3,5)) = 8$. □

Note that this proof actually shows that any integral homology sphere with nontrivial Rokhlin invariant has $\varepsilon(\Sigma(2,3,5)) \geq 8$.

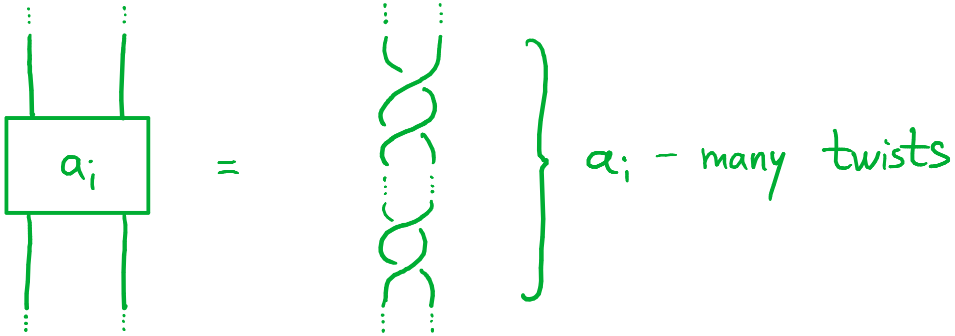
Lens spaces $L(p,q)$, p odd

appear as double branched covers over 2-bridge knots

(for p even need to branch cover over a link)



where



By 2-twist spinning 2-bridge knots: get complements fibred by punctured lens spaces $L(p,q)$, p odd

[Cochran (1983)]: Nontrivial twist-spins are not ribbon.

RIBBON KNOTS IN S^4

TIM COCHRAN

ABSTRACT

Our main results are several new obstructions to knotted 2-spheres' in S^4 being ribbon knots and the application of these to characterize fibered ribbon knots in S^4. For knots which bound punctured $S^1 \times S^2$, all of the known ribbon invariants vanish. We produce a new obstruction which detects the first known non-ribbon knots in this class. Finally, we show that ribbon knots are naturally associated with links whose groups are free, but not on their meridians.

0. *Introduction*

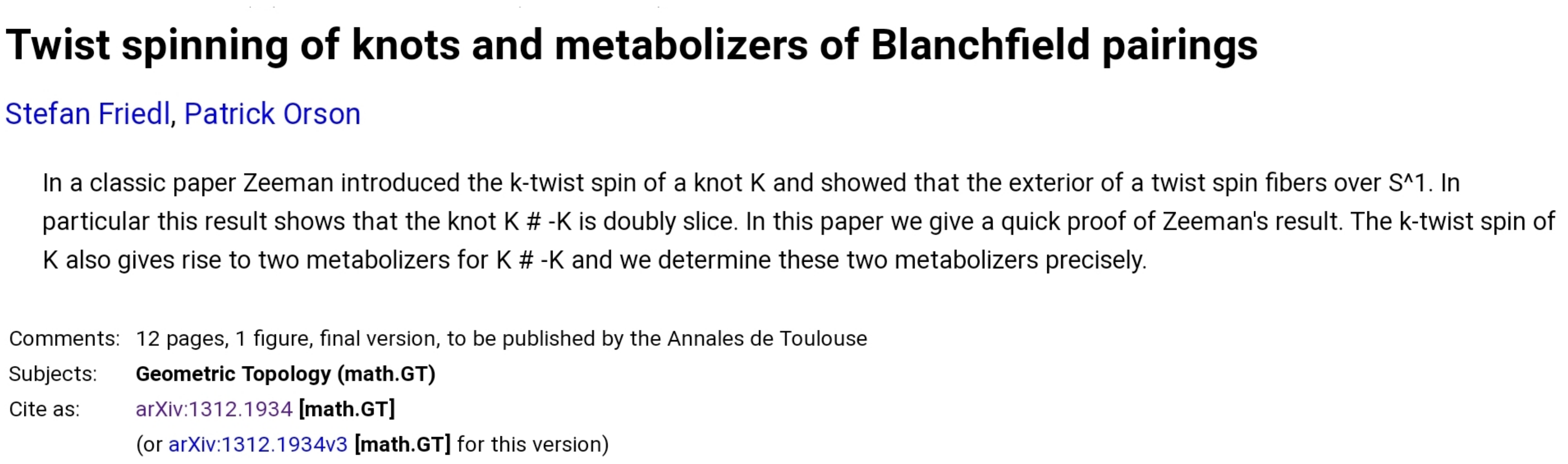
One of the outstanding questions in classical knot theory ($S^1 \hookrightarrow S^3$) asks whether every slice knot is a ribbon knot. For knotted 2-spheres in S^4, this question takes on a different flavor because, while every knot is a slice knot [**14**], it is known that the corresponding notion of *ribbon* is more restrictive [**29**, **30**]. One is left, therefore, with the question of exactly which knots in S^4 *are* ribbon knots.

This paper presents several new obstructions to knotted 2-spheres' being ribbon knots. These are applied to *characterize fibered ribbon knots* and consequently it is shown that no non-trivial twist-spun knot is ribbon. We also investigate the class of

Proof idea for Zeeman's fibration theorem

Zeeman's fibration theorem: $m \neq 0$

The m-twist spin of a classical knot k is fibred by the punctured m-fold cyclic branched cover of k.



Twist spinning of knots and metabolizers of Blanchfield pairings

Stefan Friedl, Patrick Orson

In a classic paper Zeeman introduced the k-twist spin of a knot K and showed that the exterior of a twist spin fibers over S^1. In particular this result shows that the knot K # -K is doubly slice. In this paper we give a quick proof of Zeeman's result. The k-twist spin of K also gives rise to two metabolizers for K # -K and we determine these two metabolizers precisely.

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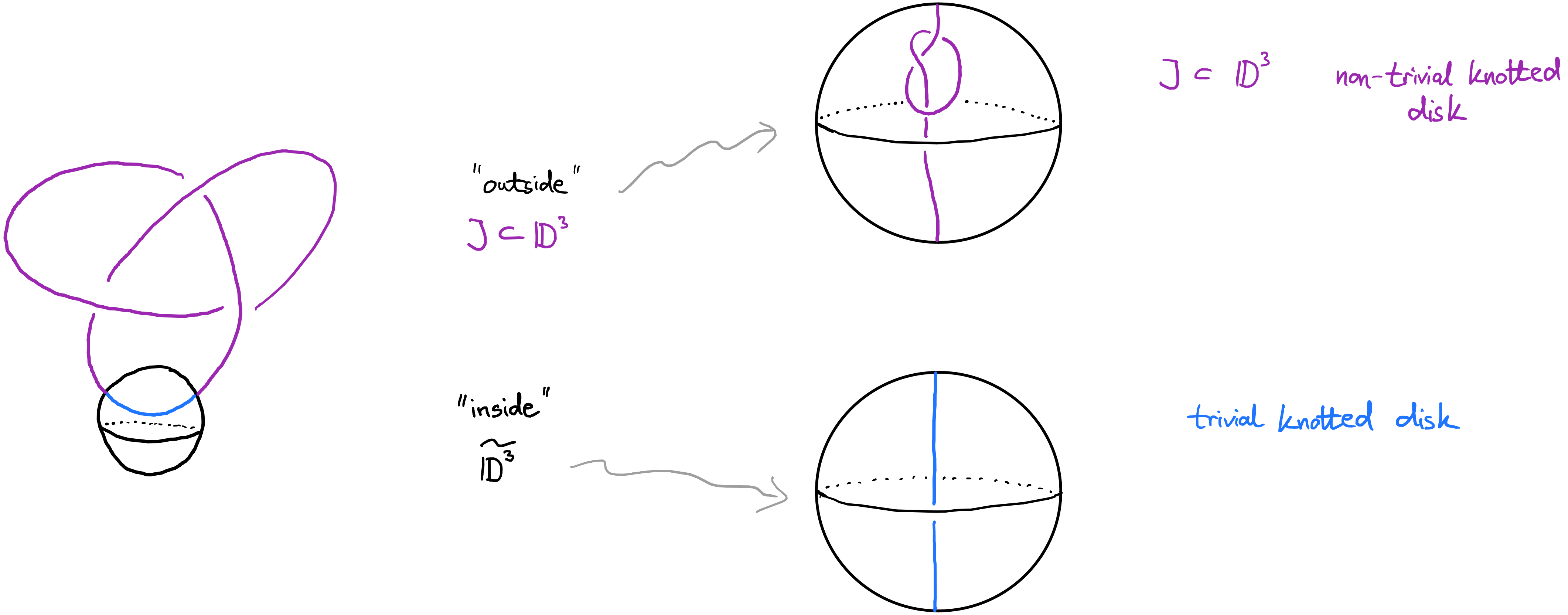
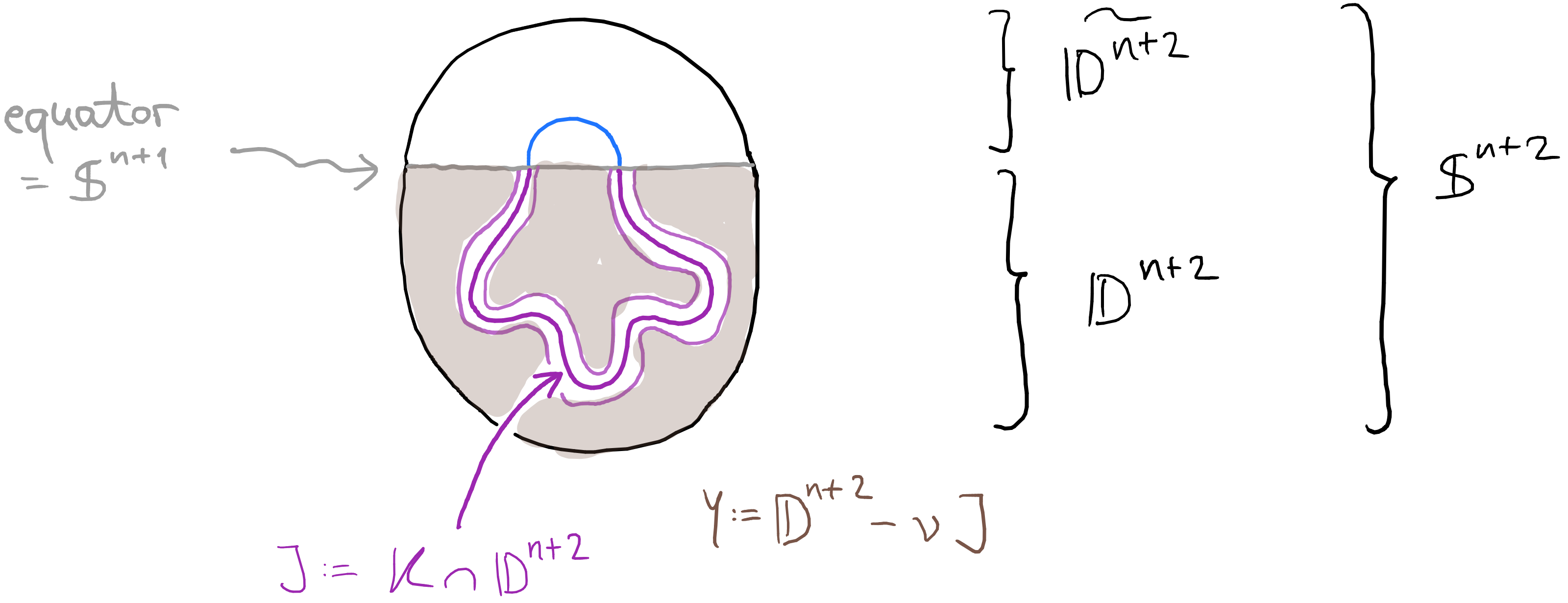
Plan:

- ·) decompose the exterior of the knot $\tau^m(k)$ into two pieces
- ·) write down fibre bundle structures on each which can be glued together


"outside"
$J \subset \mathbb{D}^3$
$J \subset \mathbb{D}^3$
non-trivial knotted disk
"inside"
$\widetilde{\mathbb{D}}^3$
trivial knotted disk


equator
$= \mathbb{S}^{n+1}$
$\widetilde{\mathbb{D}}^{n+2}$
$\mathbb{D}^{n+2}$
$\mathbb{S}^{n+2}$
$J := K \cap \mathbb{D}^{n+2}$
$Y := \mathbb{D}^{n+2} - \nu J$

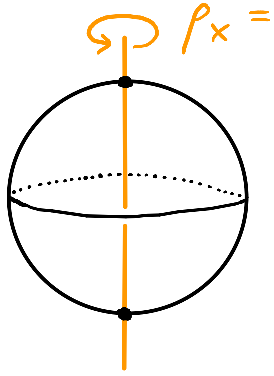
$$\Phi_m : \mathbb{S}^1 \times \mathbb{D}^3 \longrightarrow \mathbb{S}^1 \times \mathbb{D}^3$$

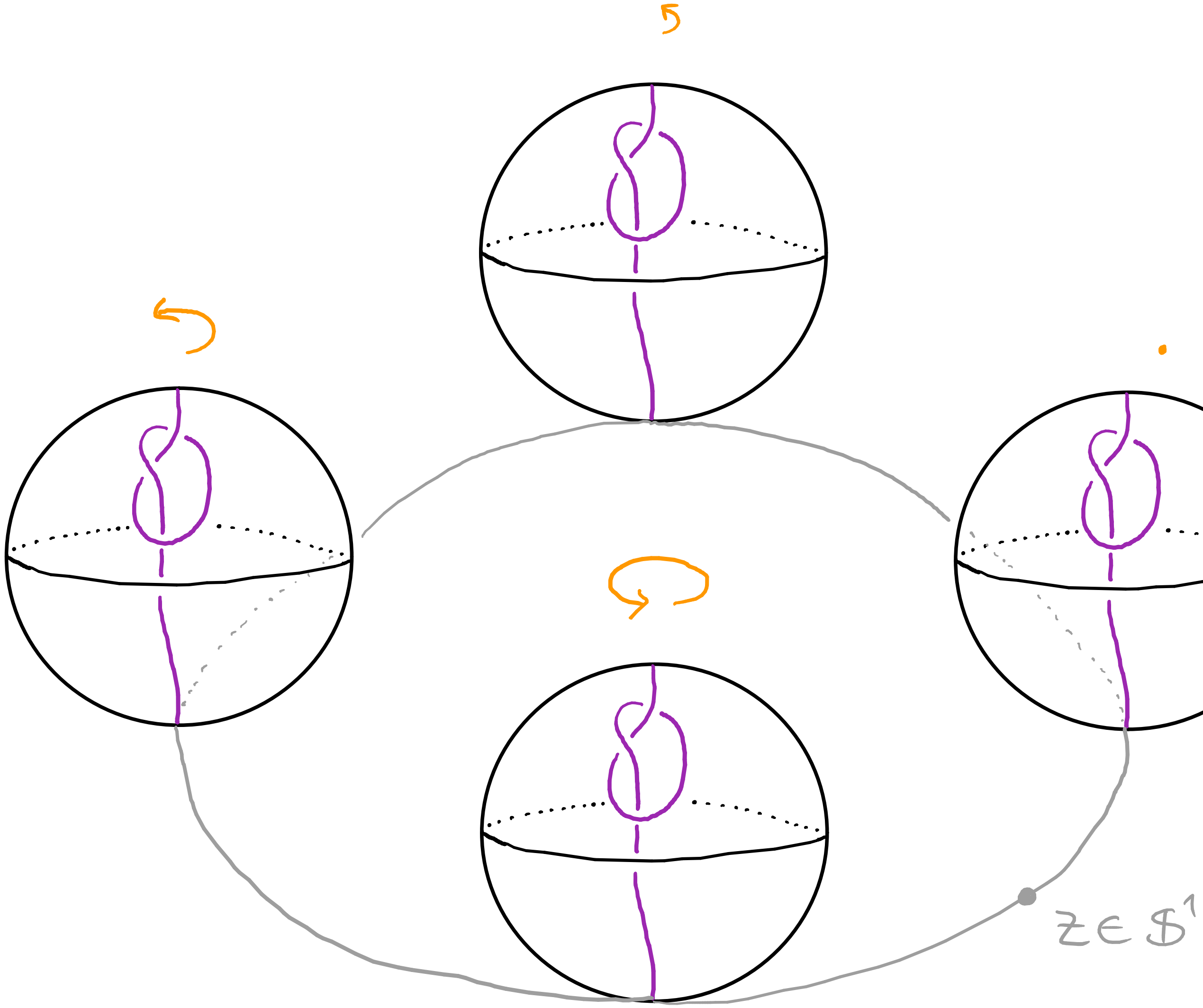
$$(z, x) \longmapsto (z, \rho_{z^m}(x))$$

where $\rho_x : \mathbb{D}^3 = \overset{\overset{\mathbb{C}}{\cup}}{\mathbb{D}^2} \times \mathbb{D}^1 \longrightarrow \overset{\overset{\mathbb{C}}{\cup}}{\mathbb{D}^2} \times \mathbb{D}^1 = \mathbb{D}^3$

$$(\alpha, \beta) \longmapsto (x \cdot \alpha, \beta)$$

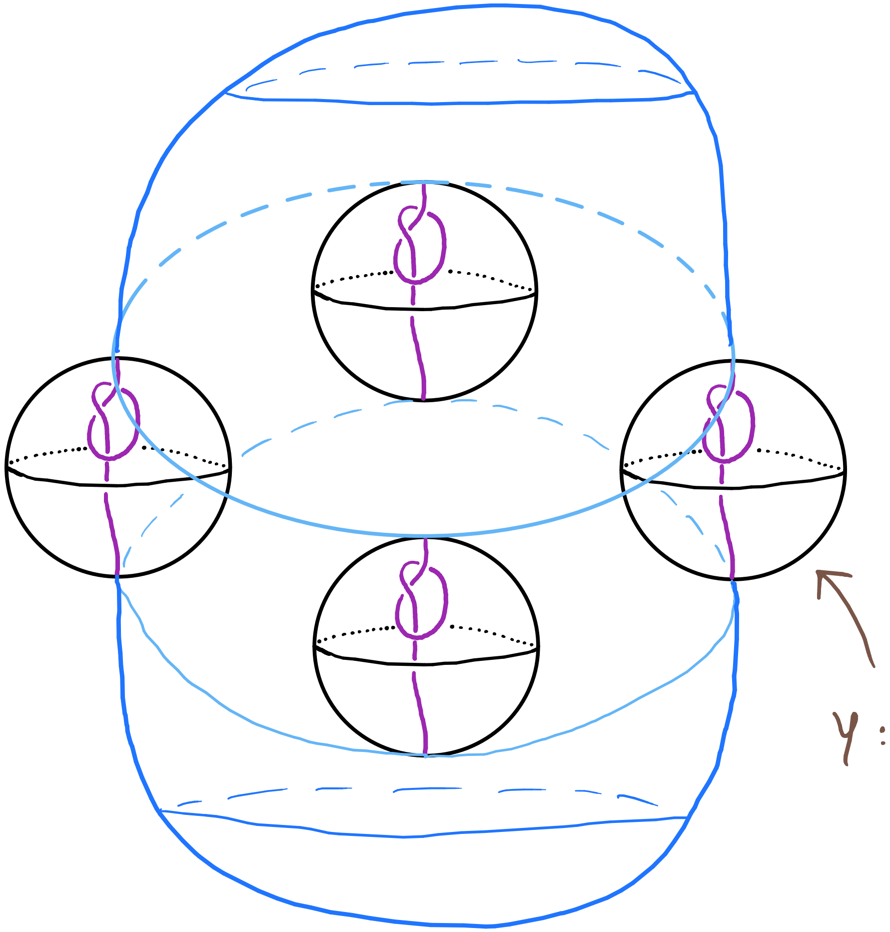
ρ_x = rotate by x





$$\Phi_m : \mathbb{S}^1 \times \mathbb{D}^3 \longrightarrow \mathbb{S}^1 \times \mathbb{D}^3$$
$$(z, x) \longmapsto (z, \rho_{z^m}(x))$$

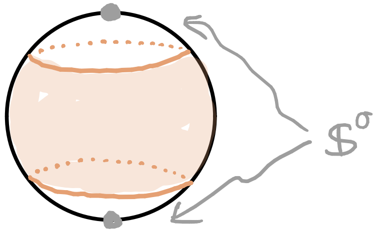
$$\tau^m k = \Phi_m(\mathbb{S}^1 \times J) \cup \mathbb{D}^2 \times \mathbb{S}^0$$
$$\cap$$
$$\mathbb{S}^4 = \mathbb{S}^1 \times \mathbb{D}^3 \cup \mathbb{D}^2 \times \mathbb{S}^2$$



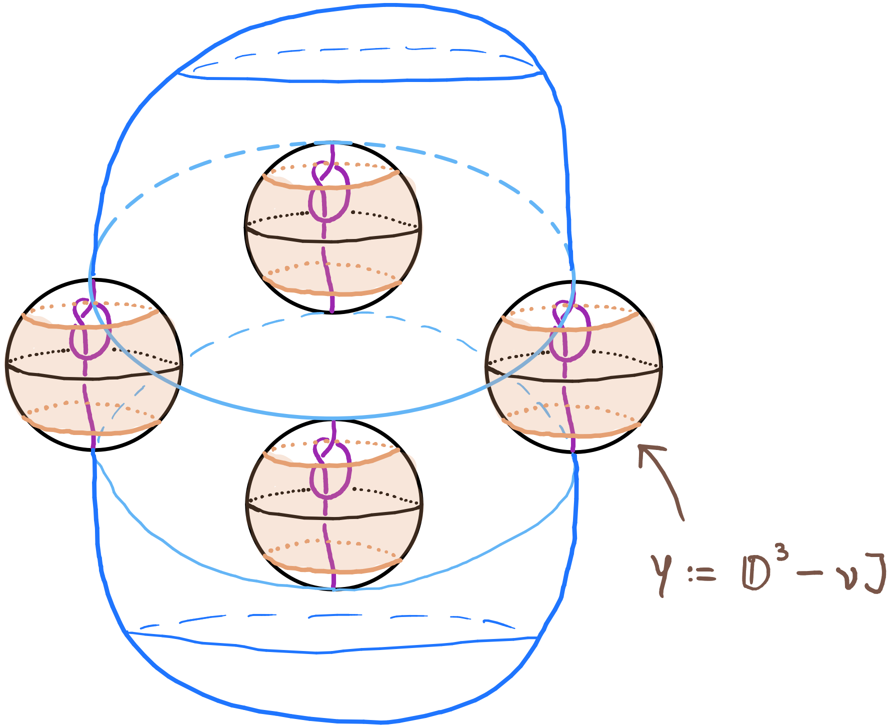
$Y := \mathbb{D}^3 - \nu J$

Knot exterior:

$$\mathbb{S}^4 - \nu(\tau^m(k)) = \mathbb{S}^1 \times \mathbb{D}^3 - \Phi_m(\mathbb{S}^1 \times \nu J) \cup \mathbb{D}^2 \times (\mathbb{S}^2 - \nu \mathbb{S}^0)$$
$$= \Phi_m(\mathbb{S}^1 \times Y) \cup \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{D}^1$$



Φ_m restricts to an automorphism of $\mathbb{S}^1 \times (Y \cap \partial\mathbb{D}^3) = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}^1$



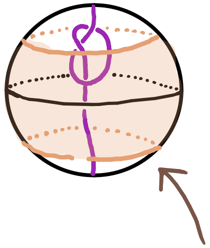
Knot exterior:

$$\mathbb{S}^4 - \nu(\tau^m(k)) = \mathbb{S}^1 \times \mathbb{D}^3 - \Phi_m(\mathbb{S}^1 \times \nu J) \cup \mathbb{D}^2 \times (\mathbb{S}^2 - \nu \mathbb{S}^0)$$

$$\begin{tikzcd} = \Phi_m(\mathbb{S}^1 \times Y) \cup \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{D}^1 \\ \mathbb{S}^1 \times Y \cup_{\Phi_m} \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{D}^1 \arrow[u, "\cong"', "\Phi_m \cup \mathrm{id}"] \end{tikzcd}$$

We will use this description to write down a fibre bundle structure over $\mathbb{S}^1$:

$$\mathbb{S}^4 - \tau^m K = \mathbb{S}^1 \times Y \cup_{\Phi_m} \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{D}^1$$



$Y := \mathbb{D}^3 - \nu J$

Obstruction theory: Extend the projection

$$\begin{tikzcd}
Y \cap \partial\mathbb{D}^3 = \mathbb{S}^2 - \nu\mathbb{S}^0 = \mathbb{S}^1 \times \mathbb{D}^1 \arrow[r, hook] \arrow[d, "pr_{\mathbb{S}^1}"'] & Y \arrow[dl, dashed, "\exists\varphi"] \\
\mathbb{S}^1 &
\end{tikzcd}$$

Check:

$$\begin{tikzcd}
\{(z,x) \in \mathbb{S}^1 \times Y \mid z^{-m}\varphi(x) = 1\} \arrow[d] & \text{fibre} \arrow[l] \\
\mathbb{S}^1 \times Y \arrow[d, "p"'] & (z,x) \arrow[d, mapsto] \\
\mathbb{S}^1 & z^{-m} \cdot \varphi(x)
\end{tikzcd}$$

$\Big\}$ is a fibre bundle for $m \neq 0$

$$\Rightarrow \quad \begin{tikzcd}
\mathbb{S}^1 \times Y \cup_{\Phi_m} \mathbb{D}^2 \times \mathbb{S}^1 \times \mathbb{D}^1 \arrow[d, "p \cup pr_{\mathbb{S}^1}"] \\
\mathbb{S}^1
\end{tikzcd}$$

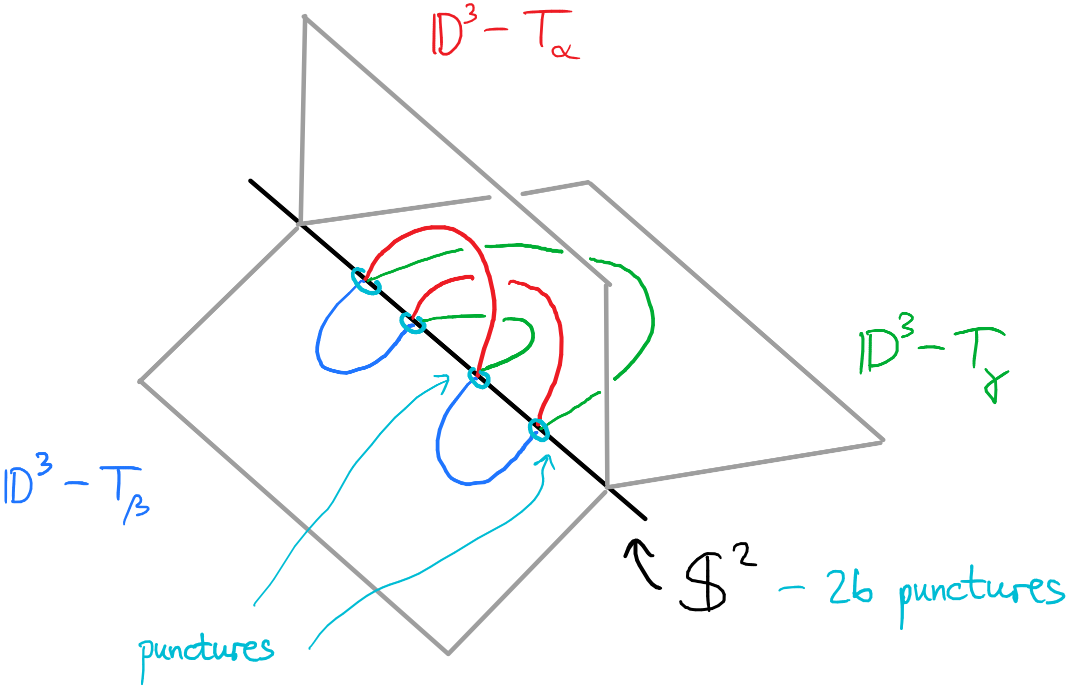
is a fibre bundle structure on $\mathbb{S}^4 - \tau^m K$

Identify the fibre:

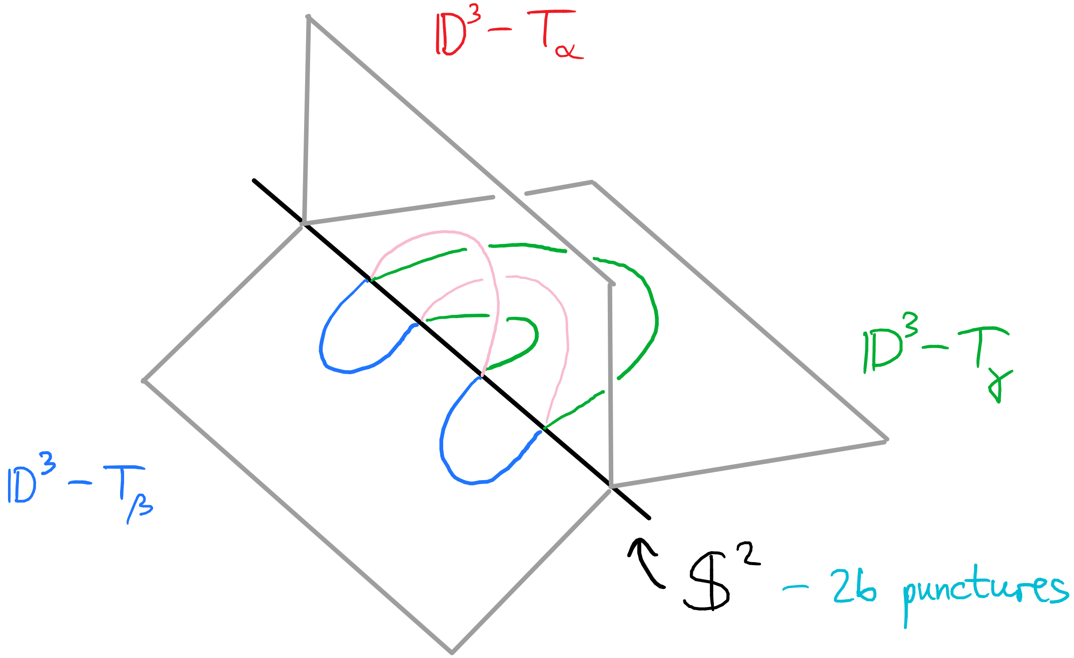
$$\begin{tikzcd}
\overbrace{\{(z,x)\in S^1\times Y \mid \varphi(x)=z^m\}}^{Y_m = \text{m-fold cyclic cover of } Y} \cup \overbrace{D^2\times\{1\}\times D^1}^{\text{2-handle attached to preimage of a meridian under the covering } Y_m\to Y} \cong \Sigma_m(k) - \operatorname{int} D^3 \arrow[d] \\
S^1\times Y \cup_{\Phi_m} D^2\times S^1\times D^1 \arrow[d, "p\,\cup\,pr_{S^1}"] \\
S^1
\end{tikzcd}$$

$\square$ (fibration theorem)

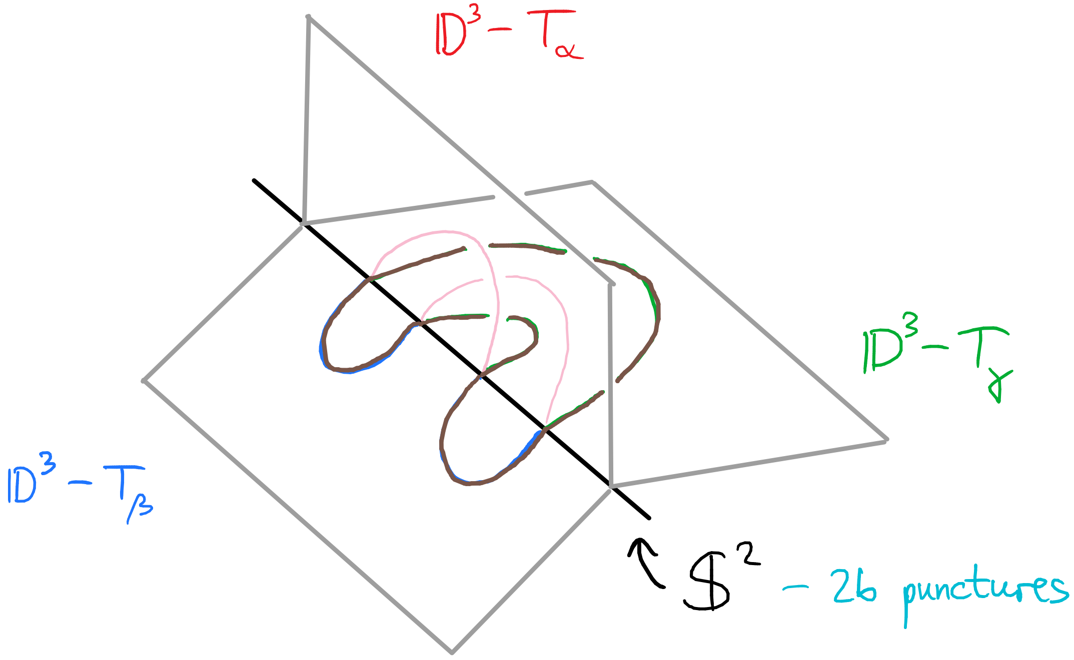
Bridge trisections
[Meier – Zupan]



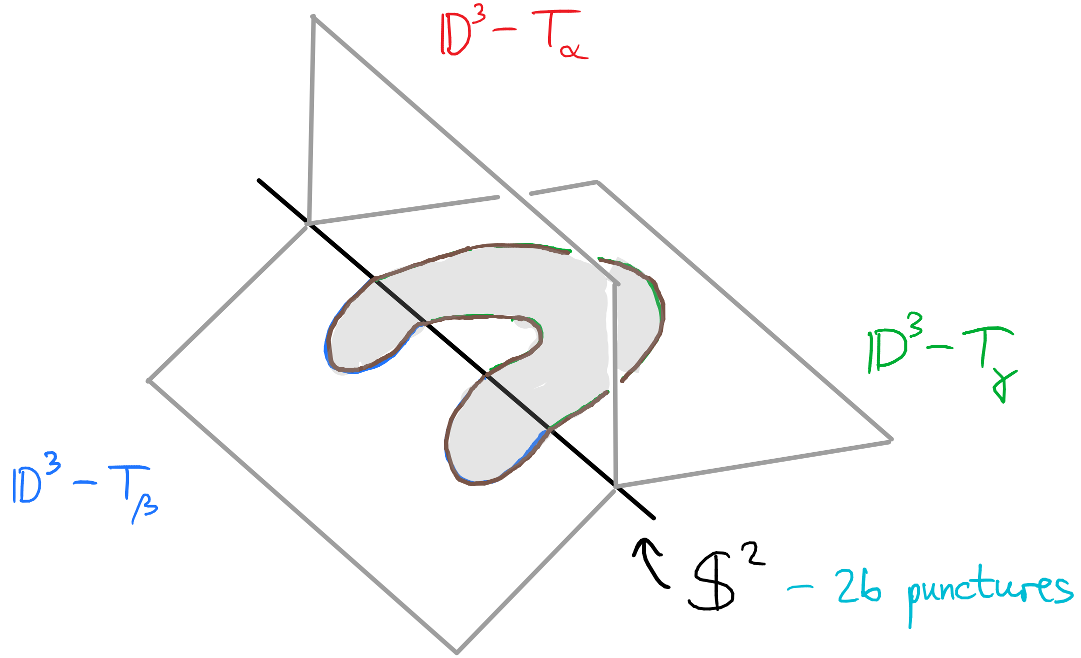
$$\begin{tikzcd}
 & \pi_1(\mathbb{D}^3 - T_\alpha) \\
\pi_1(\mathbb{S}^2 - 2b \text{ points}) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(\mathbb{D}^3 - T_\beta) \\
\pi_1(\mathbb{D}^3 - T_\gamma) &
\end{tikzcd}$$



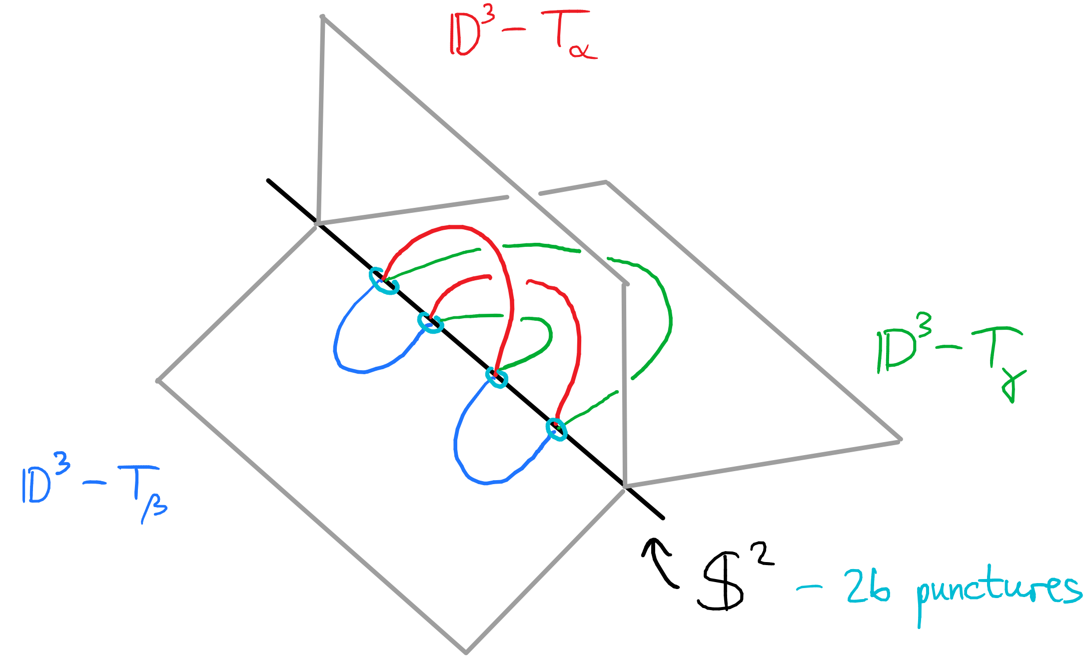
$$\begin{tikzcd}
 & \pi_1(\mathbb{D}^3 - T_\alpha) \\
\pi_1(\mathbb{S}^2 - 2b \text{ points}) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(\mathbb{D}^3 - T_\beta) \\
\pi_1(\mathbb{D}^3 - T_\gamma) &
\end{tikzcd}$$



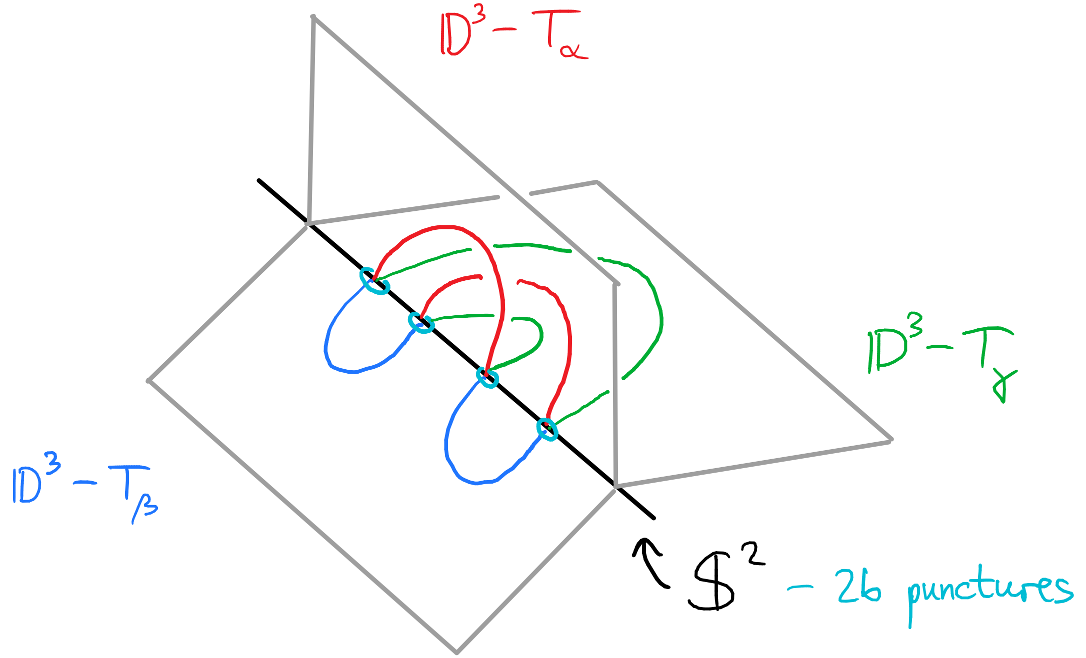
$$\begin{tikzcd}
 & \pi_1(\mathbb{D}^3 - T_\alpha) \\
\pi_1(\mathbb{S}^2 - 26 \text{ points}) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(\mathbb{D}^3 - T_\beta) \arrow[d, two heads] \\
\pi_1(\mathbb{D}^3 - T_\gamma) \arrow[r, two heads] & \pi_1(\mathbb{S}^3 - \mathcal{U}_{\alpha\cup\beta})
\end{tikzcd}$$



$$\begin{tikzcd}
 & \pi_1(\mathbb{D}^3 - T_\alpha) \\
\pi_1(\mathbb{S}^2 - 2b \text{ points}) \arrow[ur, two heads] \arrow[r, two heads] \arrow[d, two heads] & \pi_1(\mathbb{D}^3 - T_\beta) \arrow[d, two heads] \\
\pi_1(\mathbb{D}^3 - T_\gamma) \arrow[r, two heads] & \pi_1(\mathbb{S}^3 - \mathcal{U}_{\beta\cup\gamma}) \\
 & \cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\beta\cup\gamma})
\end{tikzcd}$$



$$\begin{tikzcd}
 & \pi_1(\mathbb{D}^3 - T_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \overset{\cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\alpha\cup\beta})}{\pi_1(\mathbb{S}^3 - \mathcal{U}_{\alpha\cup\beta})} \\
\pi_1(\mathbb{S}^2 - 2b \text{ points}) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(\mathbb{D}^3 - T_\beta) \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & \underset{\cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\alpha\cup\gamma})}{\pi_1(\mathbb{S}^3 - \mathcal{U}_{\alpha\cup\gamma})} & & \\
\pi_1(\mathbb{D}^3 - T_\gamma) \arrow[ur, two heads] \arrow[rr, two heads] & & \underset{\cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\beta\cup\gamma})}{\pi_1(\mathbb{S}^3 - \mathcal{U}_{\beta\cup\gamma})} &
\end{tikzcd}$$



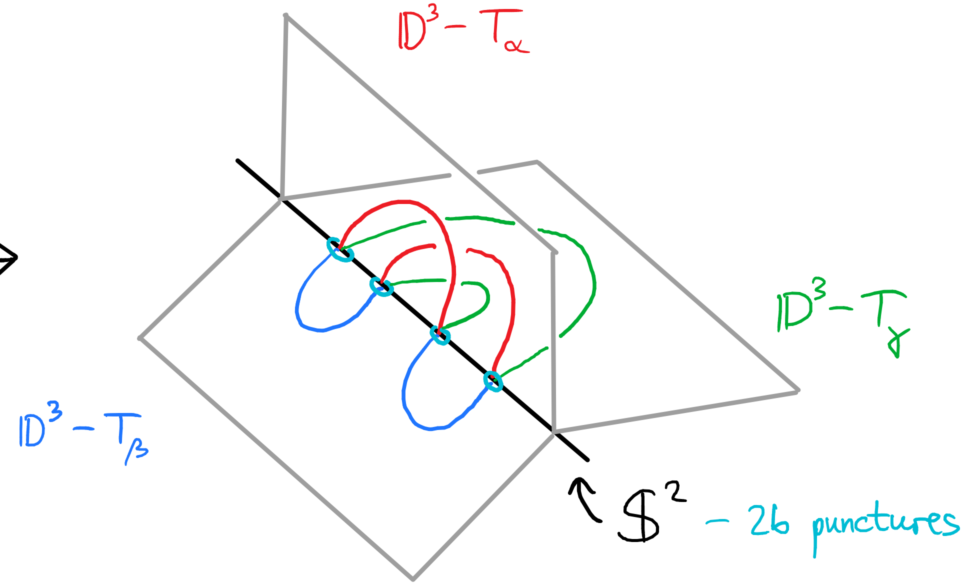
$$\begin{tikzcd}
 & & & \cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\alpha\cup\beta}) \\
 & \pi_1(\mathbb{D}^3 - T_\alpha) \arrow[rr, two heads] \arrow[dd, two heads] & & \pi_1(\mathbb{S}^3 - \mathcal{U}_{\alpha\cup\beta}) \arrow[dd, two heads] \\
\pi_1(\mathbb{S}^2 - 2b \text{ points}) \arrow[ur, two heads] \arrow[rr, two heads] \arrow[ddd, two heads] & & \pi_1(\mathbb{D}^3 - T_\beta) \arrow[ur, two heads] \arrow[ddd, two heads] & \\
 & \pi_1(\mathbb{S}^3 - \mathcal{U}_{\alpha\cup\gamma}) \arrow[rr, two heads] & & \pi_1(\mathbb{S}^4 - F) \\
 & \cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\alpha\cup\gamma}) & & \\
\pi_1(\mathbb{D}^3 - T_\gamma) \arrow[uur, two heads] \arrow[rr, two heads] & & \pi_1(\mathbb{S}^3 - \mathcal{U}_{\beta\cup\gamma}) \arrow[uur, two heads] & \text{knotted surface} \arrow[uu] \\
 & & \cong \pi_1(\mathbb{D}^4 - \mathcal{D}_{\beta\cup\gamma}) &
\end{tikzcd}$$

[Sarah Blackwell, Rob Kirby, Michael Klug, Vincent Longo, B.R.]

(*)

$$\begin{tikzcd}
 & Fr_b \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_{c_1} \arrow[dd, two heads] \\
Fr_{2b}/\langle\langle x_1, \dots, x_{2b} \rangle\rangle \arrow[ur, two heads] \arrow[rr, two heads] \arrow[dd, two heads] & & Fr_b \arrow[ur, two heads] \arrow[dd, two heads] & \\
 & Fr_{c_3} \arrow[rr, two heads] & & \pi_1(S^4 - F) \\
Fr_b \arrow[ur, two heads] \arrow[rr, two heads] & & Fr_{c_2} \arrow[ur, two heads] &
\end{tikzcd}$$

⇝



Algebra: ⇝ **Topology:**

Knotted surface group trisection ⇝ Bridge trisection

(*) conditions apply on the maps in the colored tripod

[LAST PAGE]