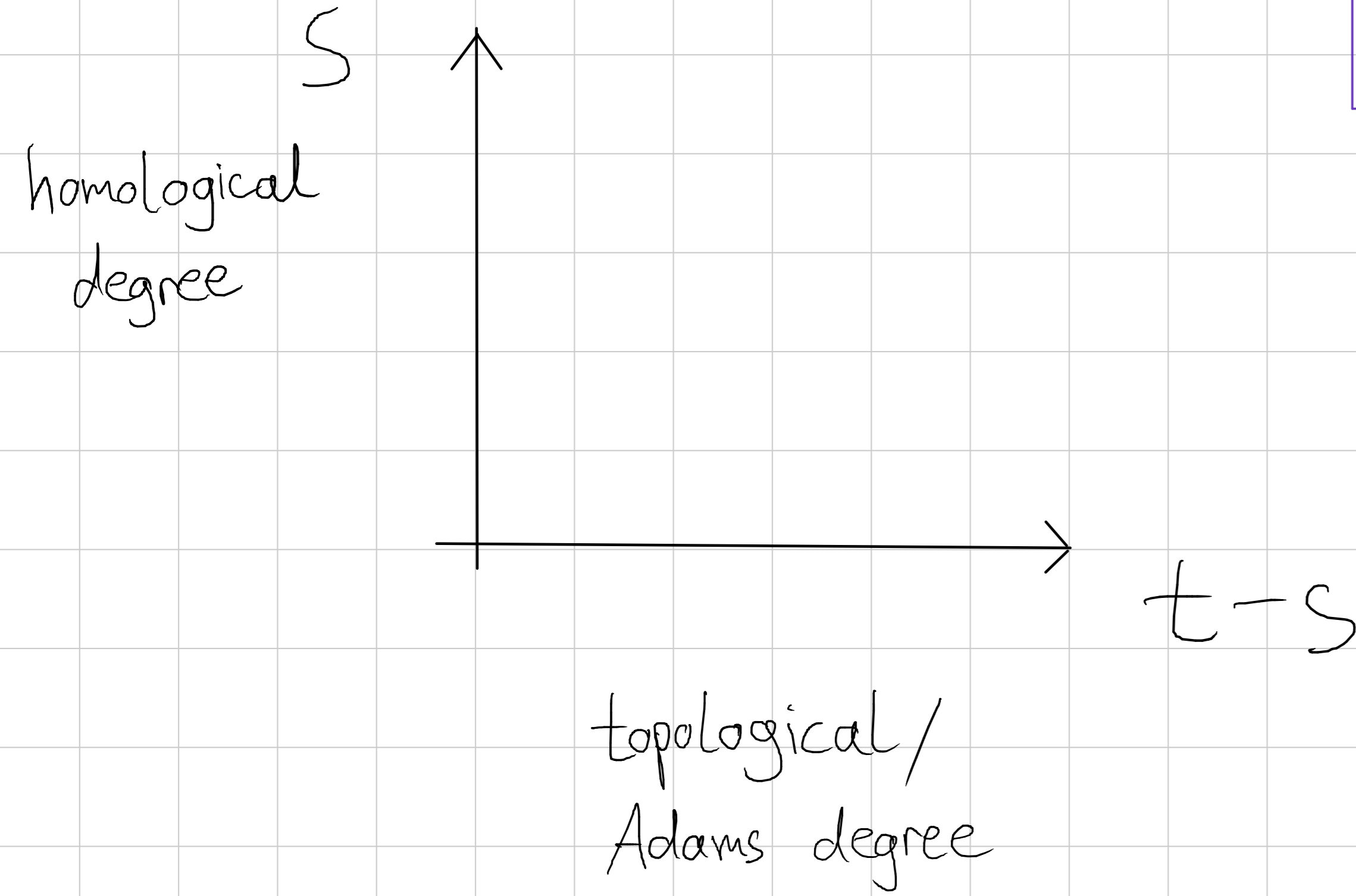
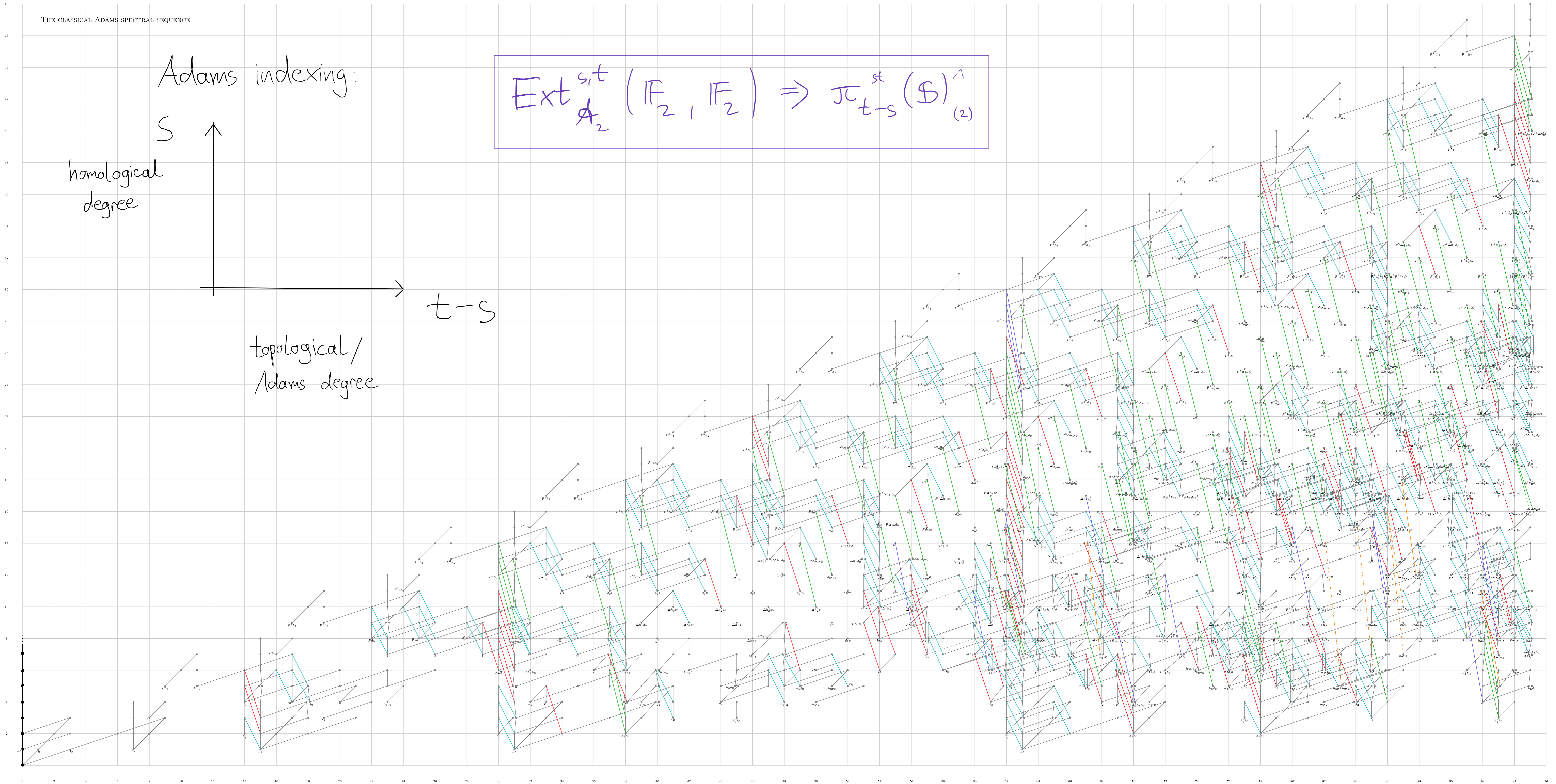




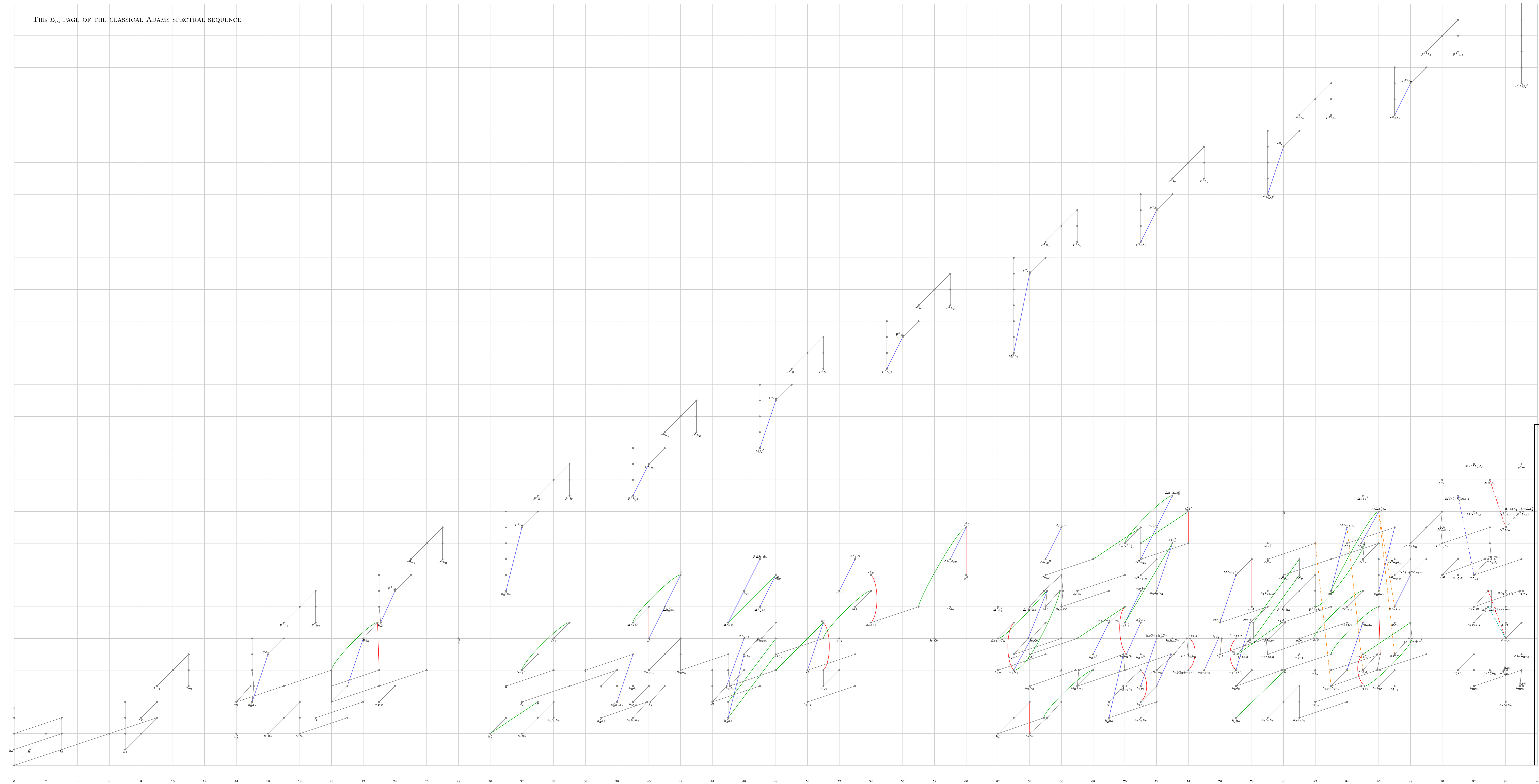
Adams indexing:



$$\text{Ext}_{A_2}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \mathcal{J}C_{t-s}^{st}(\mathcal{S})_{(2)}^{\wedge}$$







The charts are borrowed from:

more on his website

## CLASSICAL AND $\mathbb{C}$ -MOTIVIC ADAMS CHARTS

DANIEL C. ISAKSEN, GUOZHEN WANG, AND ZHOULI XU

**ABSTRACT.** This document contains large-format Adams charts that compute 2-complete stable homotopy groups, both in the classical context and in the  $\mathbb{C}$ -motivic context. The charts are essentially complete through the 90-stem and contain partial results to the 95-stem.

This document contains large-format Adams charts that compute 2-complete stable homotopy groups, both in the classical context and in the  $\mathbb{C}$ -motivic context. The charts are essentially complete through the 90-stem and contain partial results to the 95-stem.

The charts are intended to be viewed electronically. The authors can supply versions that are suitable for printing.

Justifications for these computations appear in [3], [6], [7], [9], [14], [15], and [16]. Older references that justify many of the classical computations include [1], [2], [4], [10], [11], [12], and [13].

This document supersedes [5], which included Adams charts for the cofiber of  $\tau$ . The charts associated to the cofiber of  $\tau$  now appear in the separate manuscript [8].

### 1. COHOMOLOGY OF THE CLASSICAL STEENROD ALGEBRA

This chart shows the cohomology of the classical Steenrod algebra, i.e., the  $E_2$ -page of the classical Adams spectral sequence, through the 110-stem.

- (1) Black dots indicate copies of  $\mathbb{F}_2$ .
- (2) Vertical lines indicate  $h_0$  multiplications.
- (3) Lines of slope 1 indicate  $h_1$  multiplications.
- (4) Lines of slope 1/3 indicate  $h_2$  multiplications.

### 2. THE CLASSICAL ADAMS SPECTRAL SEQUENCE

This chart shows the classical Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 95-stem.

- (1) Black dots indicate copies of  $\mathbb{F}_2$ .
- (2) Vertical lines indicate  $h_0$  multiplications.
- (3) Lines of slope 1 indicate  $h_1$  multiplications.
- (4) Lines of slope 1/3 indicate  $h_2$  multiplications.

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2010 *Mathematics Subject Classification.* 55T15, 55Q45, 14F42.

*Key words and phrases.* Adams spectral sequence, stable homotopy group, motivic stable homotopy group, cohomology of the Steenrod algebra.

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Further references: •) [Bruner: An Adams Spectral Sequence Primer]

today, 2020-11-26

Serre's method: Killing homotopy groups

Qualitative results:  $\pi_k^{st}(\mathbb{S})$  finite (for  $k > 0$ )

convergence  
spectral sequence of a filtered chain complex

Why spectral sequences?

Adams: Put filtration on the sphere spectrum

Smash & tensor products

Bar resolution

Adams resolution in spectra

Adams spectral seq. from tower of fibrations

composition & Yoneda prod.

Bertram, 2020-12-10

Adams Spectral Sequence

$$\text{Ext}_{E^*E}^{s,t}(E^*Y, E^*X) \Rightarrow [X, Y]_{t-s}^E$$

convergence

vanishing line

image of J-homomorphism

products in the Adams spectral sequence

Massey products & Toda brackets

$$\text{Ext}_{\mathcal{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}^{st}(\mathbb{S}^0)$$

Luuk, 2020-12-03

Construction of  $\mathbb{F}_2$ -Steenrod squares

Subalgebras:  $\mathcal{A}(n), \mathcal{E}(n)$

Steenrod algebra

$$H_*ko \cong \mathcal{A} // \mathcal{A}(1), \quad H_*ku \cong \mathcal{A} // \mathcal{E}(1)$$

$$\mathcal{A} = H\mathbb{F}_2^*(H\mathbb{F}_2)$$

[Milnor] dual Steenrod algebra

similar for  $E^*E$

$$E_*E = \mathcal{A}_* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots]$$

$|\xi_i| = 2^i - 1$

Adem relations

sample computations

$H\mathbb{F}_2$  Eilenberg-MacLane spectrum repr. singular

$\mathbb{F}_2$ -cohomology

$\mathbb{S}$  suspension spectrum of sphere

Recall:

Spectra,

mapping spaces  $[X, Y]$ ,

Smash products, ...

# Adams Spectral Sequence

$$\text{Ext}_{E^*E}^{s,t}(E^*Y, E^*X) \Rightarrow [X, Y]_{t-s}^E$$

•) We are interested in  $[X, Y]$  stable homotopy classes of maps from  $X$  to  $Y$

$\uparrow \quad \uparrow$   
 spectra

•) Want to calculate / approximate this with cohomology

•) Take a cohomology theory  $E^*$  ( $\nearrow$  need some assumptions for later)

$\Rightarrow$  There is a filtration  $[X, Y] = F_0 \supset F_1 \supset F_2 \supset \dots$

quotients  $F_i / F_{i+1}$  are the  $E_\infty$ -terms of a spectral sequence

$F_i = \left\{ \begin{array}{l} \text{compositions of } i\text{-many maps } f_1, \dots, f_i \\ \text{which induce the zero-map on } E\text{-cohomology} \end{array} \right\}$

•)  $E_2$ -term can be computed from  $E$ -cohomology  $E^*Y$  and  $E^*X$

considered as  $E^*E$ -modules

# Adams Spectral Sequence

$$\text{Ext}_{E^*E}^{s,t} (E^*Y, E^*X) \Rightarrow [X, Y]_{t-s}^E$$

Example: Classical Adams Spectral sequence at the prime  $p=2$

•)  $E = HF_2$  Eilenberg-MacLane spectrum representing singular  $\mathbb{Z}/2$ -cohomology

•) Take  $X, Y = \mathbb{S}$  suspension spectrum of the sphere  $\mathbb{S}^0$

$$\text{Ext}_{\mathcal{A}_2}^{s,t} (HF_2^* \mathbb{S}, HF_2^* \mathbb{S}) \Rightarrow \pi_{t-s}^{st} (\mathbb{S})^{\wedge (2)}$$

$$\mathcal{A}_2 = HF_2^* HF_2 \quad \text{Ext}_{\mathcal{A}_2}^{s,t} (F_2, F_2)$$

Steenrod algebra  
for the prime 2

stable homotopy groups of spheres  
- but only their  $p=2$ -primary part

# Adams Spectral Sequence

$$\text{Ext}_{E^*E}^{s,t}(E^*Y, E^*X) \Rightarrow [X, Y]_{t-s}^E$$

There is also a homology version

$$\text{Ext}_{E_*E}^{s,t}(E_*Y, E_*X) \Rightarrow [X, Y]_{t-s}^E$$

$\left( \quad \right)$   
considered as  
 $E_*E$ -comodules



Steenrod algebra:  $\mathcal{A}_p = H\mathbb{F}_p^* H\mathbb{F}_p = [H\mathbb{F}_p, H\mathbb{F}_p]$

graded  $\mathbb{F}_p$ -algebra of mod- $p$  stable cohomology operations

↑

product is composition of operations

degree  $n$  part of  $\mathcal{A}_p \hat{=} \text{natural transformations } \tilde{H}^i(-; \mathbb{F}_p) \longrightarrow H^{i+n}(-; \mathbb{F}_p)$

that are compatible

with suspension

**10.8. The 2-primary Steenrod algebra.** The  $i$ -th Steenrod square, for  $i \geq 0$ , is a stable mod-2 cohomology operation  $Sq^i$  of degree  $i$  with the following properties:

- (i) The operation  $Sq^0$  is the identity and  $Sq^1$  coincides with the mod-2 Bockstein operation.
- (ii) (Unstability condition) For every simplicial set  $X$  and cohomology class  $x \in H^n(X, \mathbb{F}_2)$  we have  $Sq^i(x) = x \cup x$  if  $i = n$  and  $Sq^i(x) = 0$  for  $i > n$ .
- (iii) (Cartan formula) For  $x, y \in H^*(X, \mathbb{F}_2)$  and  $i \geq 0$  we have

$$Sq^i(x \cup y) = \sum_{a=0}^i Sq^a x \cup Sq^{i-a} y.$$

- (iv) (Adem relations) The Steenrod squaring operations satisfy the following relations

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$

for all  $a < 2b$ .

The stable operation  $Sq^i$  is in fact already uniquely determined by the property that  $Sq^i(x) = x \cup x$  for all classes  $x$  of degree  $i$ .

free graded  $\mathbb{F}_2$ -algebra  
generated by symbols  $Sq^i$  of degree  $i$  for  $i \geq 0$

↓

$$\mathbb{F}_2 \langle Sq^i \rangle / (Sq^0 = 1, \text{Adem relations}) \cong \mathcal{A}_2$$

Filtration 0: degree

$$f: X \rightarrow Y \quad \mapsto \quad f^*: E^*Y \rightarrow E^*X$$

$$F_0 = [X, Y] \xrightarrow{d} \text{Hom}_{E^*E}(E^*Y, E^*X)$$

So for  $E = H\mathbb{Z}$  and  $X, Y$   $n$ -mflds,  $d(f) \in \text{Hom}(\mathbb{Z}, \mathbb{Z})$

degree of map  $f$

If  $d(f) = \sigma$ , i.e.  $f \in \text{Ker}(d)$  look at L.S. cohomology seq. from

$$X \xrightarrow{f} Y \rightarrow \text{Cofib}(f) \simeq Y \cup_f \text{Cone}(X)$$

$$\dots \leftarrow E^n X \xleftarrow[\text{zero}]{\sigma} E^n Y \leftarrow E^n(\text{Cf}) \leftarrow E^{n-1}(X) \xleftarrow[\text{zero}]{\sigma} E^{n-1}(Y) \leftarrow \dots$$

Filtration 0: degree & filtration 1: Hopf invariant

$$f: X \rightarrow Y \quad \mapsto \quad f^*: E^*Y \rightarrow E^*X$$

$$F_0 = [X, Y] \xrightarrow{d} \text{Hom}_{E^*E}(E^*Y, E^*X)$$

U1

$$F_1 = \text{Ker}(d) \xrightarrow{e} \text{Ext}_{E^*E}^1(E^*Y, E^*X)$$

$$e(f) = \{ \sigma \leftarrow E^*Y \leftarrow E^* \text{Cone}(f) \leftarrow E^*X \leftarrow \sigma \}$$

degree-shift/suspension hidden here

$$\dots \leftarrow E^n X \xleftarrow{\sigma} E^n Y \leftarrow E^n(\text{Cf}) \leftarrow E^{n-1}(X) \xleftarrow{\sigma} E^{n-1}(Y) \leftarrow \dots$$



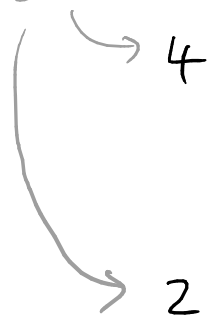
Example:

$$e(f) = \{ \sigma \leftarrow E^*Y \leftarrow E^* \text{Cone}(f) \leftarrow E^*X \leftarrow \sigma \}$$

Hopf map  $\eta: S^3 \rightarrow S^2$ , take  $E = H\mathbb{F}_2$  <sup>(reduced)</sup>  
 $\mathbb{Z}/2$ -homology

$\rightsquigarrow$  s.e.s.  $\{ \sigma \leftarrow H^*S^2 \leftarrow H^*\mathbb{C}P^2 \leftarrow H^*S^4 \leftarrow \sigma \} \in \text{Ext}_{\mathcal{A}}^{1,0}(H^*S^2, H^*S^4)$

homological degree



$$\begin{array}{ccc} \mathbb{F}_2 & \xleftarrow{\text{induced map}} & \mathbb{F}_2 \\ & \uparrow \text{Sq}^2 & \\ \mathbb{F}_2 & \xleftarrow{\quad} & \mathbb{F}_2 \end{array}$$

action of Steenrod algebra

Non-split  $\rightsquigarrow$  this element of  $\text{Ext}_{\mathcal{A}}^1$  detects  $\eta$

·) (Historic) Application: Hopf invariant 1 problem

$\mathbb{R}^n$  admits a division algebra structure only for  $n=1,2,4,8$

(later, Adams found a shorter proof using cohomology operations in K-theory)

·) Calculating the complex/oriented cobordism ring

$$\pi_k(MU) = \frac{\{\text{compact, closed almost complex } k\text{-manifolds}\}}{\{\text{complex cobordism}\}}$$

$$\text{Ext}_{MU_* MU}^{s,t} (MU_*, MU_*) \Rightarrow \pi_{t-s} \mathcal{S}$$

·) Pontryagin-Thom:  $\pi_n^{st}(\mathcal{S}) = \frac{\{\text{stably framed compact closed } n\text{-mflds.}\}}{\{\text{stably framed cobordism}\}}$

# You Could Have Invented Spectral Sequences

*Timothy Y. Chow*

## Introduction

The subject of spectral sequences has a reputation for being difficult for the beginner. Even G. W. Whitehead (quoted in John McCleary [4]) once remarked, “The machinery of spectral sequences, stemming from the algebraic work of Lyndon and Koszul, seemed complicated and obscure to many topologists.”

Why is this? David Eisenbud [1] suggests an explanation: “The subject of spectral sequences is elementary, but the notion of the spectral sequence of a double complex involves so many objects and indices that it seems at first repulsive.” I have heard others make similar complaints about the proliferation of subscripts and superscripts.

My own explanation, however, is that spectral sequences are often not taught in a way that explains how one might have come up with the definition in the first place. For example, John McCleary’s excellent text [4] says, “The user, however, needs to get acquainted with the manipulation of these gadgets without the formidable issue of their origins.” Without an understanding of where spectral sequences come from, one naturally finds them mysterious. Conversely, if one *does* see where they come from, the notation should not be a stumbling block.

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Fools rush in where angels fear to tread, so my goal below is to make you, the reader, feel that you could have invented spectral sequences (on a very good day, to be sure!). I assume familiarity with homology groups, but little more. Everything here is known to the cognoscenti, but my hope is to make the ideas accessible to more than the lucky few who are able to have the right conversation with the right expert at the right time.

Readers who are interested in the history of spectral sequences and how they were in fact invented should read [3], which gives a definitive account.

## Simplifying Assumptions

Throughout, we work over a field. All chain groups are finite-dimensional, and all filtrations (explained below) have only finitely many levels. In the “real world”, these assumptions may fail, but the essential ideas are easier to grasp in this simpler context.

## Graded Complexes

Chain complexes that occur “in nature” often come with extra structure in addition to the boundary map. Certain kinds of extra structure are particularly common, so it makes sense to find a systematic method for exploiting such features. Then we do not have to reinvent the wheel each time we want to compute a homology group.

Here is a simple example. Suppose we have a chain complex

$$\cdots \xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots$$

that is “graded”, i.e., each  $C_d$  splits into a direct sum

$$C_d = \bigoplus_{p=1}^n C_{d,p}$$

and moreover the boundary map  $\partial$  respects the grading in the sense that  $\partial C_{d,p} \subseteq C_{d-1,p}$  for all  $d$  and  $p$ . Then the grading allows us to break up the computation of the homology into smaller pieces: simply compute the homology in each grade independently and then sum them all up to obtain the homology of the original complex.

Unfortunately, in practice we are not always so lucky as to have a grading on our complex. What we frequently have instead is a *filtered complex*, i.e., each  $C_d$  comes equipped with a nested sequence of submodules

$$0 = C_{d,0} \subseteq C_{d,1} \subseteq C_{d,2} \subseteq \cdots \subseteq C_{d,n} = C_d$$

and the boundary map respects the filtration in the sense that

$$(1) \quad \partial C_{d,p} \subseteq C_{d-1,p}$$

for all  $d$  and  $p$ . (Note: The index  $p$  is called the *filtration degree*. Here it has a natural meaning only if  $0 \leq p \leq n$ , but throughout this paper, we sometimes allow indices to “go out of bounds,” with the understanding that the objects in question are zero in that case. For example,  $C_{d,-1} = 0$ .)

Although a filtered complex is not quite the same as a graded complex, it is similar enough that we might wonder if a similar “divide and conquer” strategy works here. For example, is there a natural way to break up the homology groups of a filtered complex into a direct sum? The answer turns out to be yes, but the situation is surprisingly complicated. As we shall now see, the analysis leads directly to the concept of a spectral sequence.

Let us begin by trying naïvely to “reduce” this problem to the previously solved problem of graded complexes. To do this we need to express each  $C_d$  as a direct sum. Now,  $C_d$  is certainly not a direct sum of the  $C_{d,p}$ ; indeed,  $C_{d,n}$  is already all of  $C_d$ . However, because  $C_d$  is a finite-dimensional vector space (recall the assumptions we made at the outset), we can obtain a space isomorphic to  $C_d$  by modding out by any subspace  $U$  and then direct summing with  $U$ ; that is to say,  $C_d \simeq (C_d/U) \oplus U$ . In particular, we can take  $U = C_{d,n-1}$ . Then we can iterate this process to break  $U$  itself down into a direct sum, and continue all the way down. More formally, define

$$(2) \quad E_{d,p}^0 \stackrel{\text{def}}{=} C_{d,p}/C_{d,p-1}$$

for all  $d$  and  $p$ . (Warning: There exist different indexing conventions for spectral sequences; most authors write  $E_{p,q}^0$  where  $q = d - p$  is called the *complementary degree*. The indexing convention I use here is the one that I feel is clearest pedagogically.) Then

$$(3) \quad C_d \simeq \bigoplus_{p=1}^n E_{d,p}^0$$

The nice thing about this direct sum decomposition is that the boundary map  $\partial$  naturally induces a map

$$\partial^0 : \bigoplus_{p=1}^n E_{d,p}^0 \rightarrow \bigoplus_{p=1}^n E_{d-1,p}^0$$

such that  $\partial^0 E_{d,p}^0 \subseteq E_{d-1,p}^0$  for all  $d$  and  $p$ . The reason is that two elements of  $C_{d,p}$  that differ by an element of  $C_{d,p-1}$  get mapped to elements of  $C_{d-1,p}$  that differ by an element of  $\partial C_{d,p-1} \subseteq C_{d-1,p-1}$ , by equation (1).

Therefore we obtain a graded complex that splits up into  $n$  pieces:

$$(4) \quad \begin{array}{cccccccc} \cdots & \xrightarrow{\partial^0} & E_{d+1,n}^0 & \xrightarrow{\partial^0} & E_{d,n}^0 & \xrightarrow{\partial^0} & E_{d-1,n}^0 & \xrightarrow{\partial^0} & \cdots \\ \cdots & \xrightarrow{\partial^0} & E_{d+1,n-1}^0 & \xrightarrow{\partial^0} & E_{d,n-1}^0 & \xrightarrow{\partial^0} & E_{d-1,n-1}^0 & \xrightarrow{\partial^0} & \cdots \\ & & \vdots & & \vdots & & \vdots & & \\ \cdots & \xrightarrow{\partial^0} & E_{d+1,1}^0 & \xrightarrow{\partial^0} & E_{d,1}^0 & \xrightarrow{\partial^0} & E_{d-1,1}^0 & \xrightarrow{\partial^0} & \cdots \end{array}$$

Now let us define  $E_{d,p}^1$  to be the  $p$ th graded piece of the homology of this complex:

$$(5) \quad E_{d,p}^1 \stackrel{\text{def}}{=} H_d(E_{d,p}^0) = \frac{\ker \partial^0 : E_{d,p}^0 \rightarrow E_{d-1,p}^0}{\text{im } \partial^0 : E_{d+1,p}^0 \rightarrow E_{d,p}^0}$$

(For those comfortable with relative homology, note that  $E_{d,p}^1$  is just the relative homology group  $H_d(C_p, C_{p-1})$ .) Still thinking naïvely, we might hope that

$$(6) \quad \bigoplus_{p=1}^n E_{d,p}^1$$

is the homology of our original complex. Unfortunately, this is too simple to be true. Although each term in the the complex  $(\bigoplus_p E_{d,p}^0, \partial^0)$ —known as the *associated graded complex* of our original filtered complex  $(C_d, \partial)$ —is isomorphic to the corresponding term in our original complex, this does not guarantee that the two complexes will be isomorphic as chain complexes. So although  $\bigoplus_p E_{d,p}^1$  does indeed give the homology of the associated graded complex, it may not give the homology of the original complex.



## Analyzing the Discrepancy

This is a little disappointing, but let's not give up just yet. The associated graded complex is so closely related to the original complex that even if its homology isn't exactly what we want, it ought to be a reasonably good approximation. Let's carefully examine the discrepancy to see if we can fix the problem.

Moreover, to keep things as simple as possible, let us begin by considering the case  $n = 2$ . Then the array in diagram (4) has only two levels, which we shall call the "upstairs" ( $p = 2$ ) and "downstairs" ( $p = 1$ ) levels.

The homology group  $H_d$  that we really want is  $Z_d/B_d$ , where  $Z_d$  is the space of cycles in  $C_d$  and  $B_d$  is the space of boundaries in  $C_d$ . Since  $C_d$  is filtered, there is also a natural filtration on  $Z_d$  and  $B_d$ :

$$0 = Z_{d,0} \subseteq Z_{d,1} \subseteq Z_{d,2} = Z_d$$

and

$$0 = B_{d,0} \subseteq B_{d,1} \subseteq B_{d,2} = B_d.$$

Recall that we have been trying to find a natural way of decomposing  $H_d$  into a direct sum. Just as we observed before that  $C_d$  is not a direct sum of  $C_{d,1}$  and  $C_{d,2}$ , we observe now that  $Z_d/B_d$  is not a direct sum of  $Z_{d,1}/B_{d,1}$  and  $Z_{d,2}/B_{d,2}$ ; indeed,  $Z_{d,2}/B_{d,2}$  by itself is already the entire homology group. But again we can use the same trick of modding out by the "downstairs part" and then direct summing with the "downstairs part" itself:

$$\frac{Z_d}{B_d} \simeq \frac{Z_d + C_{d,1}}{B_d + C_{d,1}} \oplus \frac{Z_d \cap C_{d,1}}{B_d \cap C_{d,1}} = \frac{Z_{d,2} + C_{d,1}}{B_{d,2} + C_{d,1}} \oplus \frac{Z_{d,1}}{B_{d,1}}.$$

Now, the naïve hope would be that

$$(7) \quad E_{d,2}^1 \stackrel{?}{\simeq} \frac{Z_{d,2} + C_{d,1}}{B_{d,2} + C_{d,1}}$$

and

$$(8) \quad E_{d,1}^1 \stackrel{?}{\simeq} \frac{Z_{d,1}}{B_{d,1}},$$

and even that the numerators and denominators in equations (7) and (8) are precisely the "cycles" and "boundaries" in the definition (equation (5) above) of  $E_{d,p}^1$ . For then expression (6) would give us a direct sum decomposition of  $H_d$ . Unfortunately, in general, neither (7) nor (8) holds. Corrections are needed.

Let us first look "downstairs" at  $E_{d,1}^1$ . The "cycles" of  $E_{d,1}^1$  are the cycles in  $E_{d,1}^0$ , and the "boundaries" are the image  $I$  of the map

$$\partial^0 : E_{d+1,1}^0 \rightarrow E_{d,1}^0.$$

The space of cycles in  $E_{d,1}^0$  is  $Z_{d,1}$ , which is the numerator in equation (8). However, the image  $I$  is not  $B_{d,1}$ , for  $B_{d,1}$  is the part of  $B_d$  that lies in  $C_{d,1}$ ,

and while this contains  $I$ , it may also contain other things. Specifically, the map  $\partial$  may carry some elements  $x \in C_{d+1}$  down from "upstairs" to "downstairs," whereas  $I$  only captures boundaries of elements that were already downstairs to begin with. Therefore,  $Z_{d,1}/B_{d,1}$  is a quotient of  $E_{d,1}^1$ .

Now let us look "upstairs" at  $E_{d,2}^1$ . In this case, the space of "boundaries" of  $E_{d,2}^1$  is  $B_{d,2} + C_{d,1}$ , which is the denominator in equation (7). However, the space of "cycles" in this case is the kernel  $K$  of the map

$$\partial^0 : E_{d,2}^0 \rightarrow E_{d-1,2}^0,$$

which, by definition of  $E^0$ , is the map

$$\partial^0 : \frac{C_{d,2}}{C_{d,1}} \rightarrow \frac{C_{d-1,2}}{C_{d-1,1}}.$$

Thus we see that  $K$  contains not only chains that  $\partial$  sends to zero but also any chains that  $\partial$  sends "downstairs" to  $C_{d-1,1}$ . In contrast, the elements of  $Z_{d,2} + C_{d,1}$  are more special: their boundaries are boundaries of chains that come from  $C_{d,1}$ . Hence

$$\frac{Z_{d,2} + C_{d,1}}{B_{d,2} + C_{d,1}}$$

is a subspace of  $E_{d,2}^1$ , the subspace of elements whose boundaries are boundaries of  $C_{d,1}$ -chains.

Intuitively, the problem is that the associated graded complex only "sees" activity that is confined to a single horizontal level; everything above and below that level is chopped off. But in the original complex, the boundary map  $\partial$  may carry things down one or more levels (it cannot carry things up one or more levels because  $\partial$  respects the filtration), and one must therefore correct for this inter-level activity.

## The Emergence of Spectral Sequences

The beautiful fact that makes the machinery of spectral sequences work is that both of the above corrections to the homology groups  $E_{d,p}^1$  can be regarded as "homology groups of homology groups"!

Notice that  $\partial$  induces a natural map—let us call it  $\partial^1$ —from  $E_{d+1,2}^1$  to  $E_{d,1}^1$ , for all  $d$ , for the boundary of any element in  $E_{d+1,2}^1$  is a cycle that lies in  $C_{d,1}$ , and thus it defines an element of  $E_{d,1}^1$ . The key claims (for  $n = 2$ ) are the following.

- *Claim 1.* If we take  $E_{d,1}^1$  and mod out by the image of  $\partial^1$ , then we obtain  $Z_{d,1}/B_{d,1}$ . To see this, just check that the image of  $\partial^1$  gives all the boundaries that lie in  $C_{d,1}$ .
- *Claim 2.* The kernel of  $\partial^1$  is a subspace of  $E_{d,2}^1$  isomorphic to

$$\frac{Z_{d+1,2} + C_{d+1,1}}{B_{d+1,2} + C_{d+1,1}}.$$

Again, simply check that the kernel consists just of those elements whose boundary equals a boundary of some element of  $C_{d+1,1}$ .

We can visualize these claims by drawing the following diagram.

$$\begin{array}{cccccc}
 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \\
 \cdots & & E_{d+1,2}^1 & & E_{d,2}^1 & & E_{d-1,2}^1 & & \cdots & & \\
 (9) & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \\
 \cdots & & E_{d+1,1}^1 & & E_{d,1}^1 & & E_{d-1,1}^1 & & \cdots & & \\
 & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \searrow^{\partial^1} & & \\
 0 & & 0 & & 0 & & 0 & & 0 & & 
 \end{array}$$

Diagram (9) is a collection of chain complexes; it's just that the chain complexes do not run horizontally as in diagram (4), but slant downwards at a 45° angle, and each complex has just two nonzero terms. (*Reminder:* Our indexing convention is different from that of most authors, whose diagrams will therefore look “skewed” relative to diagram (9).) If we now define  $E_{d,p}^2$  to be the homology, i.e.,

$$(10) \quad E_{d,p}^2 \stackrel{\text{def}}{=} H_d(E_{d,p}^1) = \frac{\ker \partial^1 : E_{d,p}^1 \rightarrow E_{d-1,p-1}^1}{\text{im } \partial^1 : E_{d+1,p+1}^1 \rightarrow E_{d,p}^1},$$

then the content of Claim 1 and Claim 2 is that  $E_{d,1}^2 \oplus E_{d,2}^2$  is (finally!) the correct homology of our original filtered complex.

For the case  $n = 2$ , this completes the story. The sequence of terms  $E^0, E^1, E^2$  is the spectral sequence of our filtered complex when  $n = 2$ . We may regard  $E^1$  as giving a first-order approximation of the desired homology, and  $E^2$  as giving a second-order approximation—which, when  $n = 2$ , is not just an approximation but the true answer.

What if  $n > 2$ ? The definitions (2), (5), and (10) still make sense, but now  $E^2$  will not in general give the true homology, because  $E^2$  only takes into account interactions between adjacent levels in diagram (4), but  $\partial$  can potentially carry things down two or more levels. Therefore we need to consider further terms  $E^3, E^4, \dots, E^n$ . For example, to define  $E^3$ , we can check that  $\partial$  induces a natural map—call it  $\partial^2$ —from  $E_{d+1,p+2}^2$  to  $E_{d,p}^2$  for all  $d$  and  $p$ . One obtains a diagram similar to diagram (9), except with  $(E^2, \partial^2)$  instead of  $(E^1, \partial^1)$ , and with each arrow going down two levels instead of one. Then  $E_{d,p}^3$  is  $(\ker \partial^2)/(\text{im } \partial^2)$  at  $E_{d,p}^2$ . In general, the picture for  $E^r$  has arrows labeled  $\partial^r$  dropping down  $r$  levels

from  $E_{d+1,p+r}^r$  to  $E_{d,p}^r$ , and  $E^{r+1}$  is defined to be the homology of  $(E^r, \partial^r)$ .

The verification that, for general  $n$ ,  $H_d \simeq \bigoplus_p E_{d,p}^n$  is a conceptually straightforward generalization of the ideas we have already seen, but it is tedious so we omit the details.

### What Good Is All This?

In analysis, the value of having a series approximation converging to a quantity of interest is familiar to every mathematician. Such an approximation is particularly valuable when just the first couple of terms already capture most of the information.

Similar remarks apply to spectral sequences. One common phenomenon is for a large number of the  $E_{d,p}^r$  and/or the boundary maps  $\partial^r$  to become zero for small values of  $r$ . This causes the spectral sequence to stabilize or collapse rapidly, allowing the homology to be computed relatively easily. We illustrate this by sketching the proof of Theorem 2 in a paper of Phil Hanlon [2]. This is far from a “mainstream” application of spectral sequences, but it has the great advantage of requiring very little background knowledge to follow. Readers who know enough topology may wish instead to proceed directly to the standard examples that may be found in any number of textbooks.

Let  $Q$  be a finite partially ordered set that is ranked—i.e., every maximal totally ordered subset has the same number of elements, so that every element can be assigned a rank (namely, a natural number indicating its position in any maximal totally ordered subset containing it)—and that is equipped with an order-reversing involution  $x \mapsto x^*$ . Let

$$(11) \quad \gamma = \{\alpha_1, \alpha_2, \dots, \alpha_t\} \quad \text{where} \\ \alpha_1 < \alpha_2 < \dots < \alpha_t$$

be a totally ordered subset of  $Q$ . We say that  $\gamma$  is isotropic if  $\alpha_i \neq \alpha_j^*$  for all  $i$  and  $j$ .

Now adjoin a minimum element  $\hat{0}$  and a maximum element  $\hat{1}$  to  $Q$ , and consider the family of all totally ordered subsets of the resulting partially ordered set. These form an abstract simplicial complex  $\Delta$ , and we can consider its simplicial homology groups  $H_d$ . We can also restrict attention to the isotropic totally ordered subsets; these form a subcomplex  $\Delta^0$ , which has its own homology groups  $H_d^0$ .

Hanlon’s Theorem 2 says that if  $Q$  is Cohen-Macaulay and its maximal totally ordered subsets have  $m$  elements, then  $H_d^0 = 0$  if  $0 \leq d < m/2$ . The definition of Cohen-Macaulay need not concern us here; it suffices to know that Cohen-Macaulay partially ordered sets satisfy a certain homological property (given in Hanlon’s paper). In particular,

knowing that  $Q$  is Cohen-Macaulay gives us information about  $H_d$ .

In order to deduce something about  $H_d^0$  from the information we have about  $H_d$ , we seek a relationship between  $H_d$  and  $H_d^0$ . Given  $\gamma$  as in equation (11), Hanlon's key idea is to let  $\rho(\gamma)$  be the rank of  $\alpha_i$ , where  $i$  is maximal subject to the condition that  $\alpha_i^* = \alpha_j$  for some  $j > i$ . Then  $\rho(\gamma) = 0$  if and only if  $\gamma$  is isotropic, but more importantly, applying the boundary map can clearly only decrease  $\rho$ , so  $\rho$  induces a filtration on  $\Delta$ . Specifically, we obtain the  $p$ th level of the filtration by restricting to those  $\gamma$  such that  $\rho(\gamma) \leq p$ . Therefore we obtain a spectral sequence! This gives a relationship between  $H_d^0 = E_{d,0}^1$  and the limit  $H_d$  of the spectral sequence.

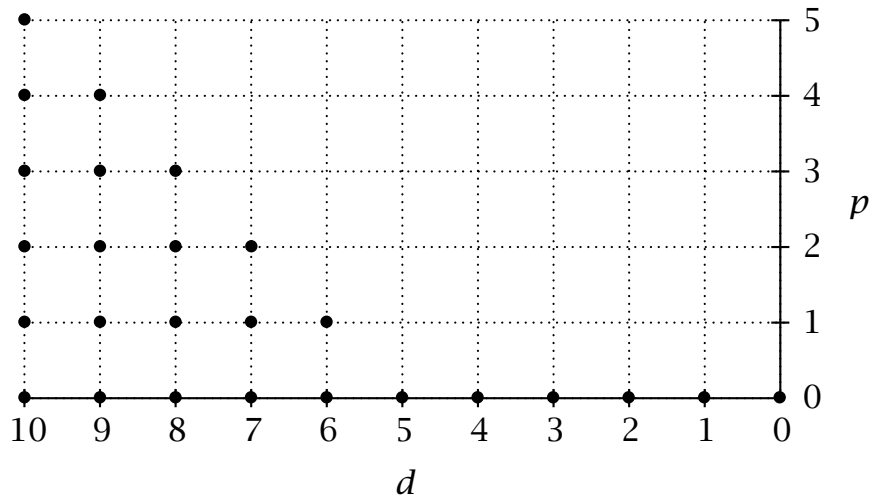
The heart of Hanlon's proof is to analyze  $E^1$ . He shows that  $E_{d,p}^1 = 0$  except possibly for certain pairs  $(d, p)$ . For instance, when  $m = 10$ ,  $E_{d,p}^1 = 0$  except possibly for the pairs  $(d, p)$  marked by dots in the diagram to the right.

If you imagine the  $45^\circ$  boundary maps, then you can see that some potentially complicated things may be happening for  $d \geq 5 = m/2$ , but for  $m/2 > d$ ,  $E_{d,p}^2$  will be isomorphic to  $E_{d,p}^1$  for all  $p$ . In fact,  $E_{d,p}^r \simeq E_{d,p}^1$  for all  $r \geq 1$  when  $m/2 > d$ ; the boundary maps slant more and more as  $r$  increases, but this makes no difference. Therefore just by computing  $E^1$ , we have computed the full homology group for certain values of  $d$ . In particular,  $H_d^0 \simeq H_d$  for  $m/2 > d$ . It turns out that the Cohen-Macaulay condition easily implies that  $H_d = 0$  for  $m/2 > d$ , so this completes the proof.

### A Glimpse Beyond

When our simplifying assumptions are dropped, a lot of complications can arise. Over an arbitrary commutative ring, equation (3) need not hold; not every short exact sequence splits, so there may be extension issues. When our finiteness conditions are relaxed, one may need to consider  $E^r$  for arbitrarily large  $r$ , and the spectral sequence may not converge. Even if it does converge, it may not converge to the desired homology. So in many applications, life is not as easy as it may have seemed from the above discussion; nevertheless, our simplified setting can still be thought of as the "ideal" situation, of which more realistic situations are perturbations.

We should also mention that spectral sequences turn out to be such natural gadgets that they arise not only from filtered complexes, but also from double complexes, exact couples, etc. We cannot even begin to explore all these ramifications here, but hope that our tutorial will help you tackle the textbook treatments with more confidence.



### Why the Adjective "Spectral"?

A question that often comes up is where the term "spectral" comes from. The adjective is due to Leray, but he apparently never published an explanation of why he chose the word. John McCleary (personal communication) and others have speculated that since Leray was an analyst, he may have viewed the data in each term of a spectral sequence as playing a role that the eigenvalues, revealed one at a time, have for an operator. If any reader has better information, I would be glad to hear it.

### Acknowledgments

This paper started life as some lecture notes accompanying a talk I gave while I was a general member of the Mathematical Sciences Research Institute in the spring of 1997, supported in part by a National Science Foundation postdoctoral fellowship. Many thanks go to Art Duval, who went through the notes with a fine-toothed comb and provided detailed suggestions for improving the exposition, as well as repeatedly encouraging me to publish a proper writeup. I thank John McCleary for pointing out references [3] and [5] to me; the latter is another exposition of spectral sequences that is targeted at the beginner. Finally, I thank Phil Hanlon for helping me understand spectral sequences in the first place. He deserves at least as much credit as I do for any merits of the approach expounded in this paper.

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- [4] ———, *A User's Guide to Spectral Sequences*, 2nd ed., Cambridge University Press, 2001.
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# Stable homotopy groups of spheres

•) Def.:  $\pi_k^{st.}(\mathbb{S}) = \pi_k(\mathbb{S}^0)$

$$= \operatorname{colim}_{n \rightarrow \infty} [S^{k+n}, S^n]_*$$

homotopy classes of maps  
 $S^{k+n} \rightarrow S^n$   
basepoint-preserving

$$= \operatorname{colim} \left( \dots \rightarrow [S^{k+n}, S^n]_* \xrightarrow{\Sigma} [S^{k+n+1}, S^{n+1}]_* \xrightarrow{\Sigma} [S^{k+n+2}, S^{n+2}]_* \rightarrow \dots \right)$$

•) Hurewicz: First non-zero homology group (with  $\mathbb{Z}$ -coeff.) agrees with the homotopy group

•) Freudenthal: The groups  $\pi_k S^j = [S^k, S^j]_*$

stabilize after enough suspensions



Hopf maps:

$$\cdot 2: S^1 \longrightarrow S^1 \xrightarrow{\quad} \mathbb{R}P^2$$

$= \mathbb{R}P^1$

$$\eta: S^3 \longrightarrow S^2 \xrightarrow{\quad} \mathbb{C}P^2$$

$= \mathbb{C}P^1$

$$\nu: S^7 \longrightarrow S^4 \xrightarrow{\quad} \mathbb{H}P^2$$

$= \mathbb{H}P^1$

$$\sigma: S^{15} \longrightarrow S^8 \xrightarrow{\quad} \mathbb{O}P^2$$

$= \mathbb{O}P^1$

Cones:

Stably essential! Detected by:

$$Sq^1 \quad (\text{degree})$$

$$Sq^2$$

$$Sq^4$$

$$Sq^8$$

## Serre's Qualitative results

•) stable homotopy groups of spheres are finite abelian  
(except for  $\pi_0^{\text{st.}} \cong \mathbb{Z}$ )

•) we know where the first  $p$ -torsion appears

→ Enough to calculate each prime at a time

For our talks, we will localize at  $p=2$

Serre's approach: Kill lowest homotopy class (Postnikov approximation)

and use Hurewicz

By taking the fiber of a map to a  $K(\mathbb{Z}, n)$

(build the Eilenberg-MacLane space by attaching cells to the original space killing homotopy in deg.  $\geq n+1$ )

Then use Serre spectral sequence to deduce (co)homology of the fibre — and repeat

Side characters: Universal coefficient thm. to deduce homology from cohomology

Try this for the sphere  $S^n$ :

Localize at  $p=2$   
 (Co)homology is usually with  $\mathbb{Z}_2$  coefficients

We will need the cohomology of EM-spaces

$$= Sq^{i_1} \cdots Sq^{i_k}$$

$$H^* \left( \underbrace{K(\mathbb{Z}_2, n)}_{\Sigma^n H\mathbb{F}_2} ; \mathbb{Z}_2 \right) \cong \mathbb{F}_2 \left[ \underbrace{Sq^{(i_1, \dots, i_k)}}_{(i_1, \dots, i_k) \text{ admissible}} (L_n) \mid \text{excess}(i_1, \dots, i_k) < n \right]$$

$L_n \in H^n(\Sigma^n H\mathbb{F}_2; \mathbb{Z}_2)$  generator

$$H^* \left( \underbrace{K(\mathbb{Z}, n)}_{\Sigma^n H\mathbb{Z}} ; \mathbb{Z}_2 \right) \cong \mathbb{F}_2 \left[ \underbrace{Sq^{(i_1, \dots, i_k)}}_{\substack{(i_1, \dots, i_k) \text{ admissible} \\ i_k > 1}} (L_n) \mid \text{excess}(i_1, \dots, i_k) < n \right]$$

"Almost" free modules over  $\mathcal{A}$ , if we take the "limit  $n \rightarrow \infty$ "

Know first nonzero homotopy group is  $\pi_n S^n \cong \mathbb{Z} \cong H^n(S^n; \mathbb{Z})$   
 (Cellular approximation + Hurewicz)

$$\begin{array}{c}
 \text{fiber} \\
 \downarrow \\
 \tau_{\geq n+1} S^n \longrightarrow S^n \xrightarrow{\text{iso. on } \pi_n} K(\mathbb{Z}, n)
 \end{array}$$

l.e.s. of homotopy groups in a fibration:

$$\pi_i(\tau_{\geq n+1} S^n) = \{0\} \quad \text{for } i \leq n$$

$$\pi_i(\tau_{\geq n+1} S^n) \cong \pi_i(S^n) \quad \text{for } i \geq n+1$$

i.e. if we compute the

$(n+1)^{\text{st}}$  homotopy group of  $\tau_{\geq n+1} S^n$ ,

we get  $\pi_{n+1}(S^n)$

↑  
 Compute  $(n+1)^{\text{st}}$  homology & Hurewicz

$$\begin{array}{ccccccc} \Omega K(\mathbb{Z}, n) & \longrightarrow & \tau_{\geq n+1} S^n & \longrightarrow & S^n & \xrightarrow{\text{iso. on } \pi_n} & K(\mathbb{Z}, n) \\ \simeq K(\mathbb{Z}, n-1) & & & & & & \end{array}$$

use cohomology Serre spectral sequence  
for this fibration

$$E_2 = H^*(S^n; H^*(K(\mathbb{Z}, n-1))) \Rightarrow H^*(\tau_{\geq n+1} S^n)$$

⏟

$$\cong H^*(K(\mathbb{Z}, n-1)) \otimes H^*(S^n)$$



$$E_2 = H^*(S^n; H^*(K(\mathbb{Z}, n-1))) \Rightarrow H^*(\tau_{\geq n+1} S^n)$$

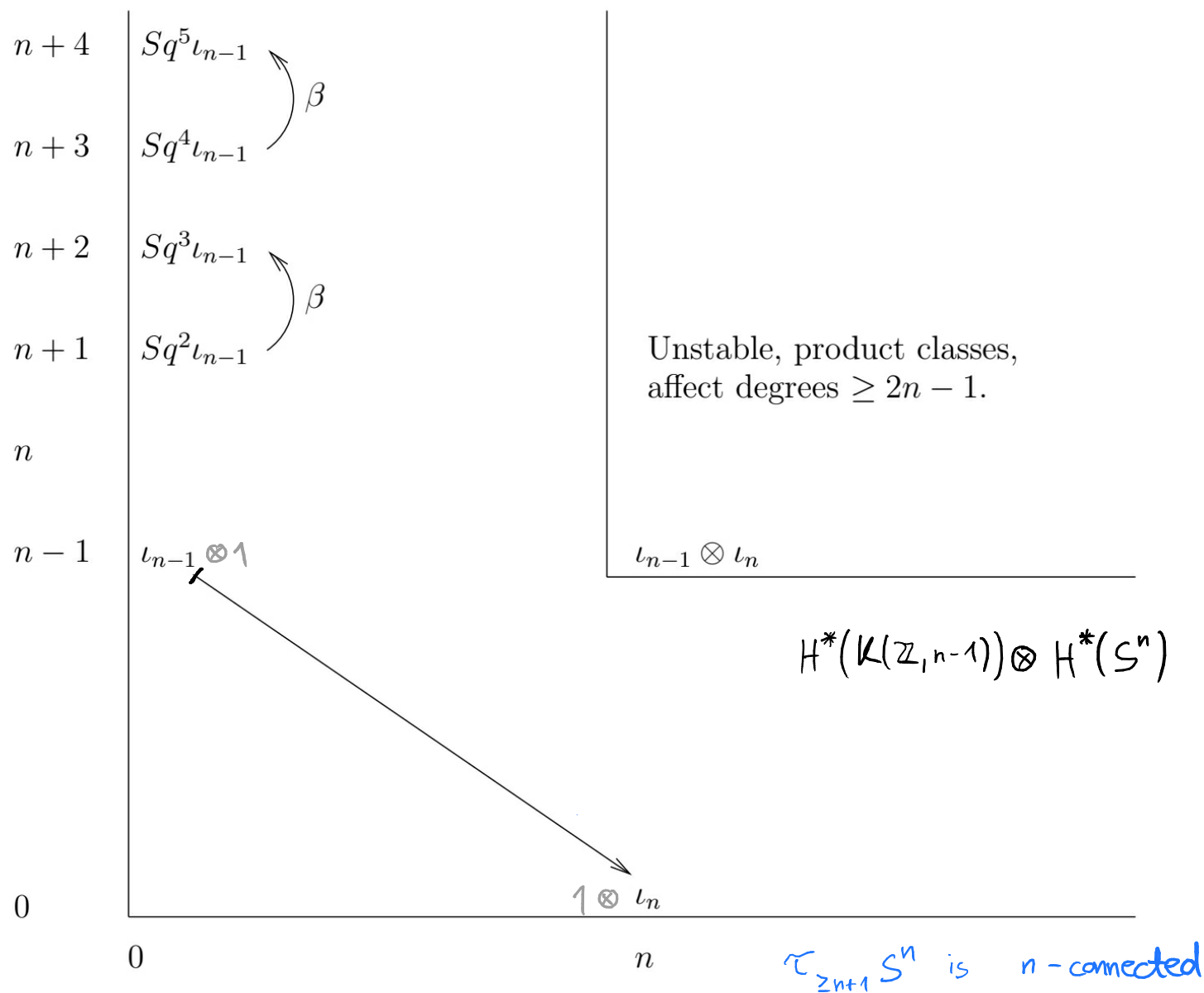


FIGURE 0.1. The cohomology Serre spectral sequence for the fibration  $K(\mathbb{Z}, n-1) \rightarrow X_1 \rightarrow S^n$

$$E_2 = H^*(S^n; H^*(K(\mathbb{Z}, n-1))) \Rightarrow H^*(\tau_{\geq n+1} S^n)$$

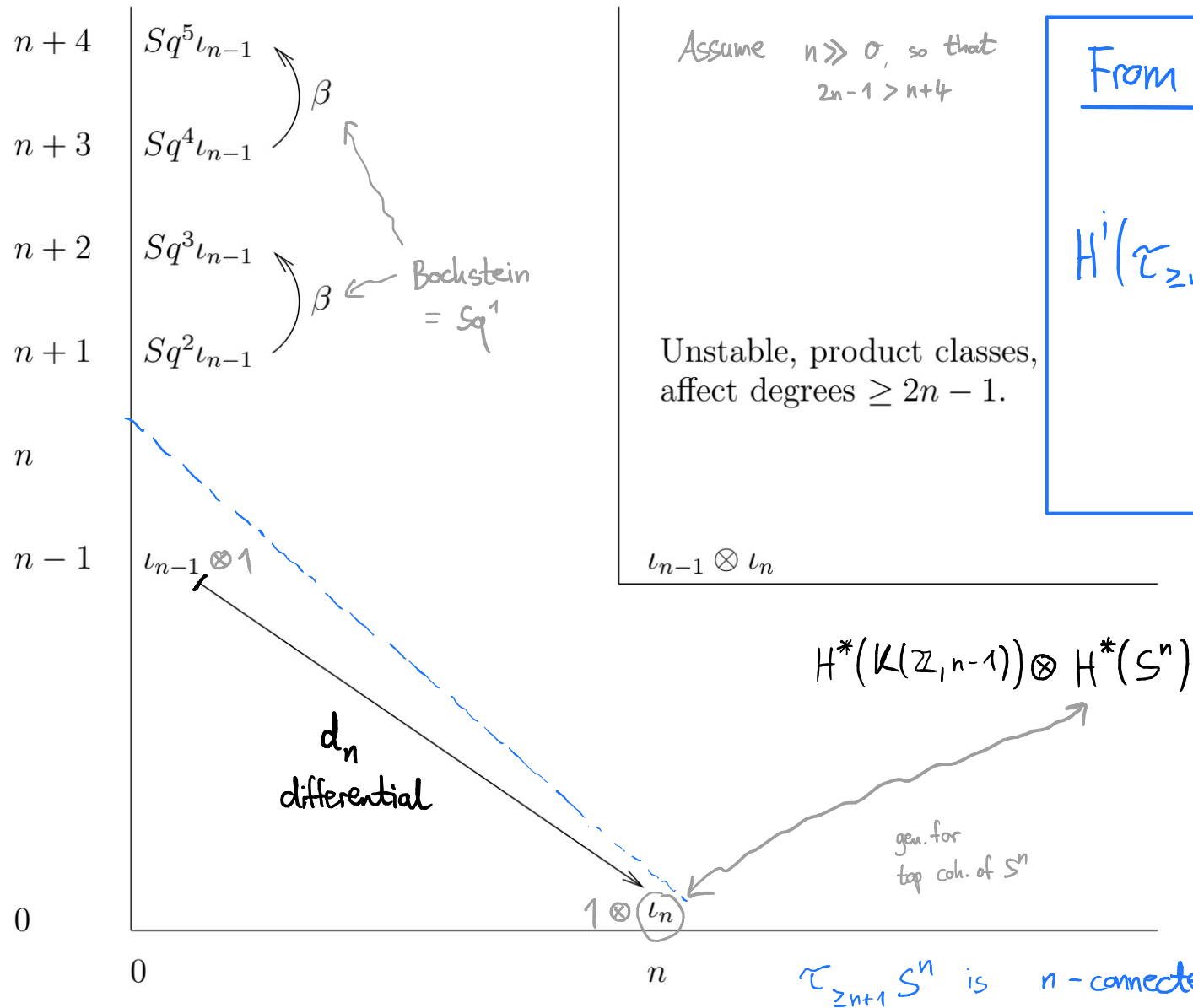


FIGURE 0.1. The cohomology Serre spectral sequence for the fibration  $K(\mathbb{Z}, n-1) \rightarrow X_1 \rightarrow S^n$

$$\Omega K(\mathbb{Z}, n) \longrightarrow \tau_{\geq n+1} S^n \longrightarrow S^n$$

$$E_2 = H^*(S^n; H^*(K(\mathbb{Z}, n-1))) \Rightarrow H^*(\tau_{\geq n+1} S^n)$$

From  $E_\infty$ -page:

$$H^i(\tau_{\geq n+1} S^n) \cong \begin{cases} 0 & i < n+1 \\ \mathbb{Z} & i = n+1 \\ \mathbb{Z} & i = n+2 \\ \mathbb{Z} & i = n+3 \\ \mathbb{Z} & i = n+4 \end{cases}$$

← Ker ( $H^*(K(\mathbb{Z}, n) \rightarrow H^*(S^n))$ )  
shifted down by 1 degree

Use relation to Bockstein:  $\beta = Sq^1$   
 $Sq^1 Sq^2 = Sq^3$

$$H_{n+2}(\tau_{\geq n+1} S^n; \mathbb{Z}/2) \xrightarrow{\cong} H_{n+1}(\tau_{\geq n+1} S^n; \mathbb{Z}/2)$$

$\underbrace{\hspace{10em}}_{\cong \mathbb{Z}/2}$

$$\pi_{n+1} S^n \cong \pi_{n+1}(\tau_{\geq n+1} S^n) \underset{\text{Hurwicz}}{\cong} H_{n+1}(\tau_{\geq n+1} S^n; \mathbb{Z}) \cong \mathbb{Z}/2$$

Keep going:

$$\begin{array}{c} \text{fiber} \\ \downarrow \\ \tau_{\geq n+2} S^n \longrightarrow \tau_{\geq n+1} S^n \xrightarrow{\text{iso. on } \pi_{n+1}} K(\mathbb{Z}/2, n+1) \end{array}$$

$$\begin{array}{c} \Omega K(\mathbb{Z}/2, n+1) \longrightarrow \tau_{\geq n+2} S^n \longrightarrow \tau_{\geq n+1} S^n \xrightarrow{\text{iso. on } \pi_{n+1}} K(\mathbb{Z}/2, n+1) \\ \simeq K(\mathbb{Z}/2, n) \end{array}$$

Use cohomology Serre spectral sequence again,

but each further step requires more work.

Serre's method stops working in degree 15, have ambiguous differentials

# Adams resolution

Assume all spectra are bounded below and of finite type

$$\leadsto \bigvee X_i \longrightarrow \mathbb{T}X_i \text{ is a homotopy equivalence}$$

Want from cofibers in the Adams resolution:

- )  $H\mathbb{F}_p$ -cohomology free module over the Steenrod algebra  $\leadsto$  take  $\bigvee \Sigma^{n_i} H\mathbb{F}_p$
- ) maps into them should be determined by effect on  $H\mathbb{F}_p$ -coh.  $\leadsto$  they are, since  $\mathbb{T} \Sigma^{n_i} H\mathbb{F}_p$

Adams resolution of  $Y$ :

$$Y = Y_0 \xleftarrow{i_0} Y_1 \xleftarrow{i_1} Y_2 \xleftarrow{i_2} \dots$$

$$\begin{array}{cccc} \downarrow & & \downarrow & & \downarrow & & \dots \\ \text{Cofib}(i_0) & & \text{Cofib}(i_1) & & \text{Cofib}(i_2) & & \dots \\ \downarrow & & \downarrow & & \downarrow & & \dots \\ \Sigma & & \Sigma & & \Sigma & & \dots \end{array}$$

s.th.

$$\bigvee \Sigma^{n_i} H\mathbb{F}_p \quad \bigvee \Sigma^{n_i} H\mathbb{F}_p \quad \bigvee \Sigma^{n_i} H\mathbb{F}_p \quad \dots$$

and

$$\begin{array}{ccc} H\mathbb{F}_p^* Y_{i_s} & & \\ \uparrow & \text{epimorphism} & \\ H\mathbb{F}_p^* \text{Cofib}(i_s) & & \end{array}$$

Adams: Put filtration on  $\mathcal{S}$

so that the associated graded is detected by  $H\mathbb{F}_2$

Kill all the homotopy which can be detected in (co)homology

$$K := \prod K(\mathbb{Z}/p_i, n_i) \longleftarrow X$$

will be cohomology of  
the fiber



$$\text{Ker} \rightarrow H^* K \longrightarrow H^* X$$



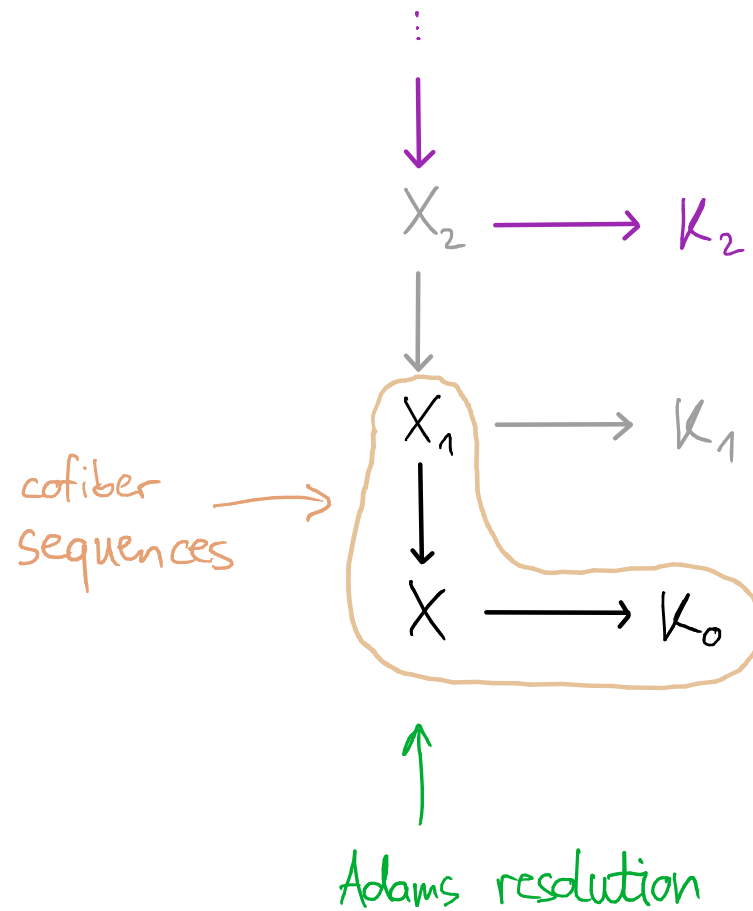
free over the  
Steenrod algebra  $\mathcal{A}$

want: induces surjection on cohomology





Keep going:



From long exact cohomology sequences can form  
a free resolution of  $H^*X$  over the Steenrod algebra  
(in the stable range)

From a tower of fibrations:

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ X_2 & \longrightarrow & K_2 \\ \downarrow & & \\ X_1 & \longrightarrow & K_1 \\ \downarrow & & \\ X & \longrightarrow & K_0 \end{array}$$

degree-wise homotopy groups form an exact couple

→ get a spectral sequence from the tower  
converging to the homotopy groups  $\pi_* X$

filtered by the images of the maps in the Adams tower

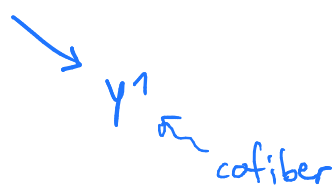
$$F^{s,t+s} = \text{im} \left( \pi_t(X_s) \rightarrow \pi_t(X) \right)$$

"filter by how far elements pull back in the Adams tower"

# Example of beginning of Adams resolution:

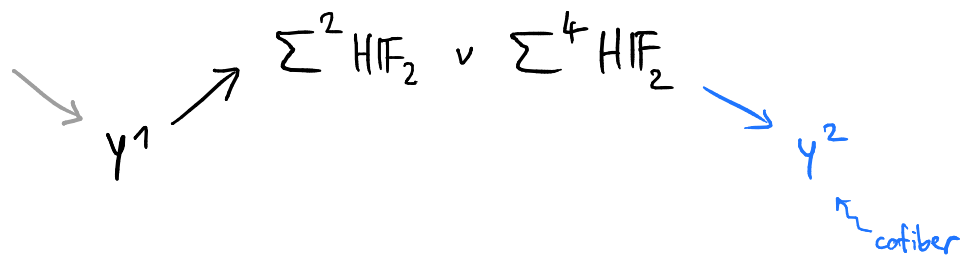
Start with

$$\mathbb{S} \rightarrow \mathbb{H}\mathbb{Z}$$



Warning: Not functorial,  
this example involves arbitrary choices to  
obtain a kind of "minimal" resolution

$$\mathbb{S} \rightarrow \mathbb{H}\mathbb{Z}$$

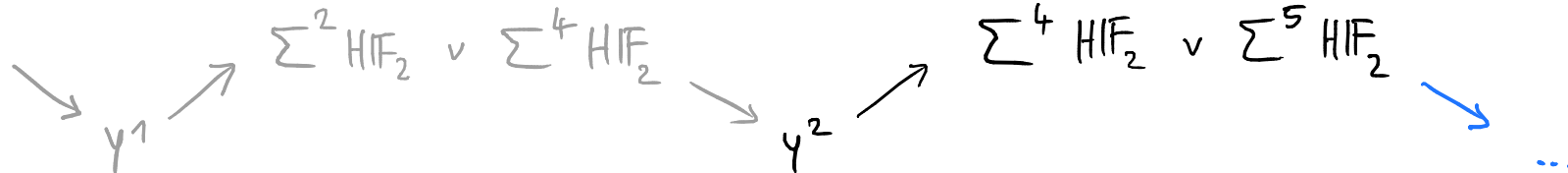


$$H^*(Y^1) \leftarrow H^*(\Sigma^2 \mathbb{H}\mathbb{F}_2 \vee \Sigma^4 \mathbb{H}\mathbb{F}_2)$$

$$Sq^2(L_1) \longleftarrow L_2$$

$$Sq^4(L_1) \longleftarrow L_4$$

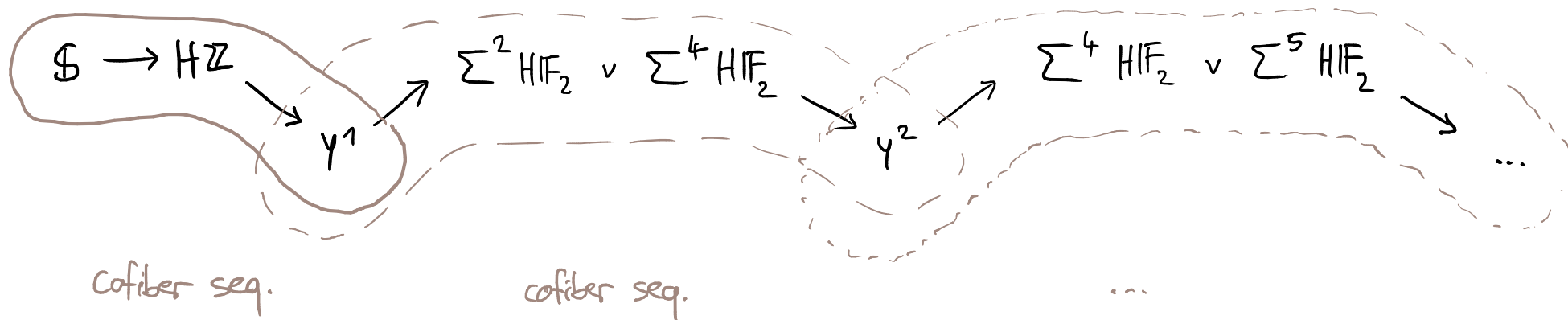
$$\mathbb{S} \rightarrow \mathbb{H}\mathbb{Z}$$



$$H^*(Y^2) \leftarrow H^*(\Sigma^4 \mathbb{H}\mathbb{F}_2 \vee \Sigma^5 \mathbb{H}\mathbb{F}_2)$$

$$Sq^2(L_2) \longleftarrow L_4$$

$$Sq^1(L_4) + Sq^2 Sq^1(L_2) \longleftarrow L_5$$



$$H^*(\$) \leftarrow H^*(H\mathbb{Z}) \leftarrow H^*(\Sigma^2 H\mathbb{F}_2 \vee \Sigma^4 H\mathbb{F}_2) \leftarrow H^*(\Sigma^4 H\mathbb{F}_2 \vee \Sigma^5 H\mathbb{F}_2) \leftarrow$$

exact

$$\begin{array}{ccccccc}
 & K^0 & & & K^1 & & K^2 \\
 & \parallel & & & \parallel & & \parallel \\
 \$ \longrightarrow & \text{Hz} & \longrightarrow & \Sigma^2 \text{HF}_2 \vee \Sigma^4 \text{HF}_2 & \longrightarrow & \Sigma^4 \text{HF}_2 \vee \Sigma^5 \text{HF}_2 & \longrightarrow \dots \\
 & \searrow & & \nearrow & \searrow & & \nearrow \\
 & \gamma^1 & & & \gamma^2 & & \dots
 \end{array}$$

Can rearrange terms (by taking fibers) to put this into tower form:

$$\begin{array}{ccc}
 \vdots & & \\
 \downarrow & & \\
 \Sigma^{-2} \gamma^2 & \longrightarrow & \Sigma^{-2} K^2 \\
 \downarrow & & \\
 \Sigma^{-1} \gamma^1 & \longrightarrow & \Sigma^{-1} K^1 \\
 \downarrow & & \\
 \$ & \longrightarrow & K^0
 \end{array}$$

## Extra topics:

- ) Products
- ) (Co) Bar resolution
- ) Adams - Novikov

Products: Adams spectral sequence is a spectral sequence of rings

Converging to the ring structure of  $[\mathbb{S}, \mathbb{S}]$  from composition product.

$$\text{Ext}_{\mathcal{A}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow [\mathbb{S}, \mathbb{S}]_{(p)}^{\wedge}$$

$$[Y, Z]_n \otimes [X, Y]_m \xrightarrow{\circ} [X, Z]_{m+n}$$

$$f \quad g \quad \mapsto f \circ (\Sigma^n g)$$

compose after stabilizing

from smash product  $[X_1, Y_1]_m \otimes [X_2, Y_2]_n \xrightarrow{\wedge} [X_1 \wedge X_2, Y_1 \wedge Y_2]$

$$\text{Ext}(H^*Y_1, H^*X_1) \otimes \text{Ext}(H^*Y_2, H^*X_2) \rightarrow \text{Ext}(H^*Y_1 \otimes H^*Y_2, H^*X_1 \otimes H^*X_2)$$

21 Künneth

$$\text{Ext}(H^*(Y_1 \wedge Y_2), H^*(X_1 \wedge X_2))$$

Every other Adams spectral sequence

$$\text{Ext}_{\mathcal{A}}(H^*Y, H^*X) \Rightarrow [X, Y]_{(p)}^{\wedge}$$

is a module over the sphere one.

Pairing on  $E_{\infty}$ -page induced by smash pairing  $[\mathbb{S}, \mathbb{S}] \otimes [X, Y] \longrightarrow [X, Y]$



# (Co) Bar complex

$$H := \mathbb{H}\mathbb{F}_2$$

Augmented cosimplicial object  $\mathbb{H}\mathbb{F}_p^\bullet X = H^\bullet X$  for every symmetric spectrum  $X$

$s^i$ : use multiplication  $H \wedge H \rightarrow H$   
on  $i^{\text{th}}$  and  $(i+1)^{\text{st}}$  factor

$$\begin{array}{c}
 \vdots \\
 \leftarrow \\
 \vdots \\
 \leftarrow \\
 \vdots
 \end{array}
 H \wedge H \wedge H \wedge X
 \begin{array}{c}
 \xrightarrow{\quad} \\
 \xleftarrow{\quad} \\
 \xrightarrow{\quad} \\
 \xleftarrow{\quad} \\
 \xrightarrow{\quad}
 \end{array}
 H \wedge H \wedge X
 \begin{array}{c}
 \xleftrightarrow{\quad} \\
 \xleftarrow{\quad} \\
 \xleftrightarrow{\quad} \\
 \xleftarrow{\quad}
 \end{array}
 H \wedge X$$

$d^i$ : insert unit  $\mathbb{S} \rightarrow H$   
into  $i^{\text{th}}$  slot

$\leadsto$  spectral sequence

$$E_1^{s,t} = \pi_{t-s} \left( H^{\wedge(s+2)} \wedge X \right) \Rightarrow \pi_{t-s} \text{Tot } H^\bullet X$$

$\Gamma$  to identify  $E_2$ -page,  
need  $H \wedge H$  flat over  $H$

$\underbrace{\hspace{10em}}$   
totalization of the  
cosimplicial spectrum

# Outlook: Adams-Novikov Spectral Sequence

$$E_*^{s,t} = H^s(\mathcal{J}C_t E^{1+t} \wedge X) \Rightarrow \mathcal{J}C_{t-s}(X \hat{E})$$

This is the  $E$ -based Adams-Novikov Spectral Sequence

Example: Classical Adams Spectral Seq. at  $p=2$

$$E = H\mathbb{F}_2 \quad \text{space } X = \mathbb{S}$$

$$\mathbb{S}_{H\mathbb{F}_2}^\wedge = \mathbb{S}_2^\wedge \quad \text{2-completion of sphere}$$

Dual Steenrod alg.:  $E_*E = \mathcal{A}^* = \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \dots]$   $|\xi_i| = 2^i - 1$

$[\xi_1^{2^i}] = h_i \in \text{Ext}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$

converge to

$$h_i \Rightarrow \text{Hopf inv. 1 elem.}$$

$$h_0 \Rightarrow 2 \in \mathcal{J}C_0^{\text{st.}}$$

$$h_1 \Rightarrow \eta \in \mathcal{J}C_1^{\text{st.}}$$

$$h_2 \Rightarrow \nu \in \mathcal{J}C_3^{\text{st.}}$$

$$h_3 \Rightarrow \sigma \in \mathcal{J}C_4^{\text{st.}}$$

$$[\xi_i^{2^i} | \xi_i^{2^i}] = h_i^2 \in \text{Ext}^2$$

$$h_i^2 \Rightarrow \theta_i \text{ Kervaire class (if it exists)}$$

$h_i^2$  permanent cycle in Adams Sp. Seq.

iff. there exists a framed  $(2^i - 2)$ -mfld. with Kervaire invariant 1

Then stop being permanent cycles

$$d_i(h_i) = h_0 h_{i+1}^2, \quad i \geq 4$$

p odd prime:  $E = H\mathbb{F}_p$

$$E_* E = A^* = \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E[\tau_0, \tau_1, \dots]$$

$$|\xi_i| = 2(p^i - 1)$$

$$|\tau_i| = 2p^i - 1$$

Important elements:

$$[\xi_i^{p^i}] = h_i \in \text{Ext}^i$$

$h_i \Rightarrow$  "odd primary Hopf invariant 1"

$h_i$  odd total degree

$$h_0 \Rightarrow \alpha_i \in \mathbb{Z}_{2p-3}^s$$

$$h_i^2 = 0$$

$$dh_i = v_0 b_{i-1} \quad i \geq 1$$

non-trivial class  $b_i \in \text{Ext}^{2ip^{i+1}}$  on the 2-line : 
$$b_i = \sum_{t=1}^{p-1} \frac{\binom{p}{t}}{p} [\xi_1^{tp^i} | \xi_i^{(p-t) \cdot p^i}]$$

[last slide]