2020-11-09, Exotic 4-mflds. study group (2h talk)

Some of Freedman's results on topological 4-manifolds

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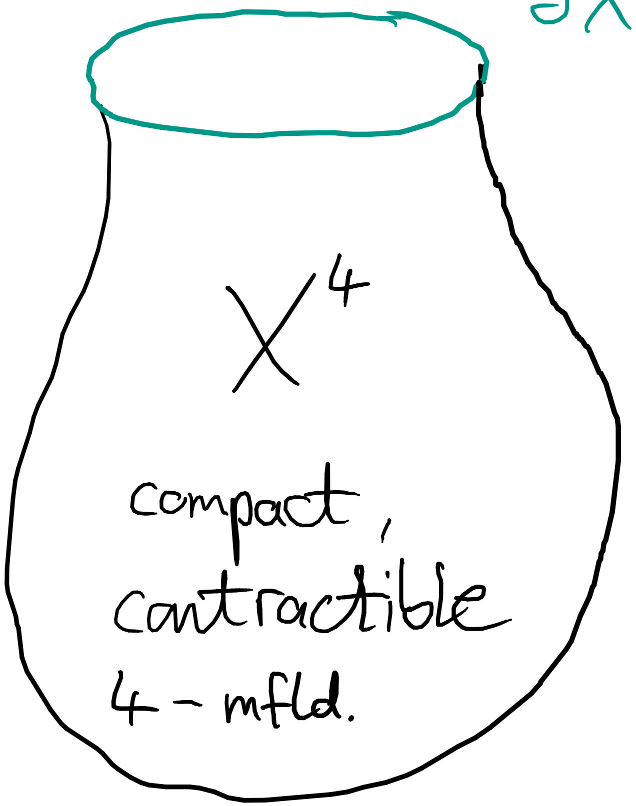
Plan: We will discuss two results which under the assumptions give topological conclusions

(1)



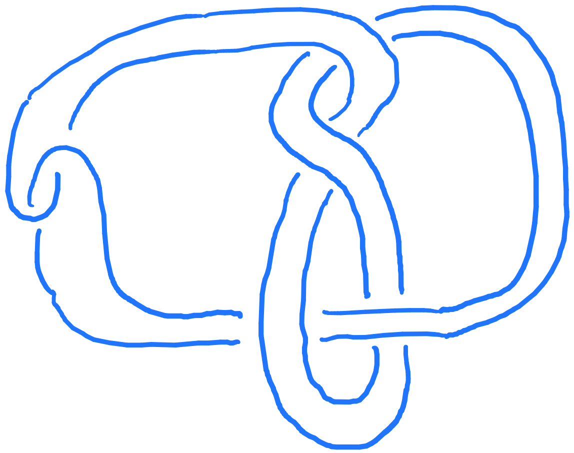
$\partial X \circlearrowleft^{\cong}$ diffeo. f of ∂X

$\circlearrowleft$ **Thm:** $\exists$ homeo. F extending f

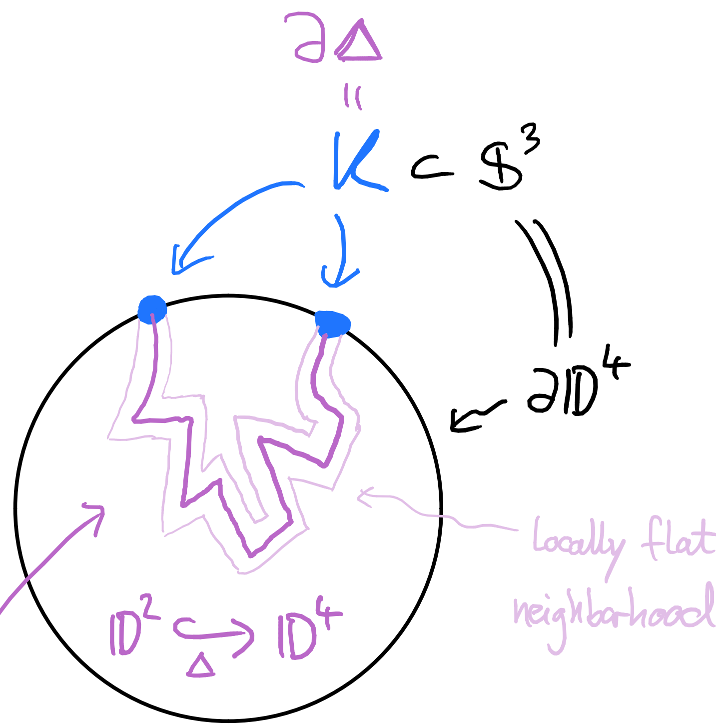
②

Knot $K \subset S^3$ with $\Delta_K(t) \doteq 1$ is topologically slice in the 4-ball

<u>Ex.:</u>



Untwisted Whitehead doubles are TOP slice



$\exists$ topologically locally flat slice disk Δ

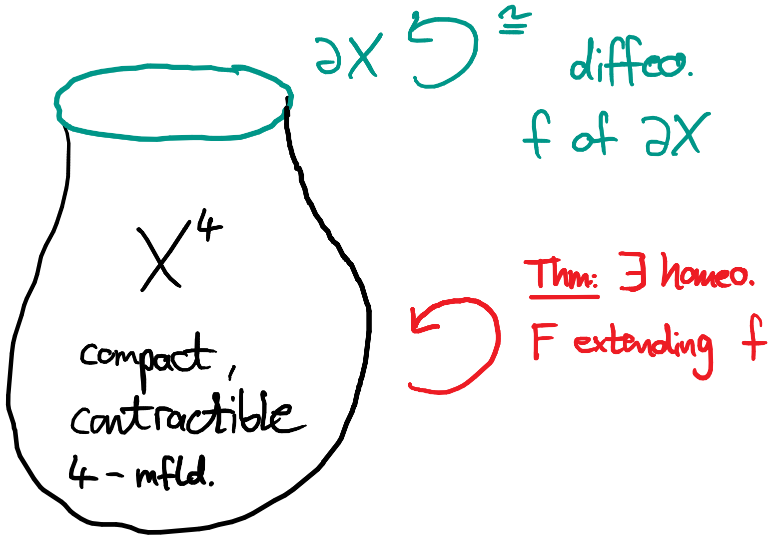
① Extending diffeomorphisms of the boundary to homeomorphisms

Thm.: $f: \partial X \xrightarrow{\cong} \partial X$ diffeo

X^4 compact, contractible 4-mfld.

can be extended to a homeomorphism $X \underset{F}{\xrightarrow{\cong}} X$ of X

(that is, $F\restriction_{\partial X} = f$)



Application / Example: $\boxed{\text{Mazur cork}}$

↗ Outlook on constructing exotic mflds. via cork twists

Theorem 9.3. *([A11]) Let W be the contractible manifold given in Figure 9.1, and let $f : \partial W \to \partial W$ be the involution induced from the obvious involution of the symmetric link in S^3, with $f(\gamma) = \gamma'$, where γ, γ' are the circles in ∂W, as shown in the figure. Then $f : \partial W \to \partial W$ does not extend to a diffeomorphism $F : W \to W$ (but it does extend to a homeomorphism).*

Proof. ([AM1]) From Theorem 8.11, we see that W is Stein (for this we use the description of W given in the second picture of Figure 9.1). Since γ' is slice and $f(\gamma') = \gamma$, if f extended to a diffeomorphism $F : W \to W$ then γ would also be slice in W. But this violates the inequality of Theorem 9.1 (here $F = D^2$, $n = 0$ and $TB(\gamma) = 0$). The fact that f extends to a homeomorphism of W follows from the Freedman theorem [F]. □

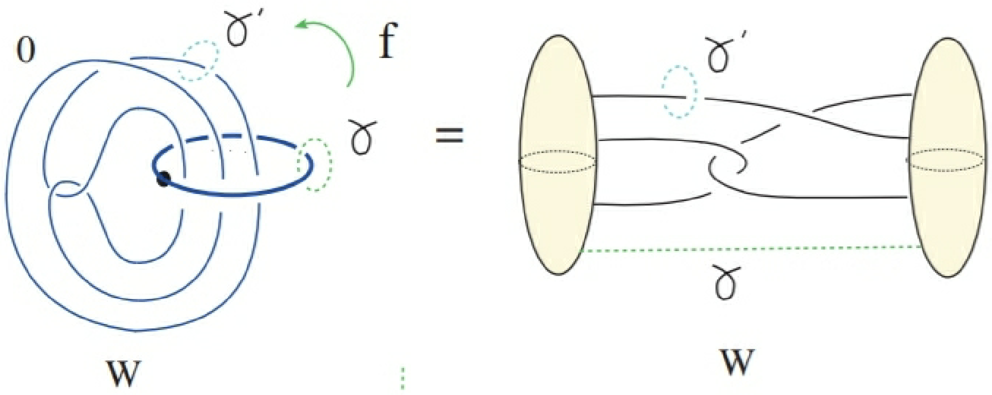
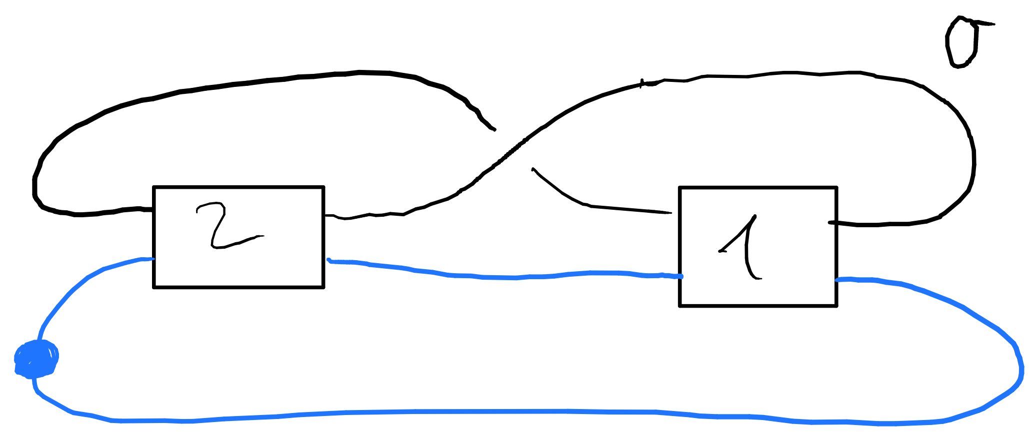


Figure 9.1

Symmetrical picture of Mazur cork:



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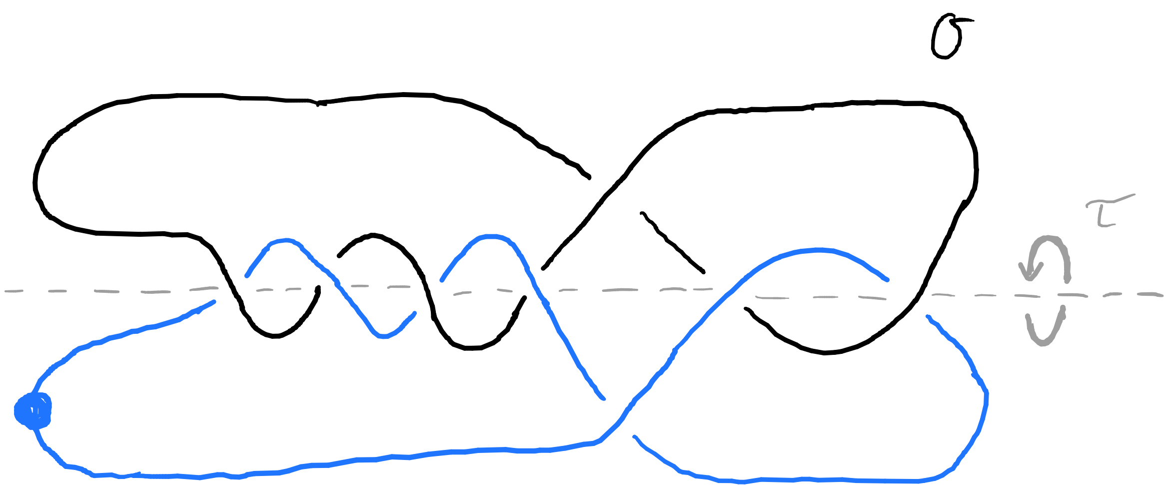
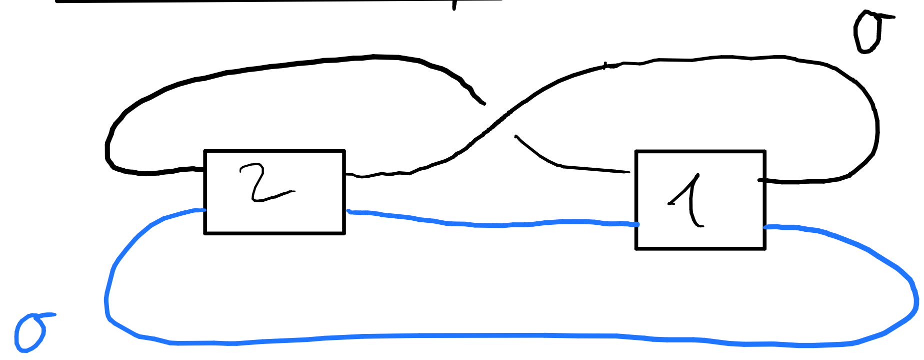
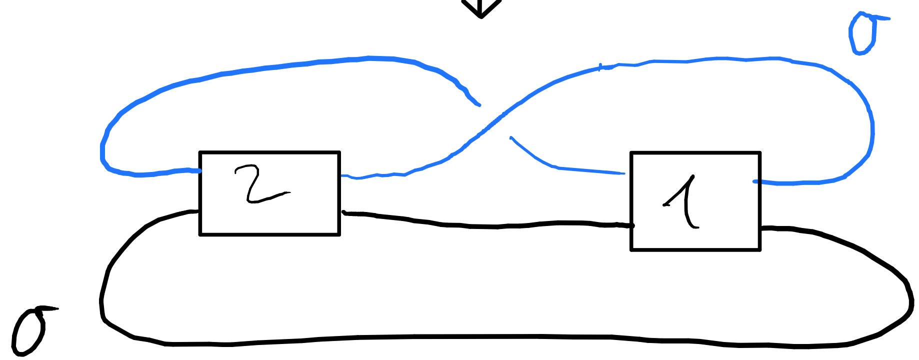
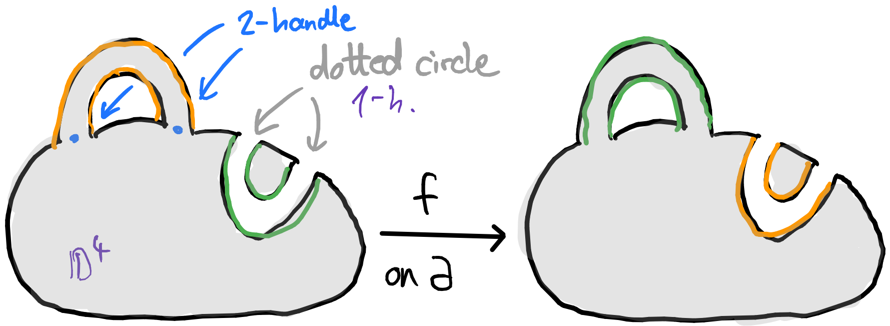


On boundary:



$\big\downarrow f$ interchange components

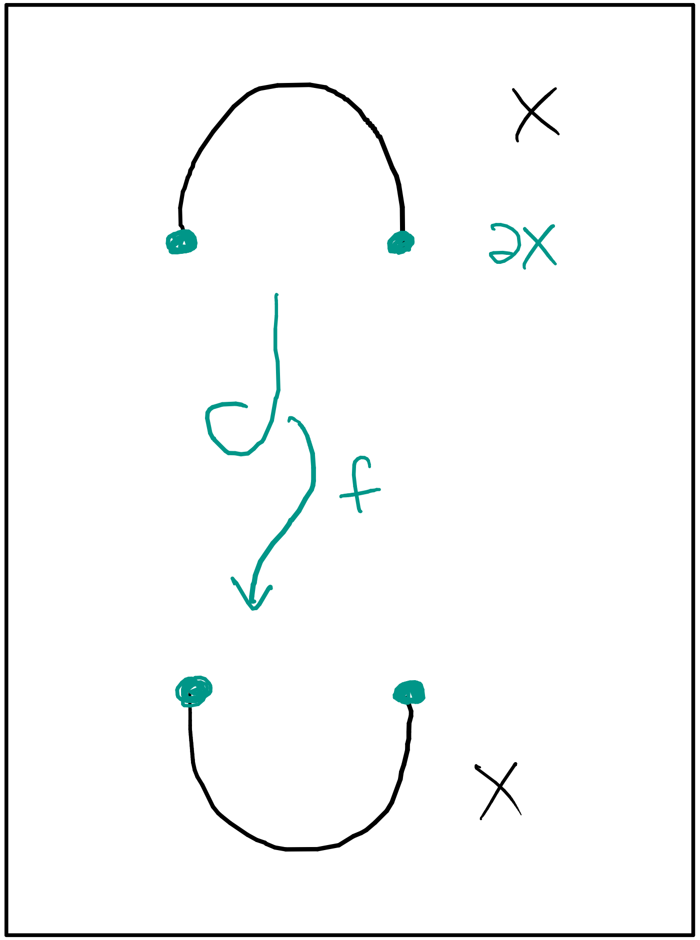


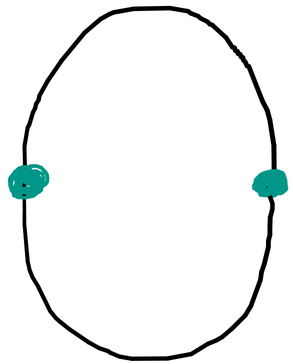


Proof idea

following [Gompf: Infinite order corks via handle diagrams, Remark on page 2]

Glue two copies of X via f: $\boxed{X \cup_{(\partial X \xrightarrow{f} \partial X)} X}$

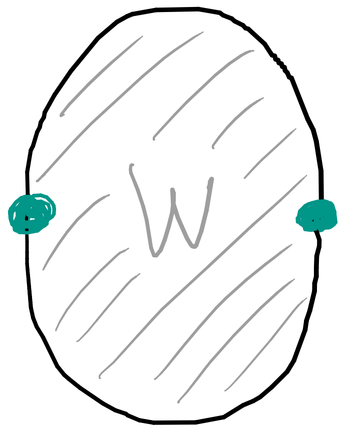




← homotopy 4-sphere!

$\Downarrow$ topological 4-dim. Poincaré conjecture

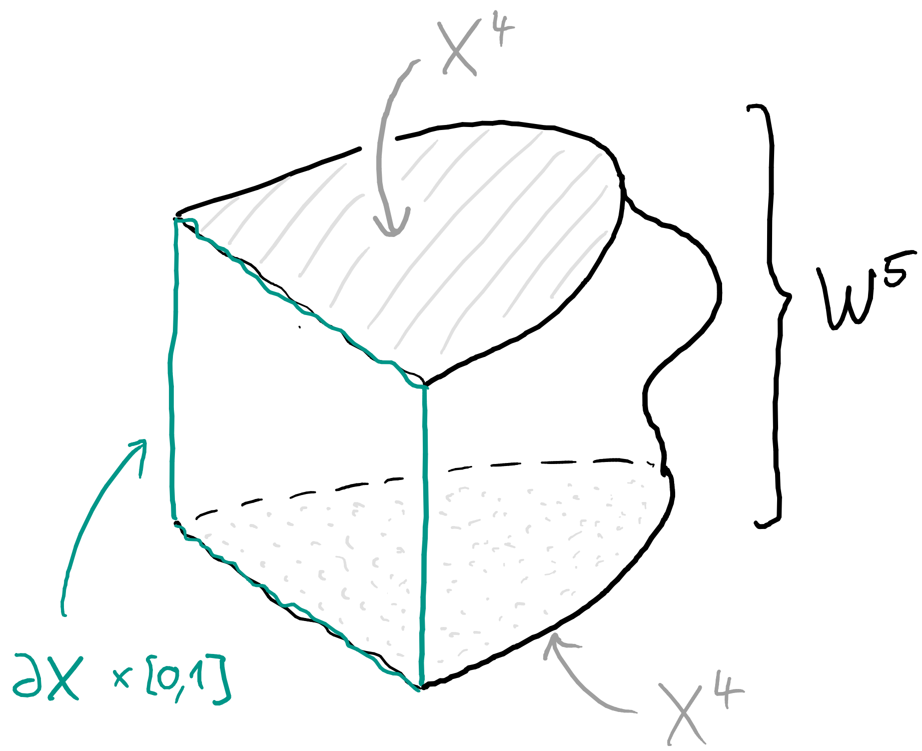
(or use h-cob. to S^4)



$X \cup_f X$ bounds contractible W^5

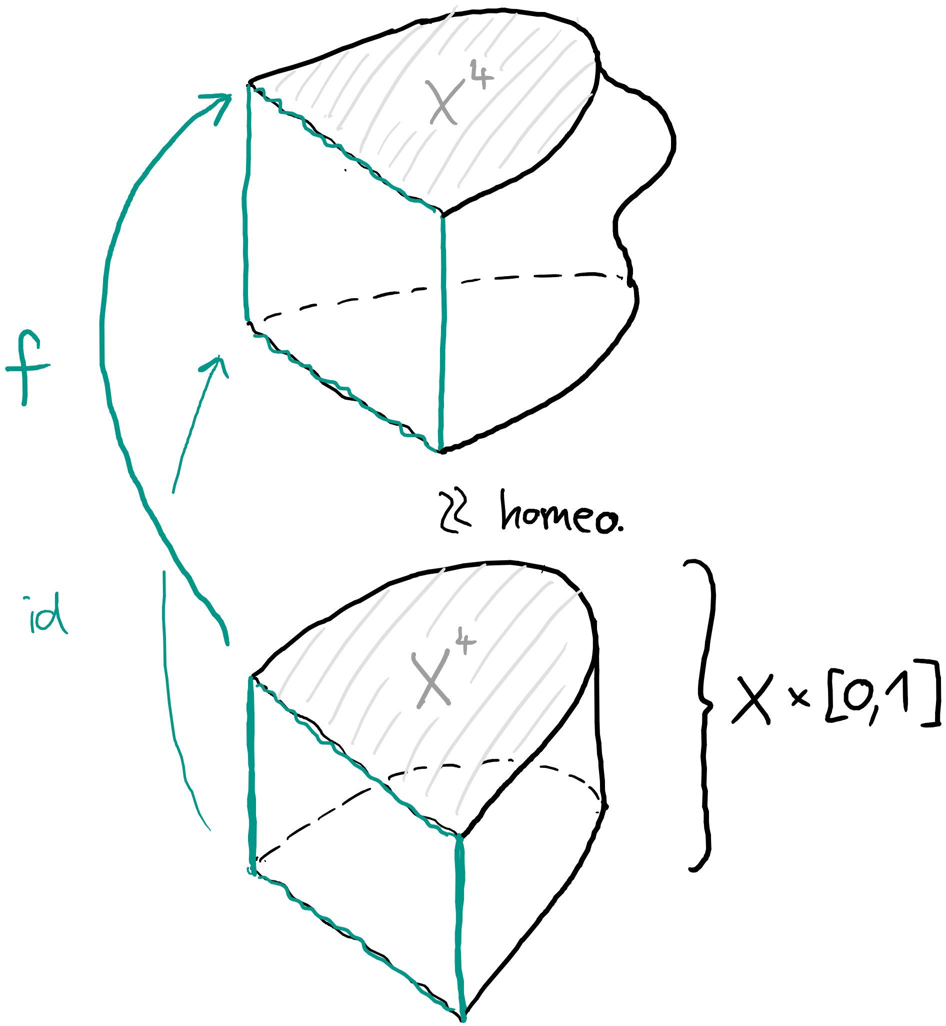
(can take $W^5 = \mathbb{D}^5$)

Can view W as a relative h-cobordism:



$\Rightarrow$

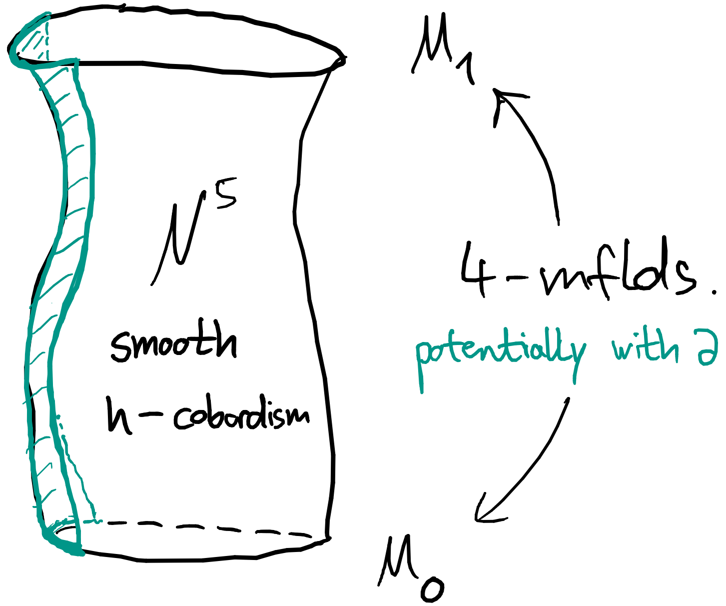
W^5 homeo. to product



"Project" this to the bottom X^4 to obtain the extension F of f

$\square$

(Relative) topological 4-dim. s-cobordism thm.:



- vanishing Whitehead torsion $\tau(N, M_0)$
- $\pi_1(N)$ good group

$\Rightarrow$ N homeomorphic to product $M_0 \times [0,1]$
relative to boundary

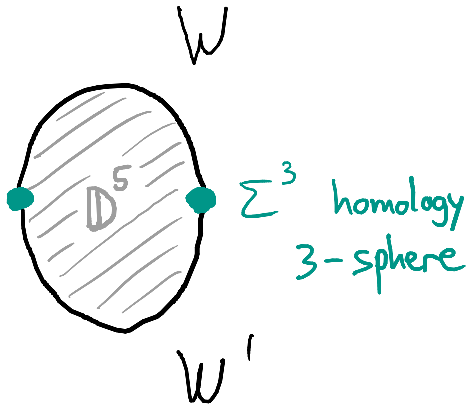
Neither s- nor h-cobordism theorem holds in dim. 4 in the smooth category

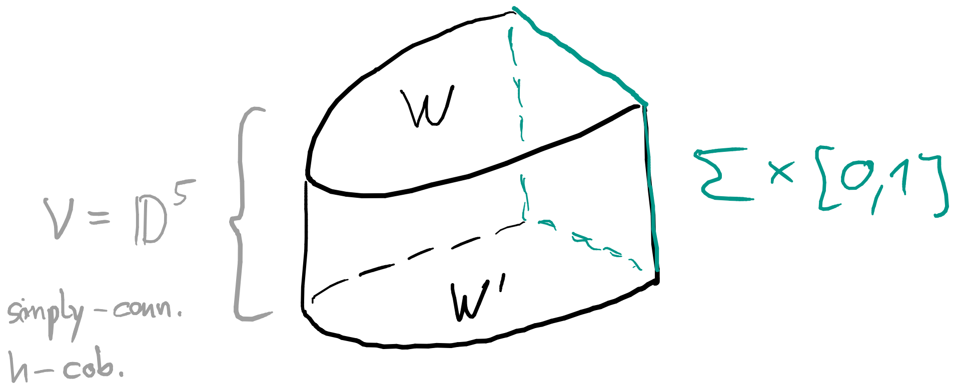
Another application:

[DET book, Remark 21.2]

> The topological Freedman ball bounding an integral homology 3-sphere is unique.

Remark 21.2. For a fixed homology 3-sphere Σ, the contractible 4-manifold constructed above is unique up to homeomorphism relative to the boundary. This requires further ingredients. Here is a sketch of the proof. Let W and W' be two contractible 4-manifolds with boundary a homology 3-sphere Σ. By the topological input Poincaré conjecture (Section 21.6.2), the union $W \cup_\Sigma -W'$ is homeomorphic to S^4, so bounds a 5-ball $V = D^5$. Decompose the boundary as $W \cup \Sigma \times [0,1] \cup -W'$ to view V as a simply connected h-cobordism relative boundary. The category preserving compact h-cobordism theorem (Section 21.5) implies that V is homeomorphic to $W \times [0,1]$ and thus W is homeomorphic to W' relative to the boundary.



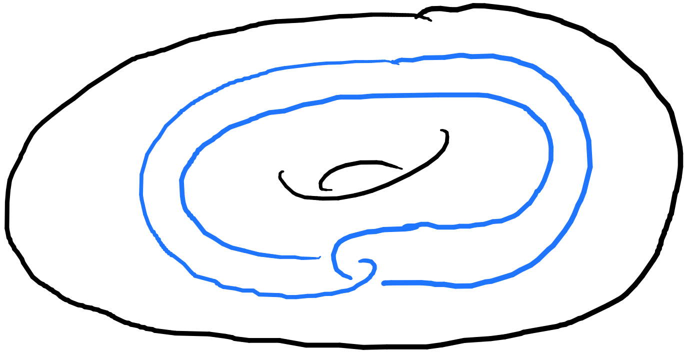


$$\underset{\substack{\text{CAT-preserving} \\ h\text{-cob. thm.}}}{\Longrightarrow} \quad V \underset{\text{homeo}}{\approx} W \times [0,1] \quad \Longrightarrow \quad W \underset{\text{homeo}}{\approx} W' \ \text{rel}\ \partial = \Sigma$$

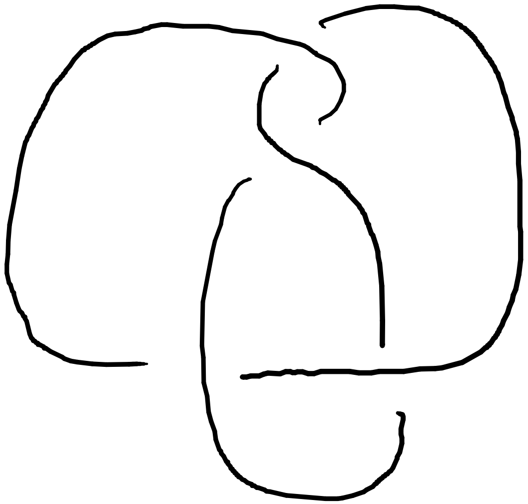
$\square$

② $\Delta_K = 1 \Rightarrow K$ is topologically $\mathbb{Z}$-slice [Freedman, Quinn]

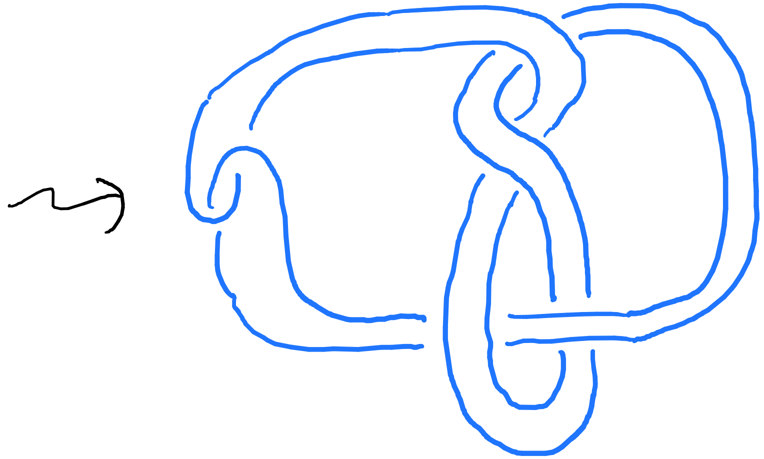
For example, for any K its Whitehead double is TOP-slice



Whitehead pattern
in solid torus



companion K

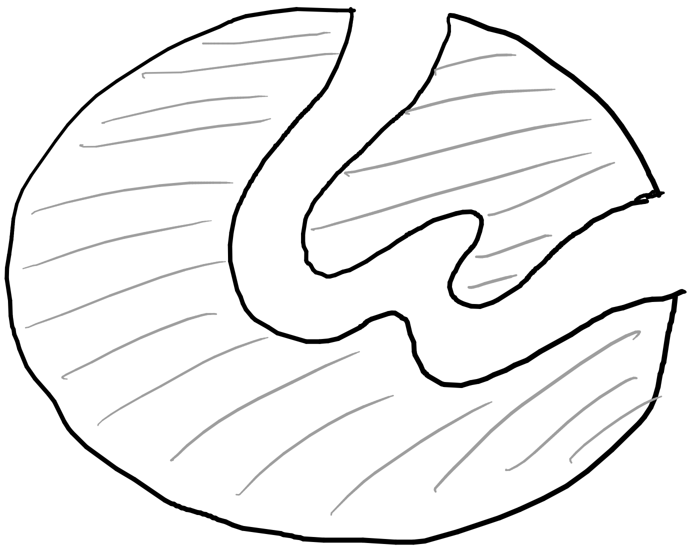


$Wh_0^+(K)$

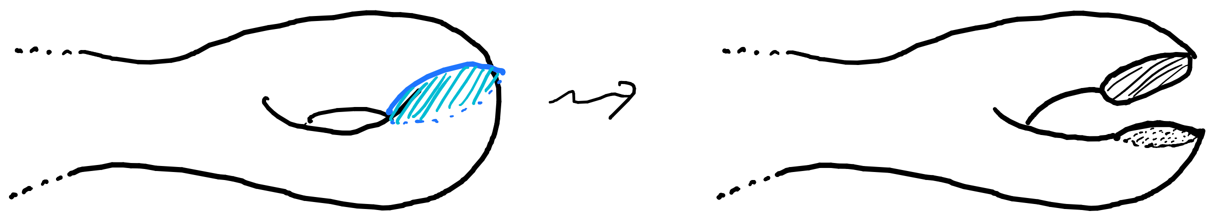
satellite of K
with Whitehead pattern

Two proof sketches

(A) Build a slice disk complement

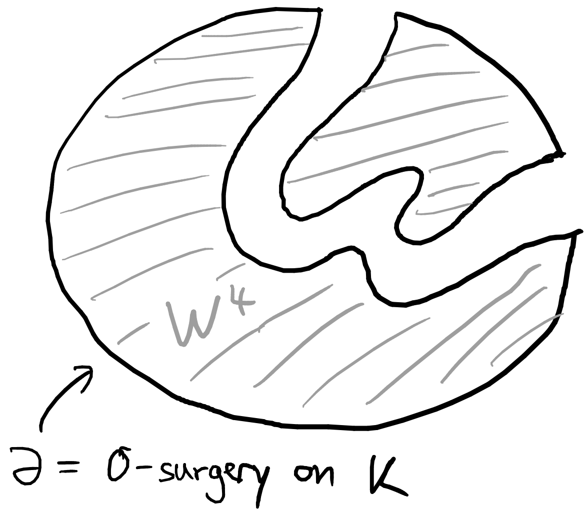


(B) Start with a higher genus slice surface and ambiently surger it into a disk



(A) Build a slice disk complement

Surgery theory proof in [DET book, Thm. 1.14]

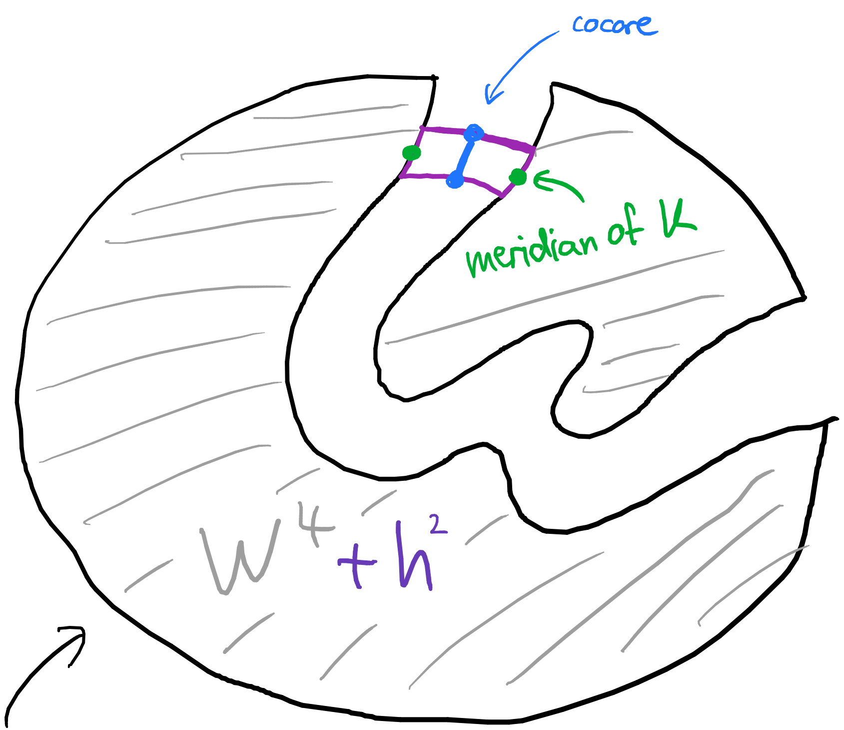


<u>Claim:</u> $K \subset S^3$ is top. loc. flat slice

iff. there is a compact 4-mfld W^4 s.th.

- $\partial W = M_K \leftarrow$ 0-surgery on K and
- W is a homology circle (where the inclusion $\partial W \hookrightarrow W$ induces iso. on H_*) and
- $\pi_1(W)$ is normally generated by a meridian of K

Pf.: Glue 2-handle to meridian



∂ was 0-surgery on K

now $= S^3$

> **Claim:** $K \subset S^3$ is top. loc. flat slice
> iff. there is a compact 4-mfld W^4 s.th.
> - $\partial W = M_K \leftarrow$ 0-surgery on K
> - W is a homology circle (where the inclusion $\partial W \hookrightarrow W$ induces iso on H_*)
> - $\pi_1(W)$ is normally generated by a meridian of K

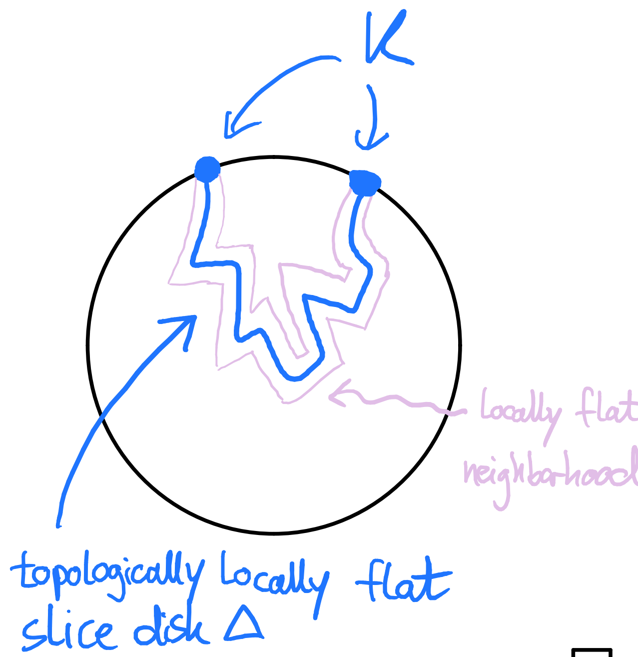
$$H_*(W^4 + h^2) \cong 0$$

$$\pi_1(W^4 + h^2) \cong \{e\}$$

$$\partial(W^4 + h^2) \cong S^3$$

[Freedman]

$$\Rightarrow (W^4 + h^2) \approx_{\text{homeo.}} \mathbb{D}^4$$



$\square$

Strategy: Build such a W^4 for a knot K with $\Delta_K = 1$ using the surgery exact sequence in dimension 4

Claim: $K \subset S^3$ is top. loc. flat slice iff. there is a compact 4-mfld W^4 s.th.

- $\partial W = M_K \leftarrow$ 0-surgery on K
- W is a homology circle (where the inclusion $\partial W \hookrightarrow W$ induces iso on H_*)
- $\pi_1(W)$ is normally generated by a meridian of K

In order to construct W, observe that the spin bordism group $\Omega_3^{spin}(S^1) \cong \Omega_2^{spin} \cong \mathbb{Z}/2$ is detected by the Arf invariant of K. The Arf invariant can be computed from the Alexander polynomial, and so vanishes. Thus there exists a compact, spin 4-manifold V with boundary M_K and a map to S^1 extending the map to S^1 on M_K corresponding to a generator of $H^1(M_K;\mathbb{Z})$ and sending a positively oriented meridian to 1.

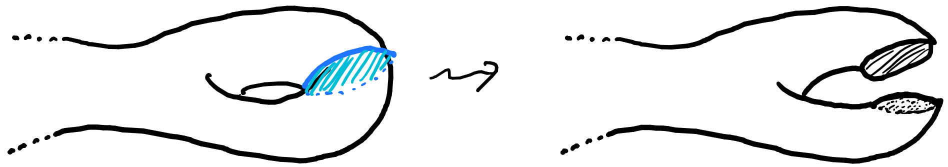
Perform surgery on circles in V to obtain V' with $\pi_1(V') \cong \mathbb{Z}$. The spin condition on V implies that for every element of $\pi_2(V)$ there is a fixed regular homotopy class of immersions of S^2 having trivial normal bundle: the Euler number of the normal bundle can be changed by ± 2 by adding local kinks. The $\mathbb{Z}$-equivariant intersection form on $\pi_2(V')$ is nonsingular and thus defines a surgery obstruction in $L_4(\mathbb{Z}[\mathbb{Z}])$. Here for nonsingularity we use the fact that $H_1(M_K;\mathbb{Z}[\mathbb{Z}]) = 0$, since $\Delta_K(t)$ is a unit in $\mathbb{Z}[\mathbb{Z}]$. Moreover, we are using surgery for manifolds with boundary. It is crucial here that the relevant fundamental group is $\mathbb{Z}$, which is a good group. We have that $L_4(\mathbb{Z}[\mathbb{Z}]) \cong 8\mathbb{Z}$ with generator the E_8 form. Take the connected sum of V' with copies of the E_8-manifold to produce V'' with vanishing surgery obstruction. This implies, by the exactness of the surgery sequence for manifolds with boundary, that there exists a half-basis of $H_2(V'')$ consisting of framed embedded spheres with geometric duals (see the sphere embedding theorem in Chapter 20) on which we can perform surgery to obtain a 4-manifold W. By construction, W is homotopy equivalent to S^1, and so satisfies the desired conditions. □

Ingredients:

- existence of E_8-mfld.
- topological surgery for $\pi_1 = \mathbb{Z}$
- top. input Poincaré conj.

(B) Ambient surgery to reduce genus

Proof uses a single application of Freedman's Disk Embedding Theorem (DET) with smooth input



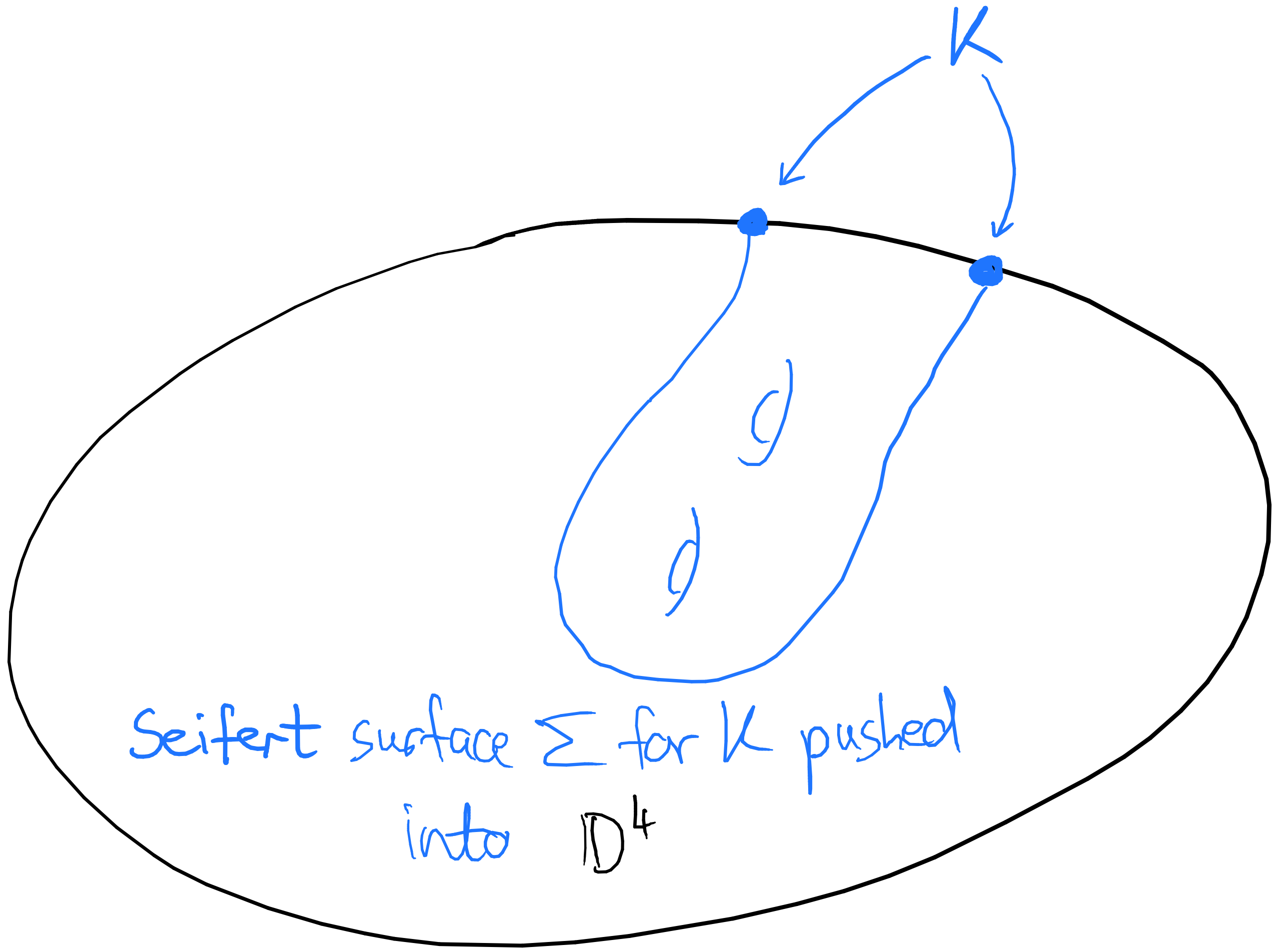
[DET book, Sec. 21.6.3]

summary of the proof in

[Garoufalidis - Teichner: On knots with trivial Alexander poly.]

⚠ Possible error in claim about triangular form of Seifert matrix in this paper

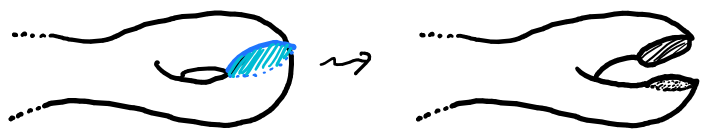
↗ Version which uses the Freedman-Matsumoto-form instead to find the disks and dual spheres



$\Sigma \subset \mathbb{D}^4$ is an example of a $\underbrace{\mathbb{Z}\text{-surface}}$

i.e. $\pi_1(\mathbb{D}^4 - \Sigma) \cong \mathbb{Z}$

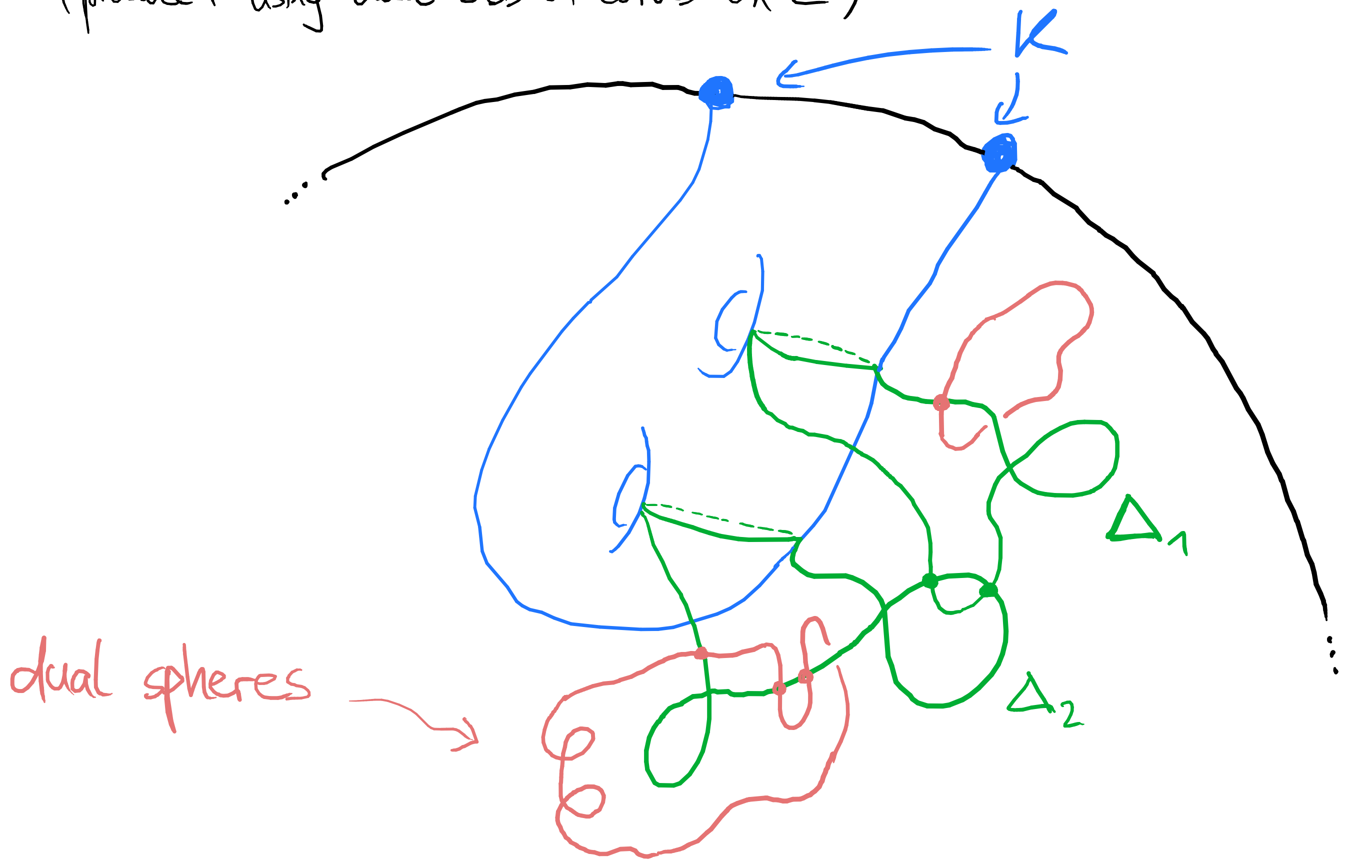
Want to ambiently surger Σ to a disk

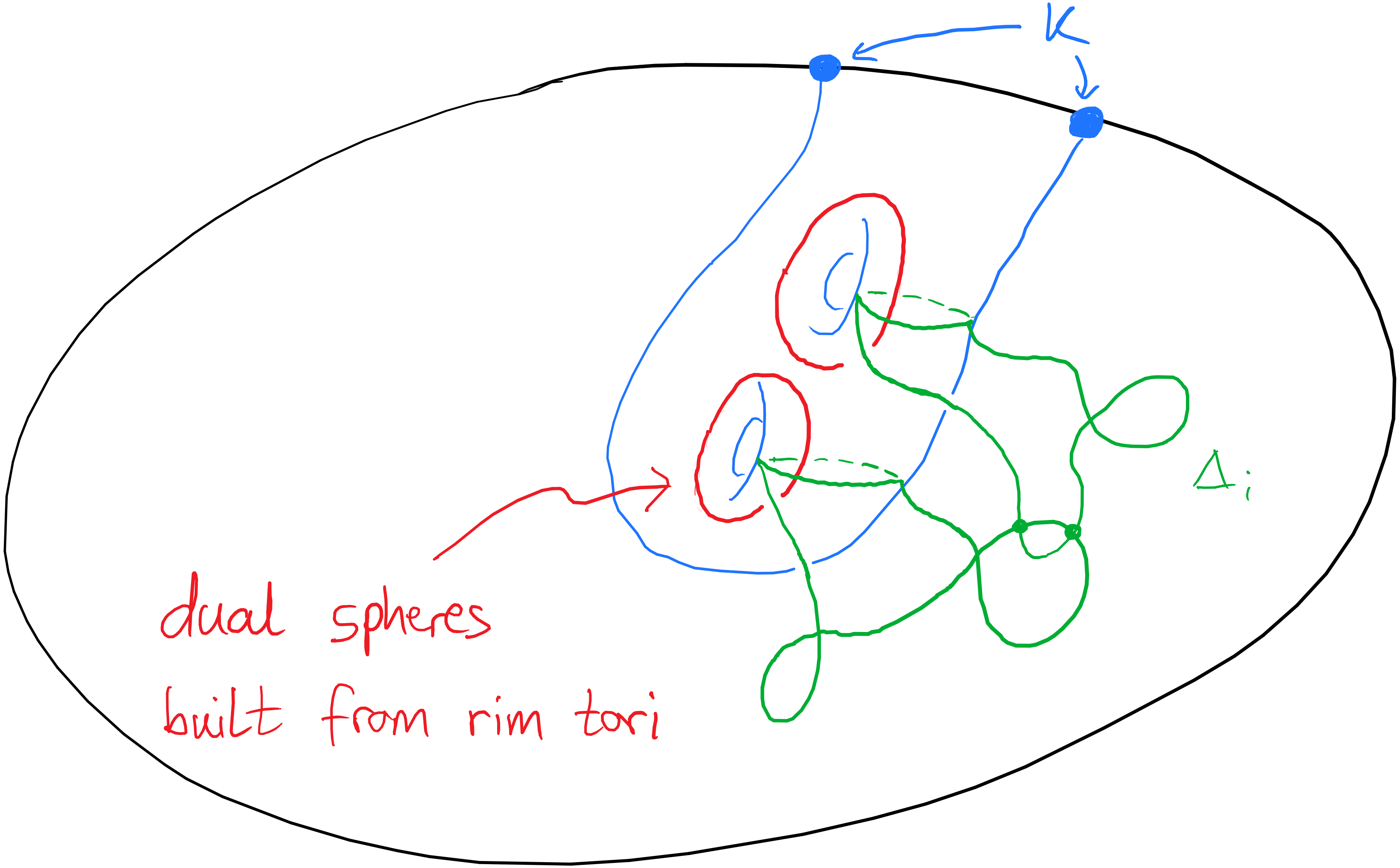


Use $\Delta_K = 1$ to find:

- half-basis of curves on Σ bounding immersed disks $\{\Delta_i\}$ with $\operatorname{int} \Delta_i \subset D^4 - \Sigma$
- disks $\{\Delta_i\}$ are equipped with algebraically dual spheres (produced using dual basis of curves on Σ)

} dual spheres let us conclude that the new surface still has group $\mathbb{Z}$





DET in $\mathbb{D}^4 - \Sigma$ $\rightsquigarrow$ replace Δ_i by mutually disjoint, flat embedded disks in $\mathbb{D}^4 - \Sigma$ (same ∂)

$\rightsquigarrow$ surger Σ to locally flat disk $\rightsquigarrow$ Done!

Disk - embedding theorem:

(version with geometrically dual spheres in the output)

·) works in 4-manifolds with <u>good</u> fundamental group

Theorem A (Disc embedding theorem cf. [FQ90, Theorem 5.1A]). *Let M be a connected 4-manifold with good fundamental group. Consider a continuous map*

$$F = (f_1, \dots, f_k) \colon (D^2 \sqcup \cdots \sqcup D^2, S^1 \sqcup \cdots \sqcup S^1) \longrightarrow (M, \partial M)$$

that is a locally flat embedding on the boundary and that admits algebraically dual spheres $\{g_i\}_{i=1}^k$ satisfying $\lambda(g_i, g_j) = 0 = \widetilde{\mu}(g_i)$ for all i, j. Then there exists a locally flat embedding

$$\overline{F} = (\overline{f}_1, \dots, \overline{f}_k) \colon (D^2 \sqcup \cdots \sqcup D^2, S^1 \sqcup \cdots \sqcup S^1) \hookrightarrow (M, \partial M)$$

such that $\overline{F}$ has the same boundary as F and admits a generically immersed, geometrically dual collection of framed spheres $\{\overline{g}_i\}_{i=1}^k$, such that $\overline{g}_i$ is homotopic to g_i for each i.

Moreover, if f_i is a generic immersion, then it induces a framing of the normal bundle of its boundary circle. The embedding $\overline{f}_i$ may be assumed to induce the same framing.

·) Assumption on existence of dual spheres is important!
(this is where the $\Delta_K = 1$ assumption comes in again, otherwise the proof would show that algebraically slice knots are slice)

Comment on Good groups:

- finite, abelian, solvable, ... groups are good
- Not known whether all groups are good (big open question: Is Fr_2 good?)

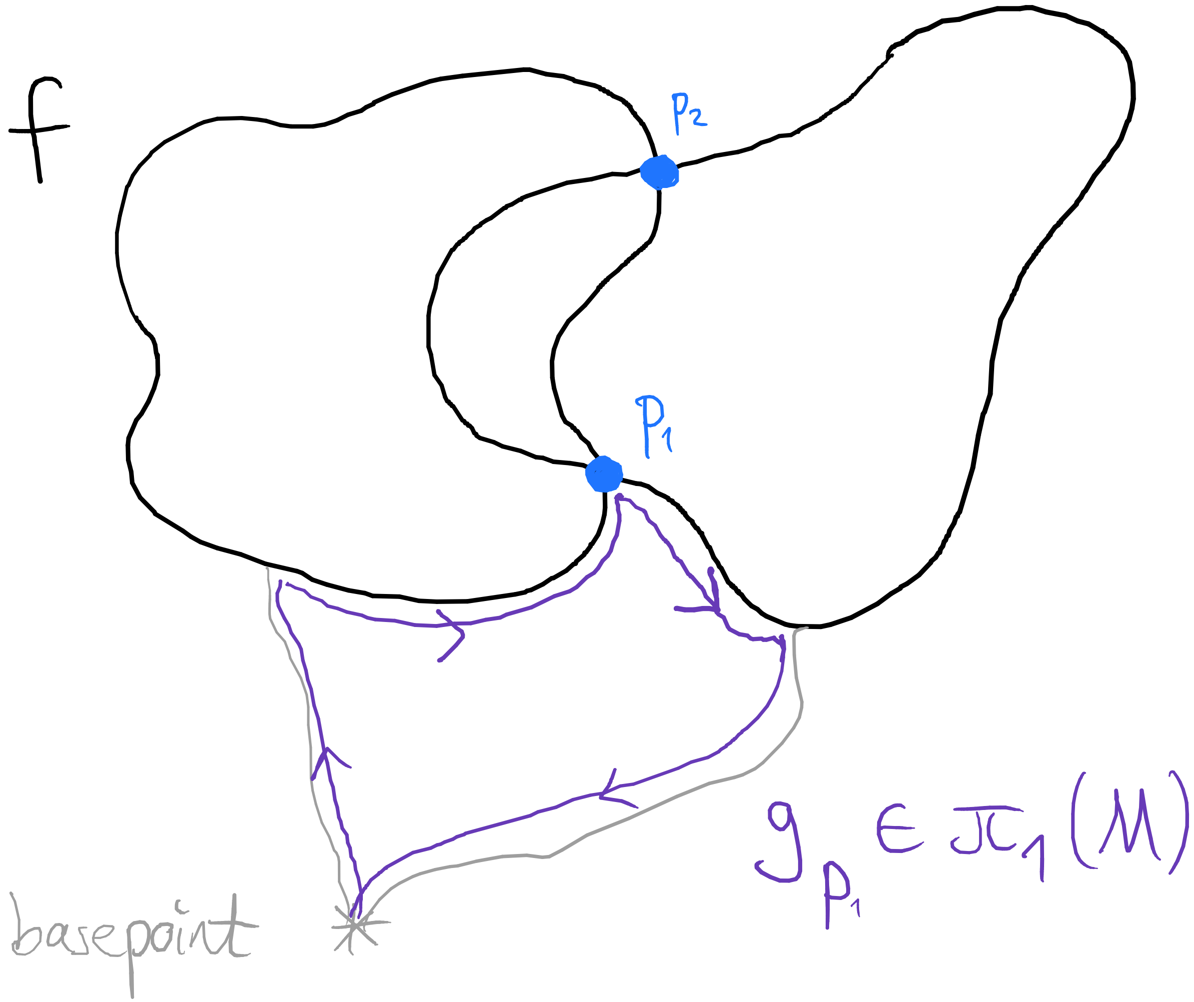
In the following,
we only need the DET for
manifolds where π_1 is a
cyclic group, which are
known to be good

Intersection numbers:

$$\lambda(f,g) = \sum_{p \in f \pitchfork g} \varepsilon_p \cdot g_p \in \mathbb{Z}[\pi_1 M]$$

sign $\in \{\pm 1\}$ of the intersection pt.

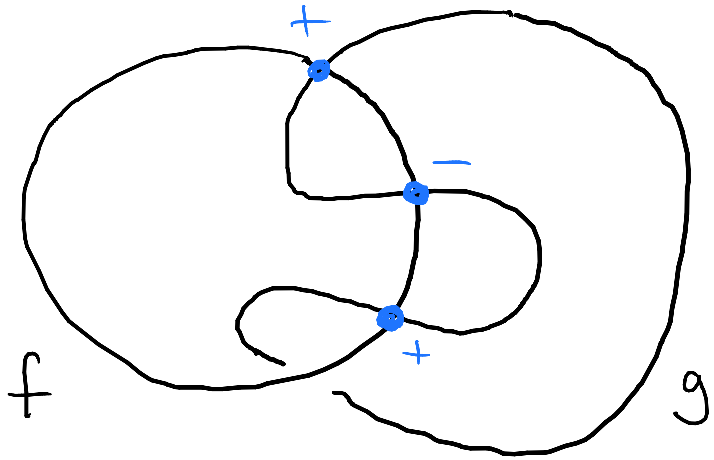
double point loop for intersection pt.



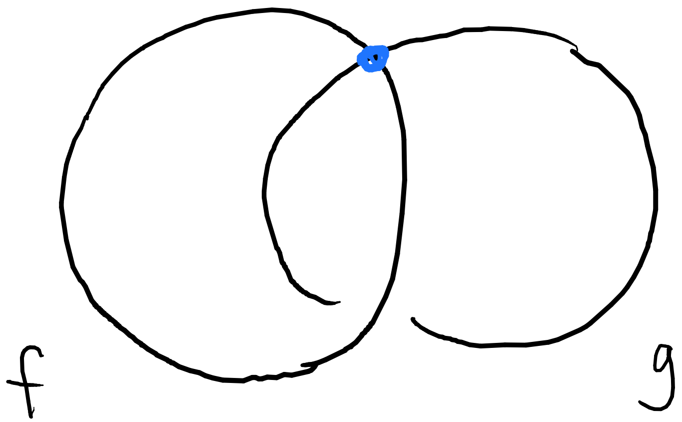
•) self-intersection number μ

$\mu(f) = 0 \iff$ all self-intersections of $f \pitchfork f$ are paired by generic collection of Whitney disks

•) f, g are algebraically dual if $\lambda(f,g) = 1 \iff$ all but one point in $f \pitchfork g$ are paired by a generic collection of Wh disks



•) geometrically dual:



Disk embedding theorem summary:

[Freedman 1982, Quinn 1990]

M^4 connected, topological manifold with $\pi_1 M$ good

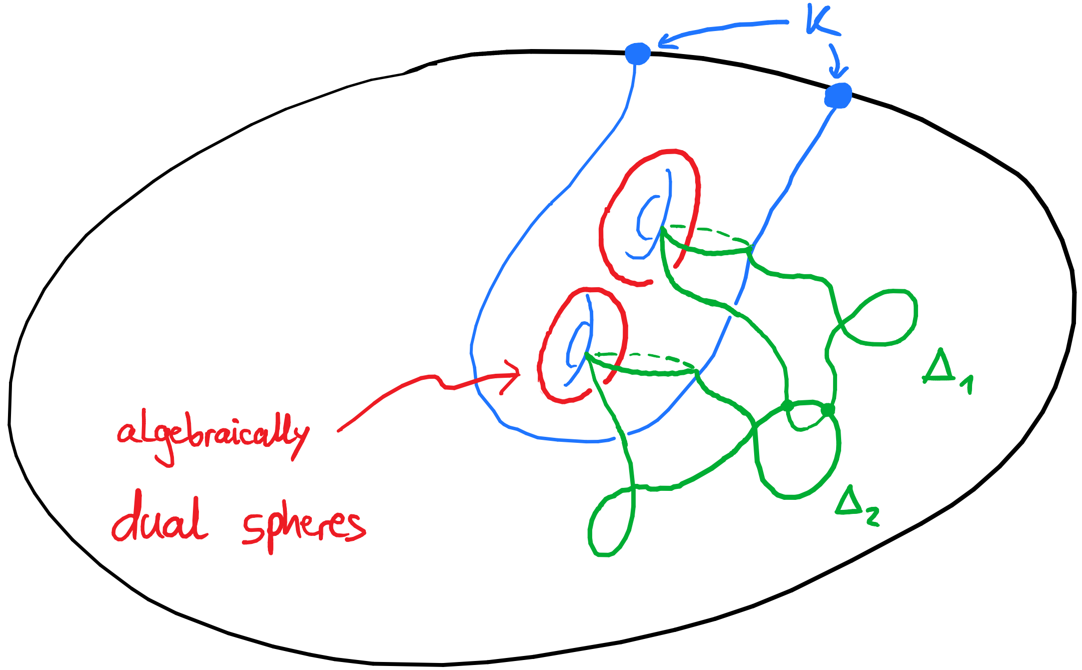
$$
\begin{array}{ccc}
F : \coprod D^2 & \looparrowright & M \\
\cup & & \cup \\
\partial = \coprod S^1 & \hookrightarrow & \partial M
\end{array}
\qquad \text{generic immersion}
$$

such that:

- algebraic intersection numbers of F vanish
- $\exists\, G : \coprod S^2 \looparrowright M$ framed, algebraic dual to F

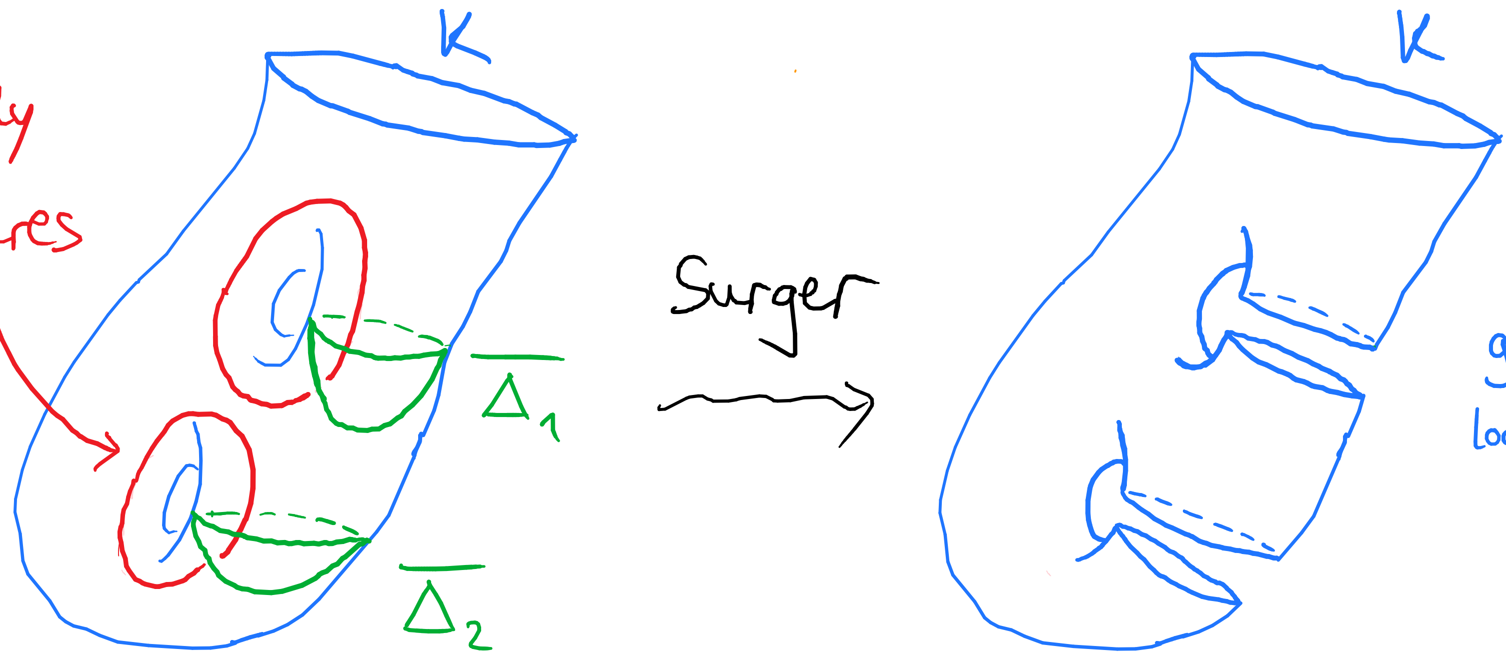
Then F is (reg.) homotopic rel. ∂ to a locally flat emb. $\overline{F}$ with geometrically dual spheres $\overline{G}$ with $G \simeq \overline{G}$ } [Powell-Ray-Teichner]

Back to our application:



Now we apply the DET to our situation:

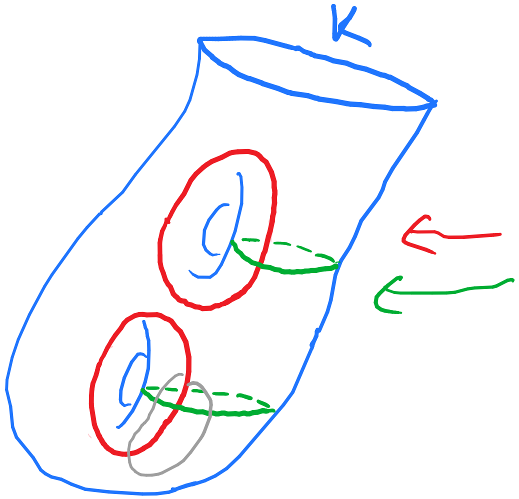
geometrically
dual spheres



get topologically locally flat slice disk for K

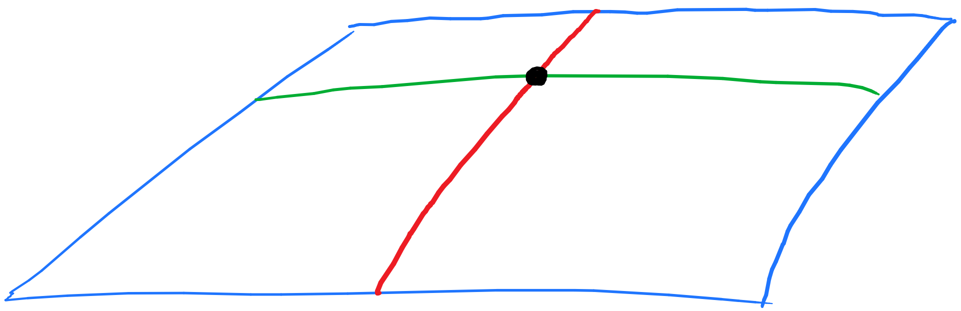
$\square$

We got the dual spheres by surgery on rim tori:

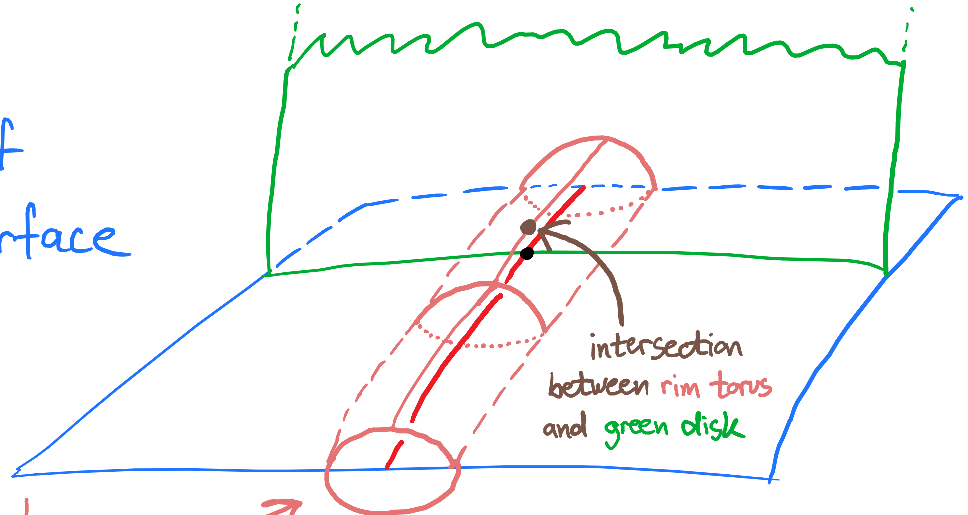


symplectic basis of curves on surface

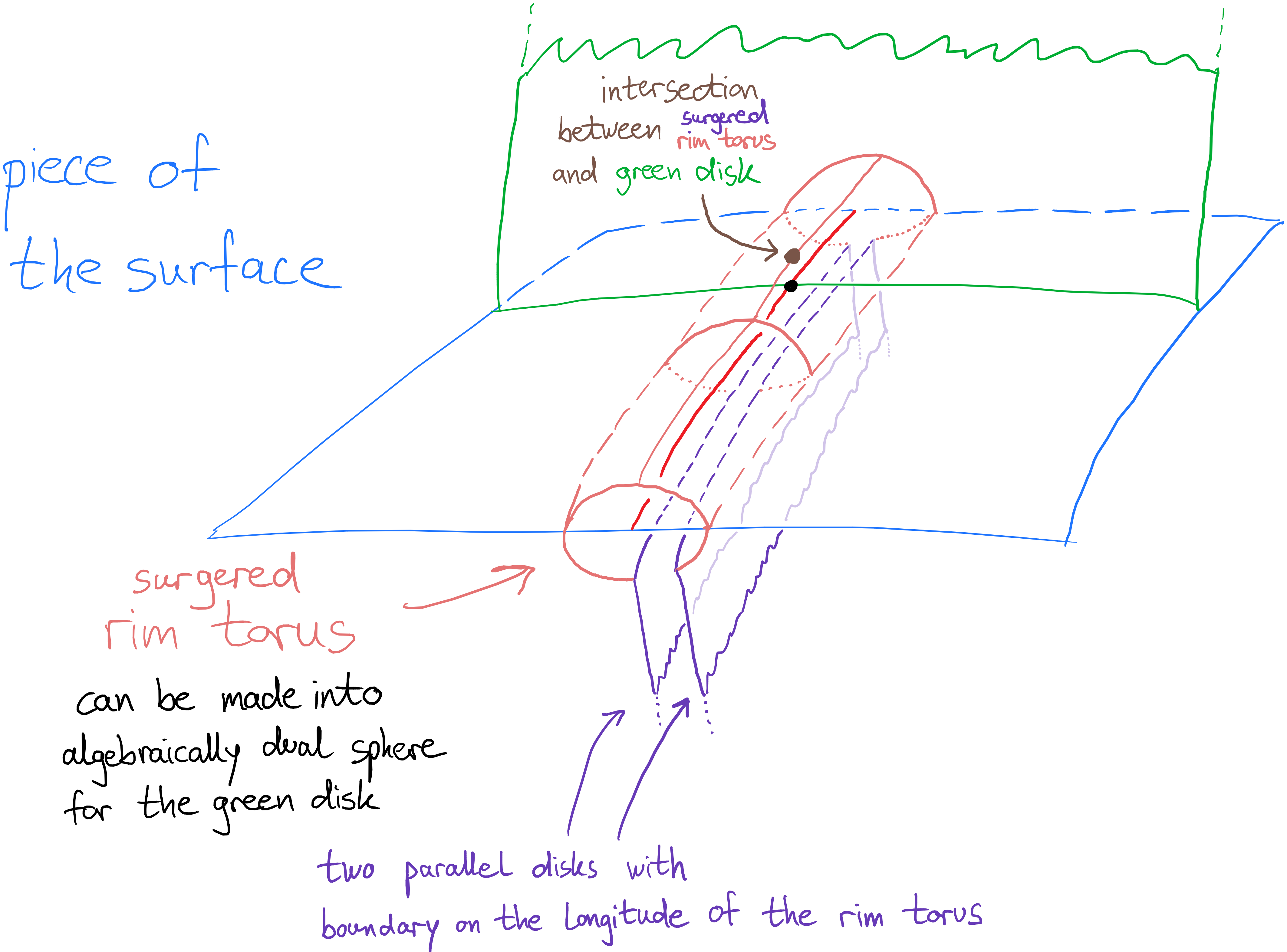
zoom in



piece of the surface



rim torus around red curve dual to green disks


piece of the surface
intersection between surgered rim torus and green disk
surgered rim torus
can be made into algebraically dual sphere for the green disk
two parallel disks with boundary on the longitude of the rim torus

References to recent developments:

- [Conway, Powell: Characterization of homotopy ribbon disks (2019)]

  show that $\mathbb{Z}$-slice disks for K are unique up to topological isotopy rel. ∂

- [Conway, Powell: Embedded surfaces with infinite cyclic knot group (2020)]

  any two $\mathbb{Z}$-slice surfaces of genus $g \neq 1,2$ for an Alex. poly. $=1$ knot are topologically isotopic rel. ∂ in $\mathbb{D}^4$

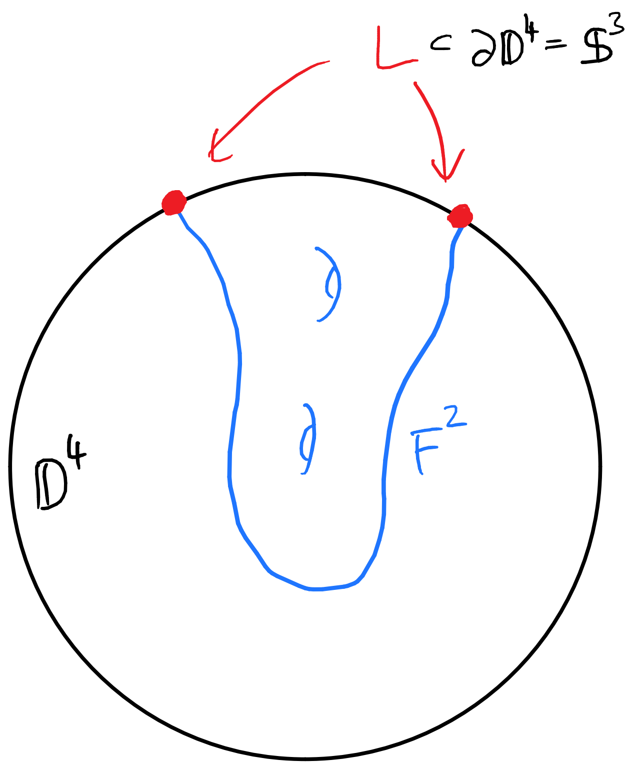
- [Hayden: Exotic ribbon disks and symplectic surfaces in the 4-ball (2020)]

  constructed pair of $\mathbb{Z}$-ribbon disks for a knot $\overset{\text{[Conway-Powell]}}{\Longrightarrow}$ topologically isotopic rel. ∂ but there is no diffeomorphism of $\mathbb{D}^4$ mapping one to the other

- For finding the disks we needed for the DET:

  Freedman-Matsumoto form controls the intersection data [Kreck-Teichner]

Outlook: Freedman-Matsumoto form



$$C := \mathbb{D}^4 - \nu F$$

F (properly embedded) $\mathbb{Z}$-surface, i.e. $\pi_1(\underbrace{\mathbb{D}^4 - F}) \cong \mathbb{Z}$

(ex.: Seifert surface for L whose interior is pushed into the 4 ball)

Freedman-Matsumoto form

$$\Lambda := \mathbb{Z}[t, t^{-1}] = \mathbb{Z}[\pi_1 C]$$

defined on $H_2(\mathbb{D}^4 - \nu F, \partial(\mathbb{D}^4 - \nu F); \Lambda)$

$$\Uparrow$$

$a_1, a_2 \in \pi_2(\mathbb{D}^4 - \nu F, \partial(\mathbb{D}^4 - \nu F), *)$

$a_1, a_2 : (\mathbb{D}^2, \mathbb{S}^1) \hookrightarrow (C, \partial C)$

$$j_* : \pi_1(\partial C) \to \pi_1(C)$$

$$\lambda_{FM}(a_1, a_2) := (t-1) \cdot \Big(\sum_{p \in a_1(\mathbb{D}^2) \pitchfork a_2(\mathbb{D}^2)} \underset{\{\pm 1\}}{\underset{\cup\!\!\!|}{\varepsilon_p}} \cdot \underset{\pi_1(C)}{\underset{\cup\!\!\!|}{g_p}} \Big) + \sum_{q \in \mathrm{pr}(a_1(\mathbb{S}^1)) \pitchfork \mathrm{pr}(a_2(\mathbb{S}^1))} \underset{\{\pm 1\}}{\underset{\cup\!\!\!|}{\varepsilon_q}} \cdot j_* \underset{\pi_1(\partial C)}{\underset{\cup\!\!\!|}{g_q}}$$

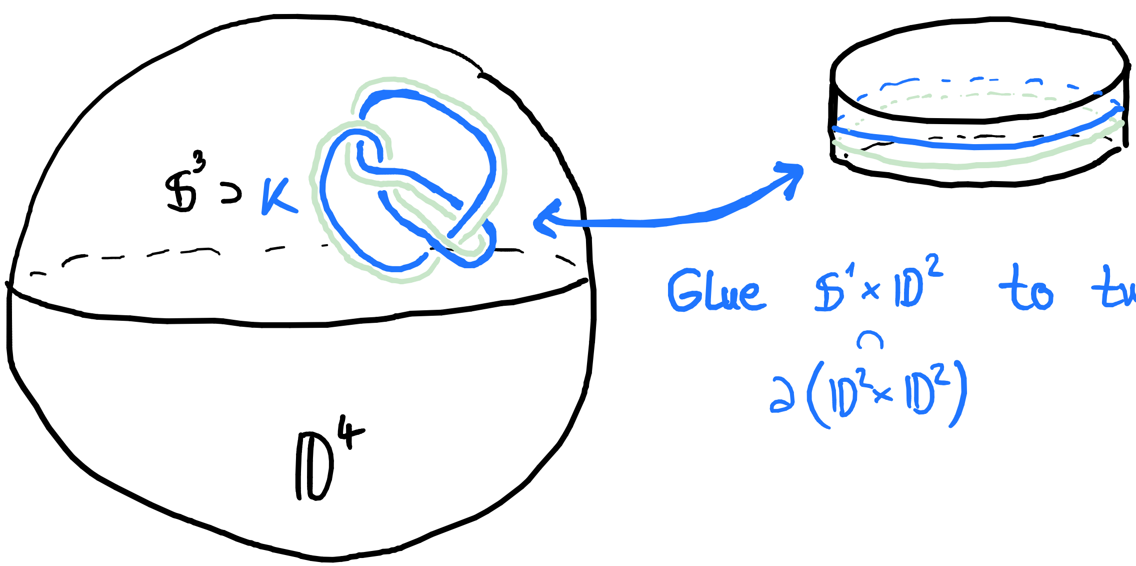
Exotic $\mathbb{R}^4$'s from topologically slice, non smoothly slice knots

0-trace of a knot $K: S^1 \hookrightarrow S^3$ is

[for example can use gauge theory or the s-invariant from Khovanov homology to find such knots]

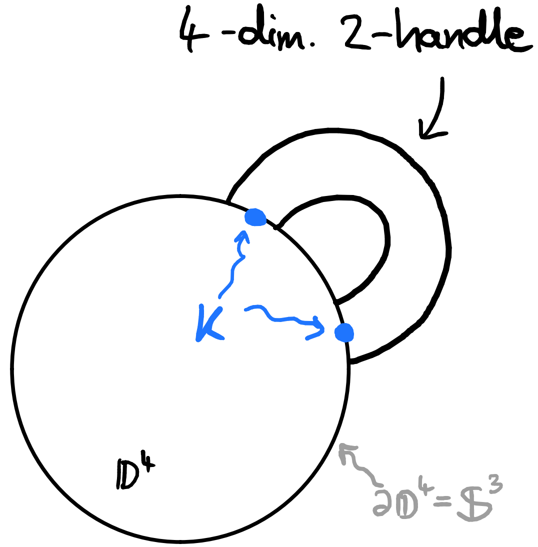
$$\boxed{X_0(K) = D^4 \cup_{\substack{K \times D^2 \\ \cap \\ \partial D^4}} D^2 \times D^2}$$

$D^2 \times D^2$



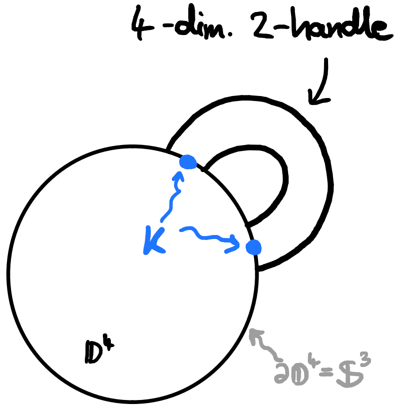
Glue $\underset{\partial(D^2 \times D^2)}{\overset{}{S^1 \times D^2}}$ to tubular neighborhood of K

"Half-dim. schematic":



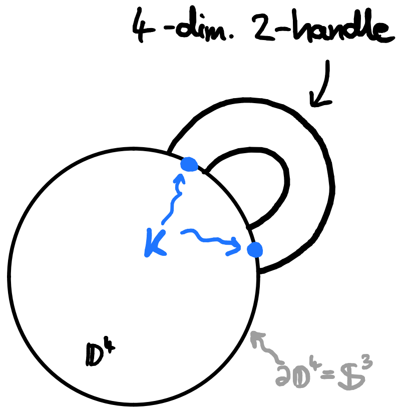
Trace embedding lemma:

K smoothly slice

$\Longleftrightarrow$ 

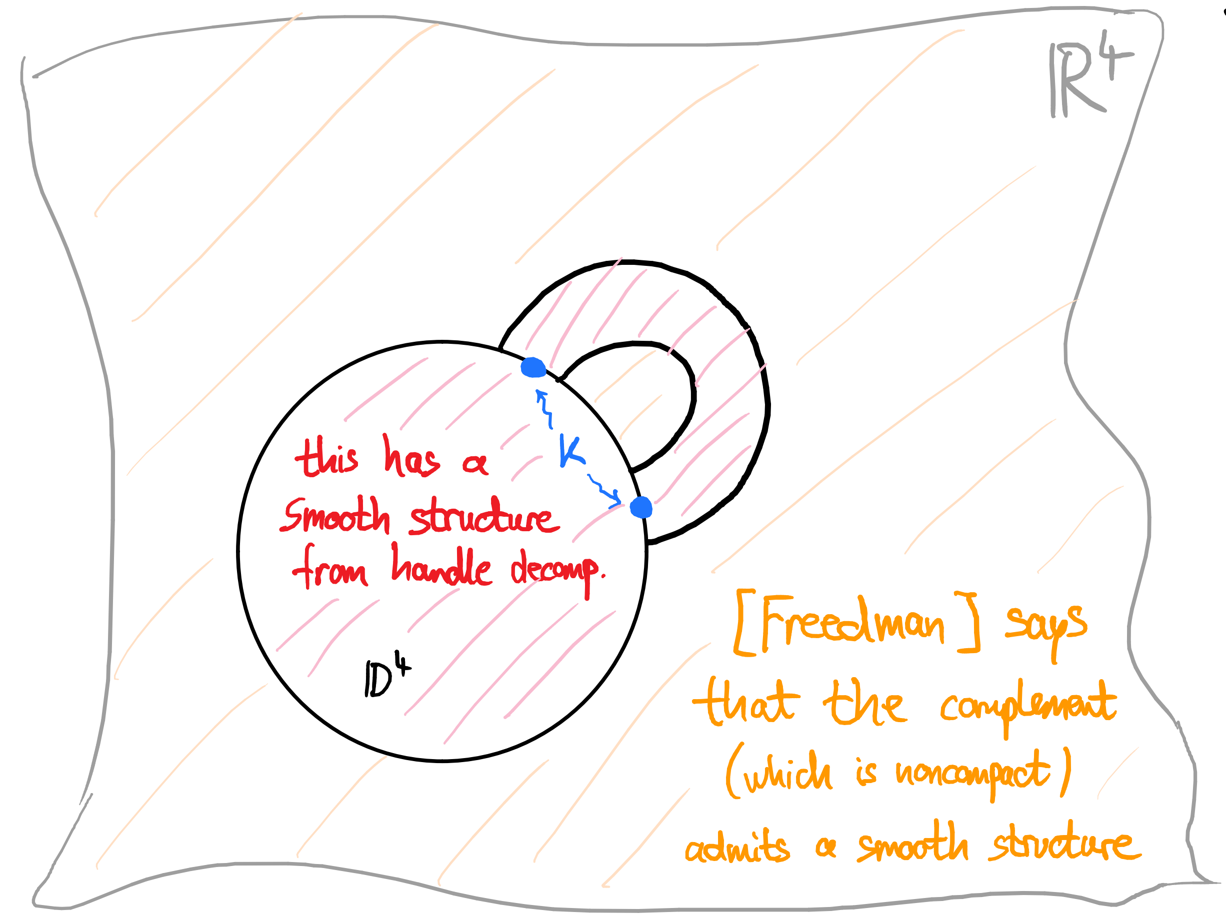
$= X_0(K) \xhookrightarrow{\text{smoothly}} \mathbb{R}^4_{\text{std.}}$

K topologically slice

$\Longleftrightarrow$ 

$= X_0(K) \xhookrightarrow{\text{topologically}} \mathbb{R}^4_{\text{std.}}$

Construction: Start with topologically slice, non-smoothly slice knot K, and a topological embedding in $\mathbb{R}^4$



Red & Orange together give smooth structure $\mathcal{R}$ on $\mathbb{R}^4$ ...

... which can't be diffeomorphic to $\mathbb{R}^4_{std.}$ because otherwise we would have a smoothly embedded $X_0(K)$