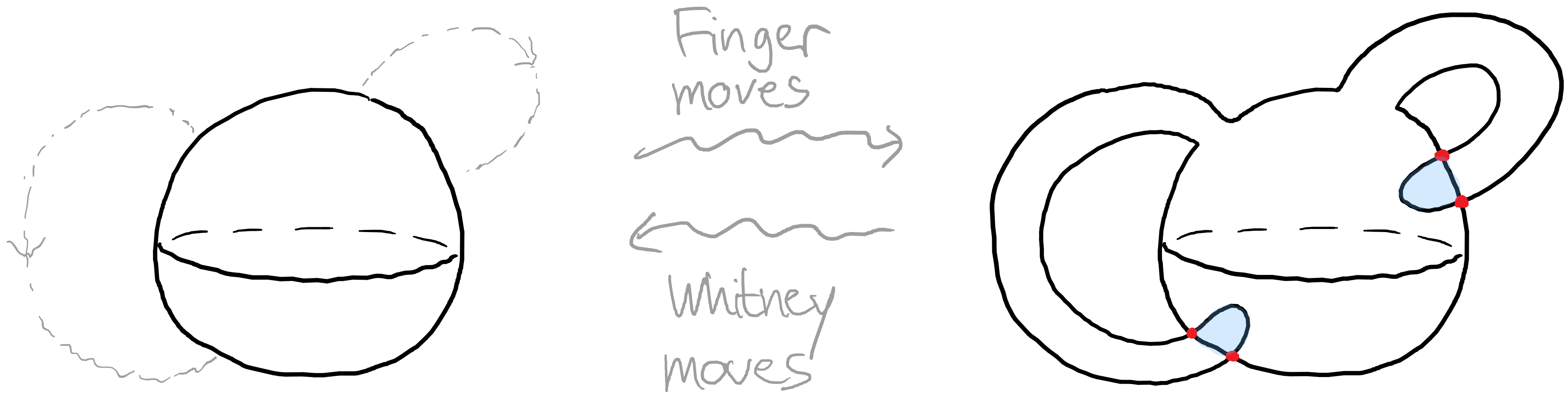
2020-11-05 Iowa topology seminar (1h talk)

Casson-Whitney unknotting numbers of 2-knots in the 4-sphere

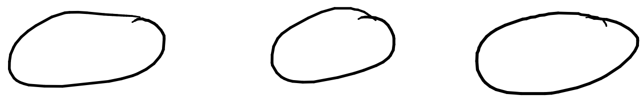
with Jason Joseph, Michael Klug & Hannah Schwartz



Benjamin Matthias Ruppik, PhD-student @ Max-Planck Institute for Mathematics, Bonn, Germany

Classical ribbon knots in S^3

Start with unlink

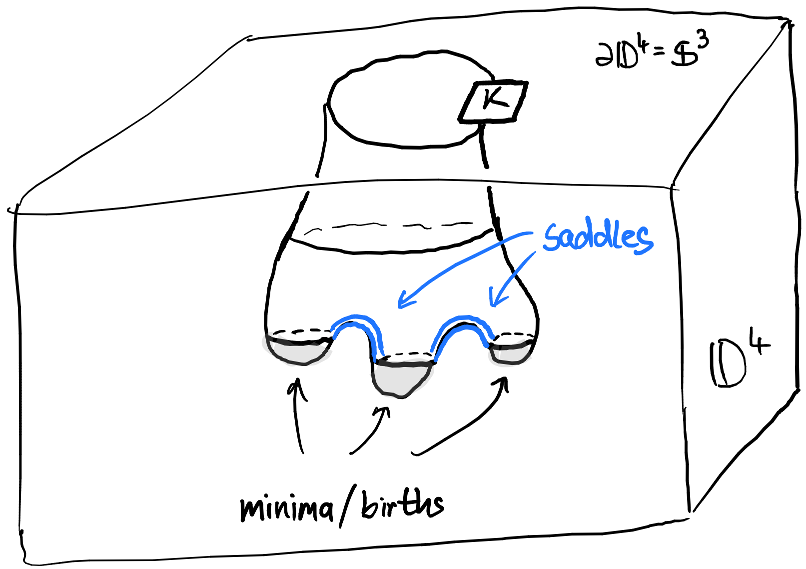


Classical ribbon knots in S^3

Start with unlink

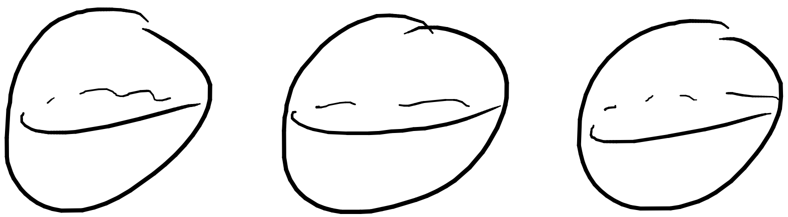
Join components with fusion bands

Ribbon disk:



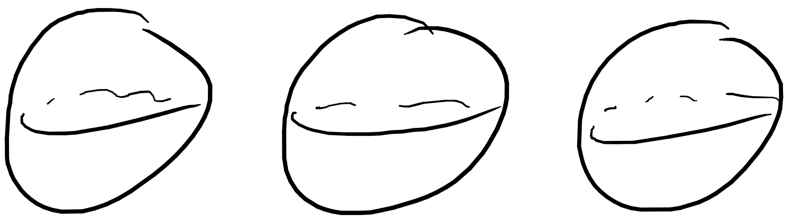
Ribbon 2-knots in S^4

Start with an unlink of 2-spheres

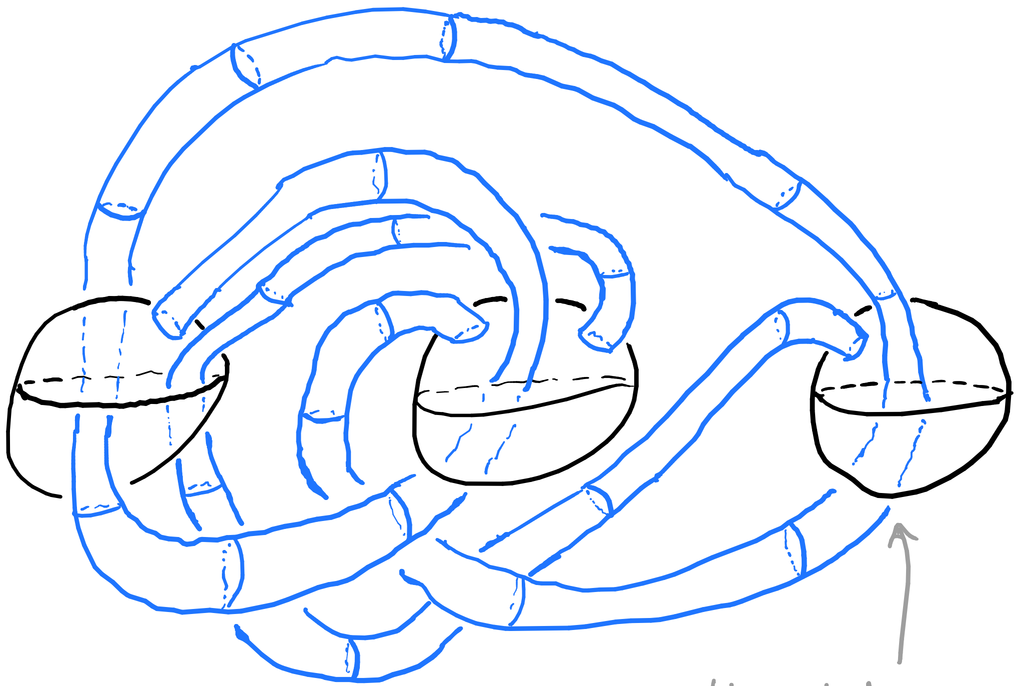


Ribbon 2-knots in S^4

Start with an unlink of 2-spheres

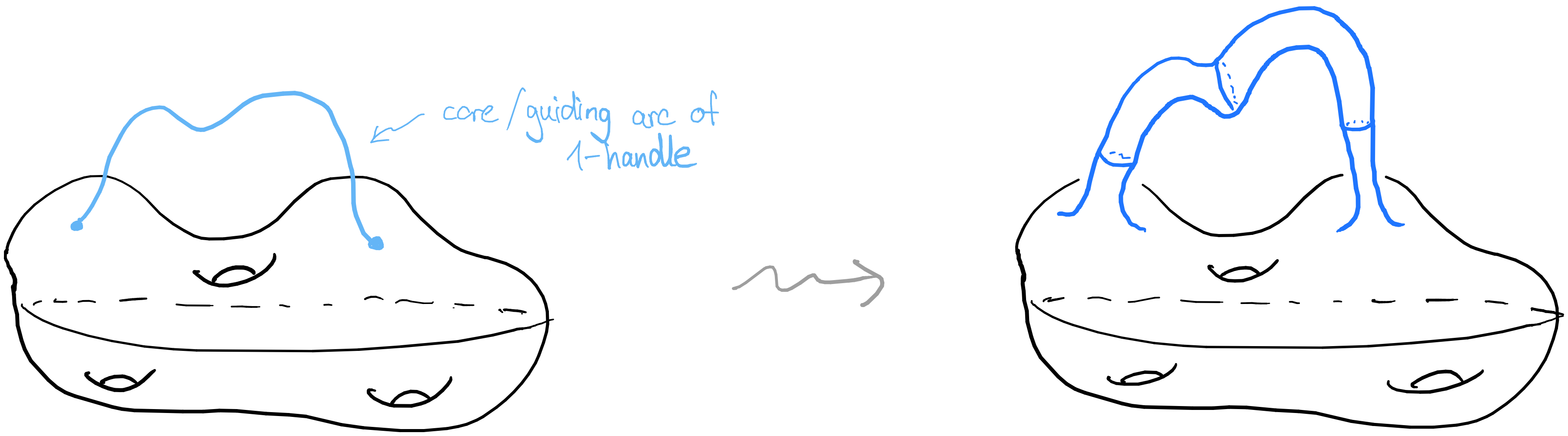


Attach fusion tubes



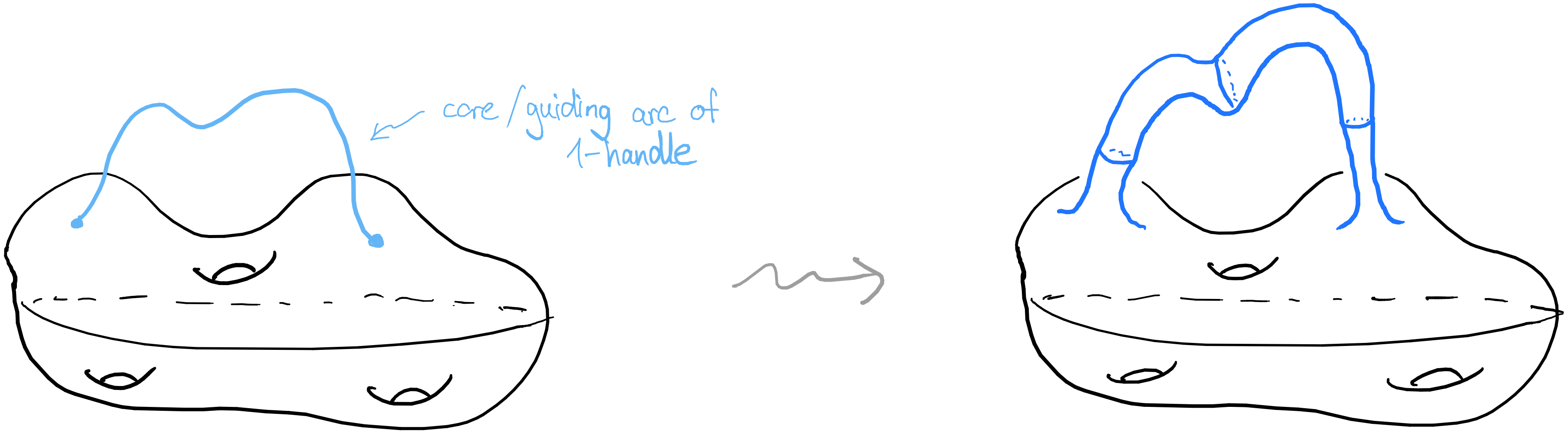
the tubes usually link with the spheres

1-handle stabilization of a surface



S $\qquad\qquad$ $S + h^1$

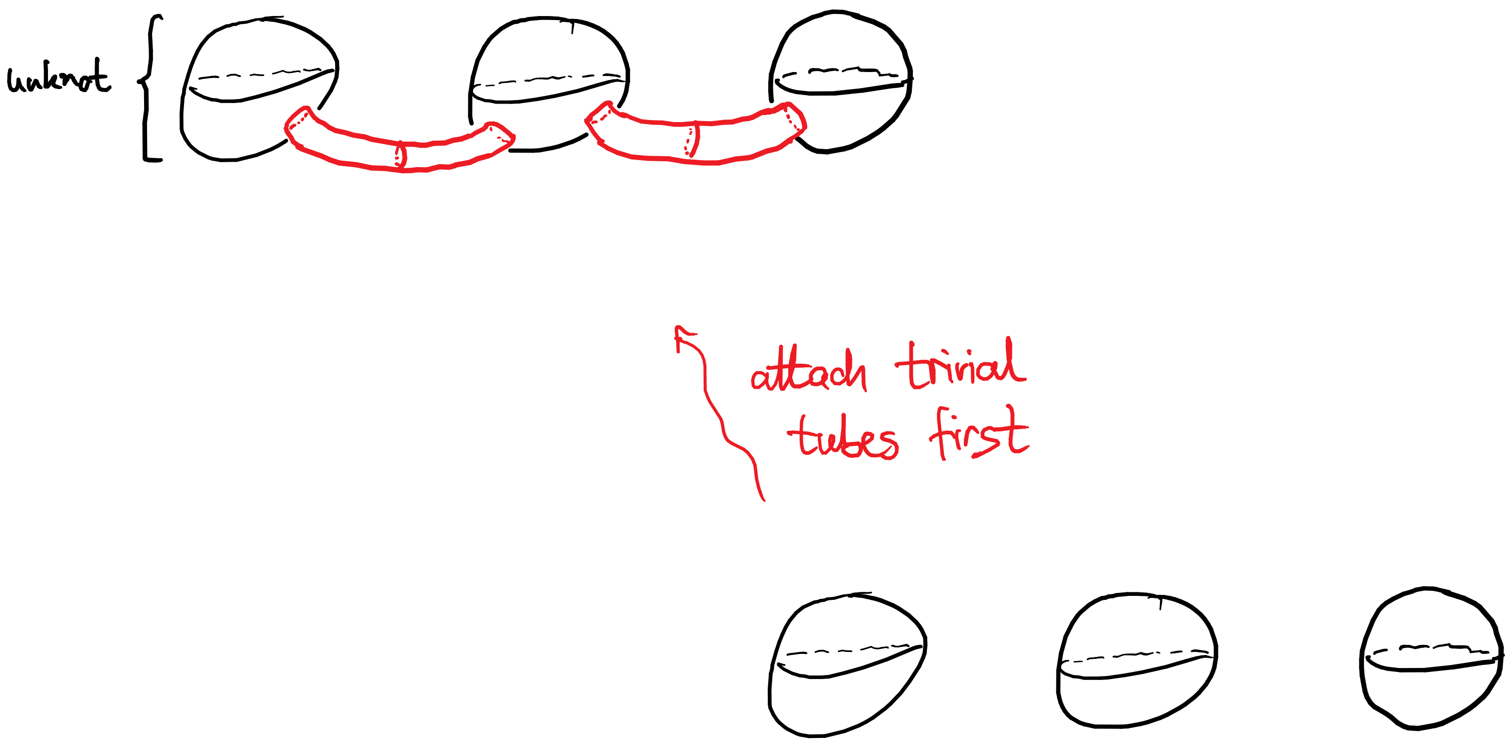
1-handle stabilization of a surface



S $\qquad\qquad\qquad\qquad$ $S + h^1$

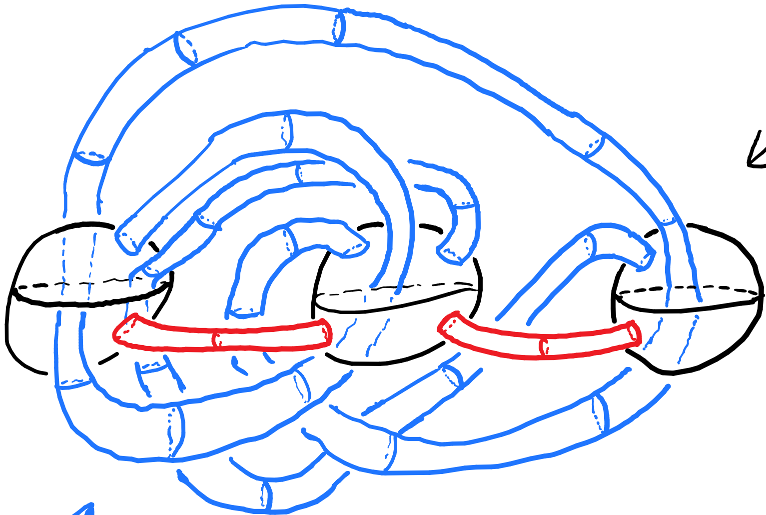
Fact: Any surface $S \subset S^4$ can be unknotted with enough 1-handle stabilizations.

A surface $S : \Sigma_g \hookrightarrow S^4$ is unknotted if it bounds a handle body



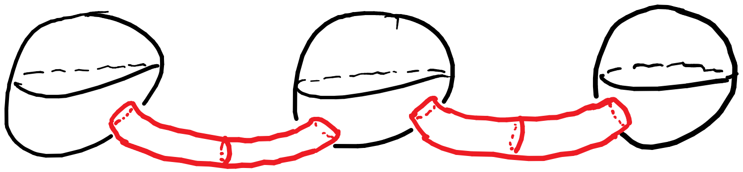
[Miyazaki]

This is unknotted!

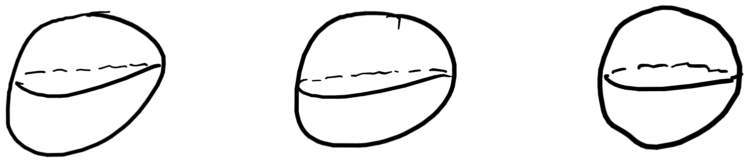


then the blue
tubes are trivial, since
there is only the trivial stabilization
of an unknotted surface

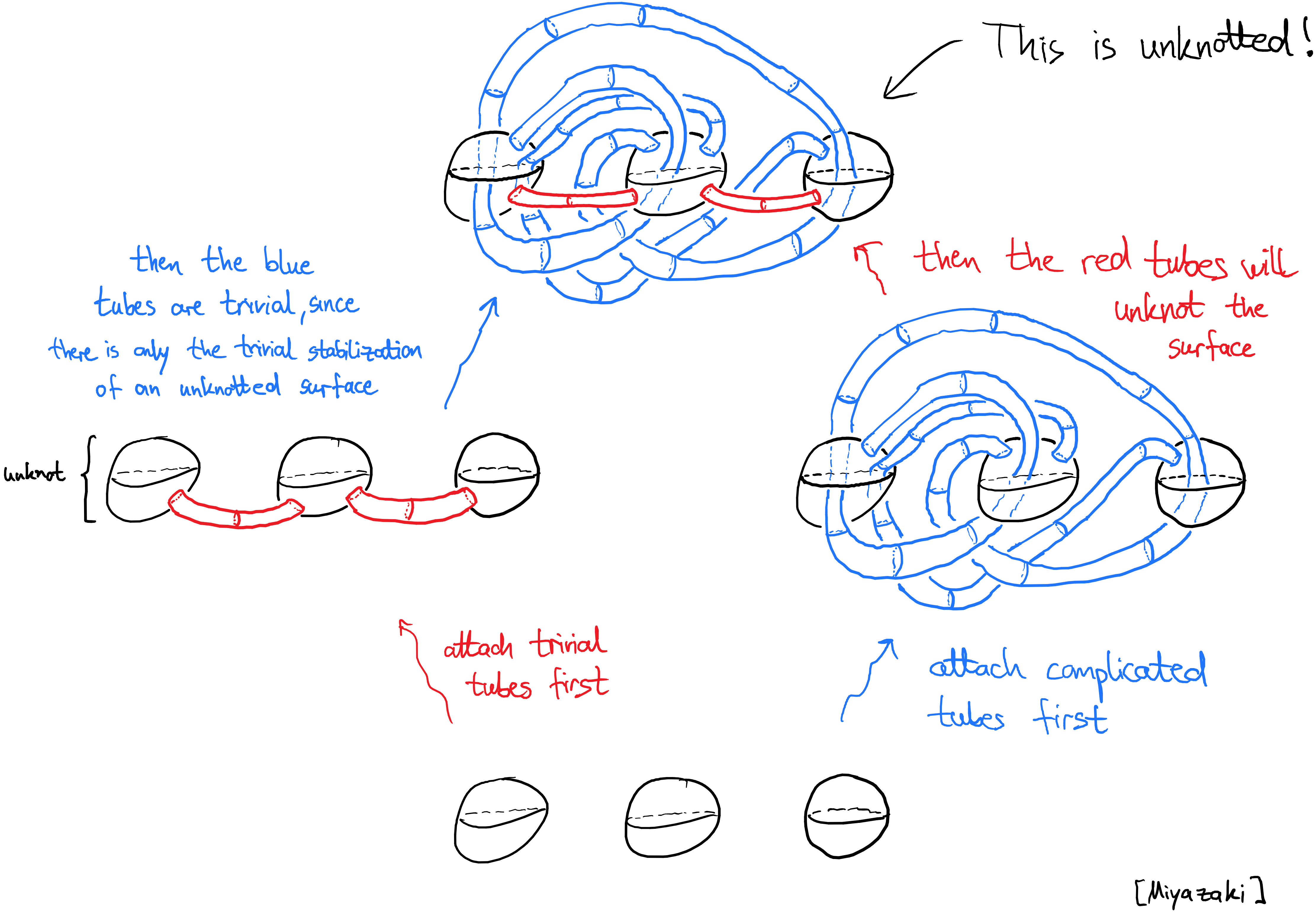
unknot



attach trivial
tubes first



[Miyazaki]


This is unknotted!
then the blue
tubes are trivial, since
there is only the trivial stabilization
of an unknotted surface
then the red tubes will
unknot the
surface
unknot
attach trivial
tubes first
attach complicated
tubes first
[Miyazaki]

Thm. [Miyazaki, 1985]: For a ribbon 2-knot $K \subset S^4$:

$$u_{1\text{-handle}}(K) \leq \text{fusion-}\#(K)$$

minimal # of fusion band in a ribbon presentation of K

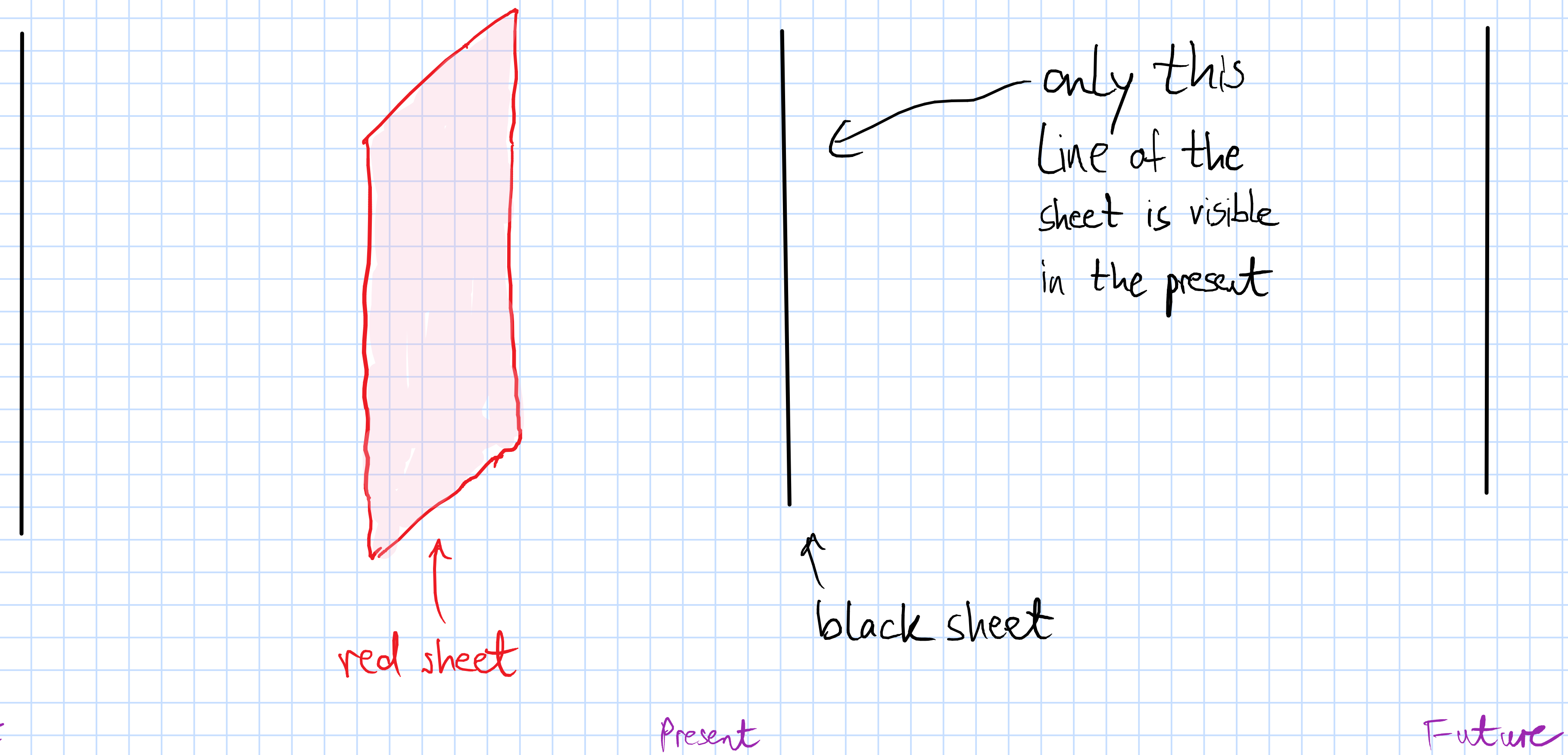
Here, we would like to present a similar lower bound for another "unknotting number" for 2-knots define by looking at the "length" of a regular homotopy

Regular homotopies of surfaces in 4-manifolds

(homotopies through immersions)

Sequence of $\boxed{\text{Finger-moves}}$ and Whitney-moves

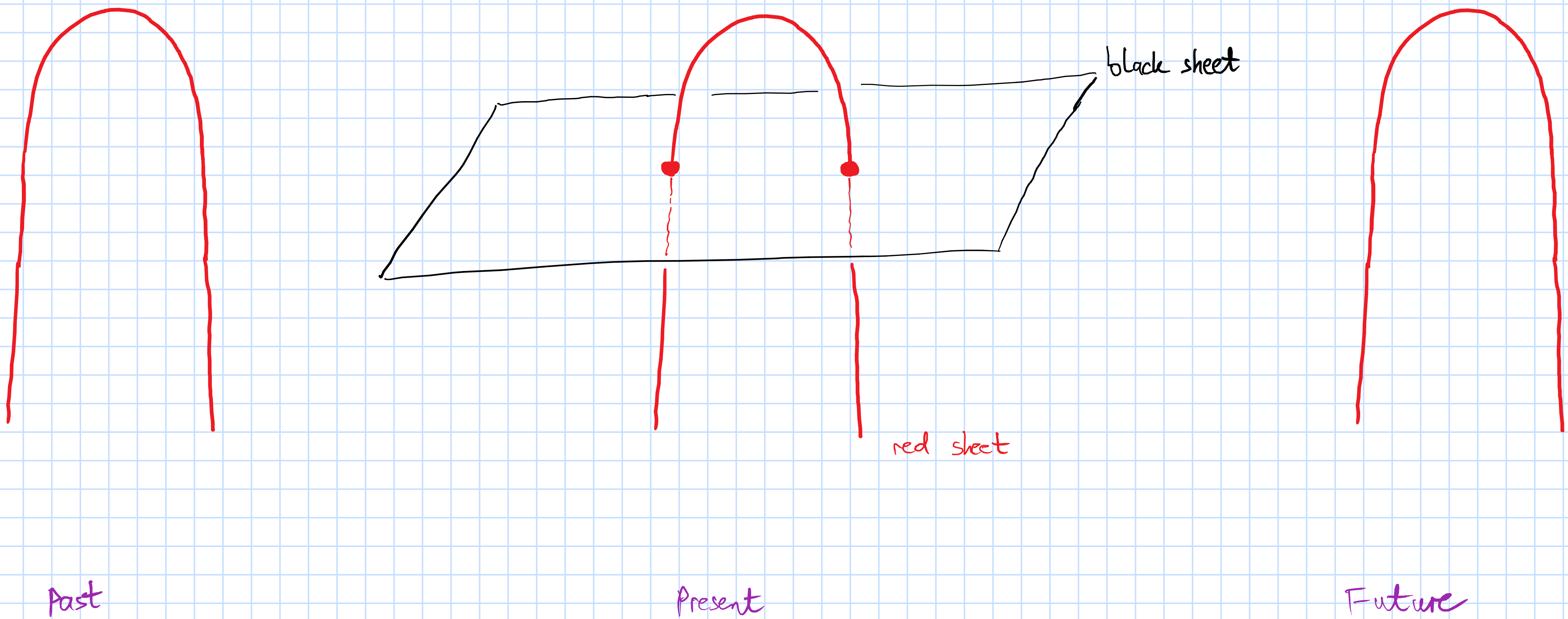
(each introduces a pair of double points)



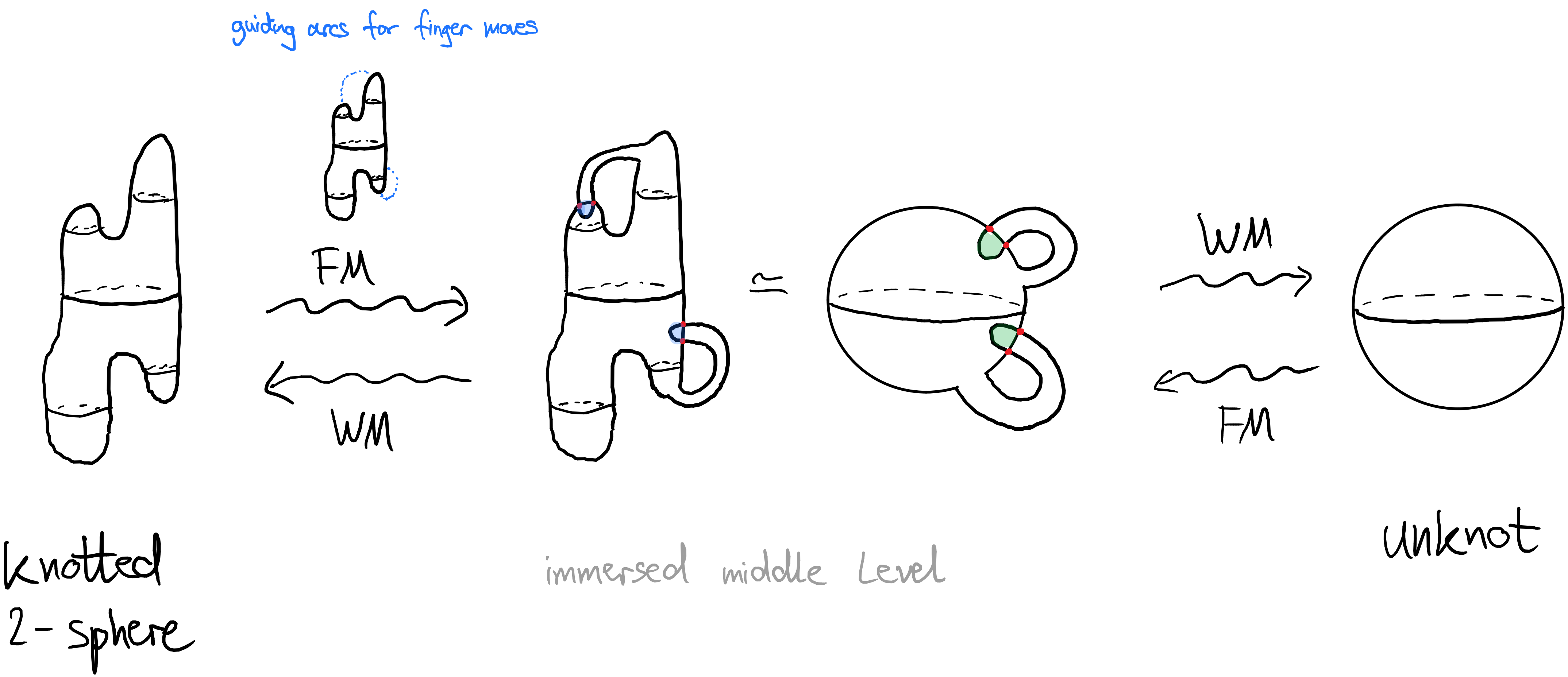
Regular homotopies of surfaces in 4-manifolds

(homotopies through immersions)

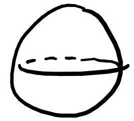
Sequence of Finger-moves and $\boxed{\text{Whitney-moves}}$
(removes two double points)



Schematic of a regular homotopy



$\pi_2(S^4) = \{0\}$ $\rightsquigarrow$ any knotted 2-sphere $K: S^2 \hookrightarrow S^4$

is (regularly) homotopic to the unknot 

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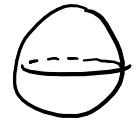
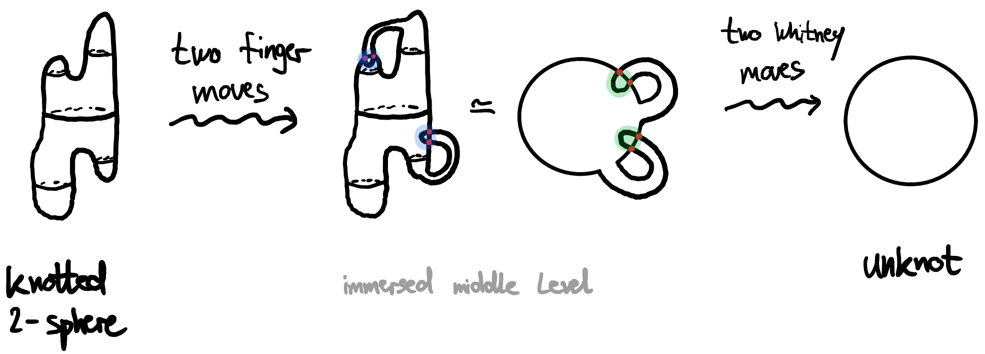


We define the Cassan-Whitney number

$$u_{cw}(K)$$

as the minimal number of Finger moves in a regular homotopy from K to the unknot



If this is the shortest seq. then here: $u_{cw}(K) = 2$

Algebraic effect of stabilization:

$$\boxed{\pi_1\left(S^4 - (S + h^1)\right) \cong \pi_1(S^4 - S) \big/ \langle\!\langle w^{-1} a w = b \rangle\!\rangle}$$

So a stabilization can make two meridians equal

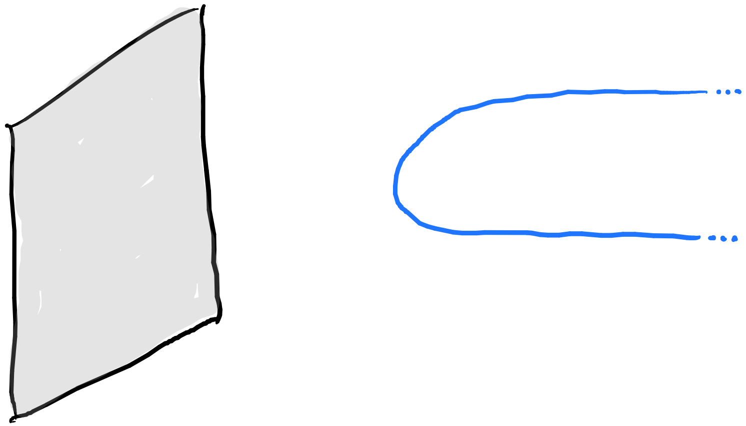
Algebraic effect of finger move:

$$\pi_1(B^4 - S^{\text{fing.}}) \cong \pi_1(B^4 - S) \Big/ \langle\langle [w^{-1} a w, b] \rangle\rangle$$

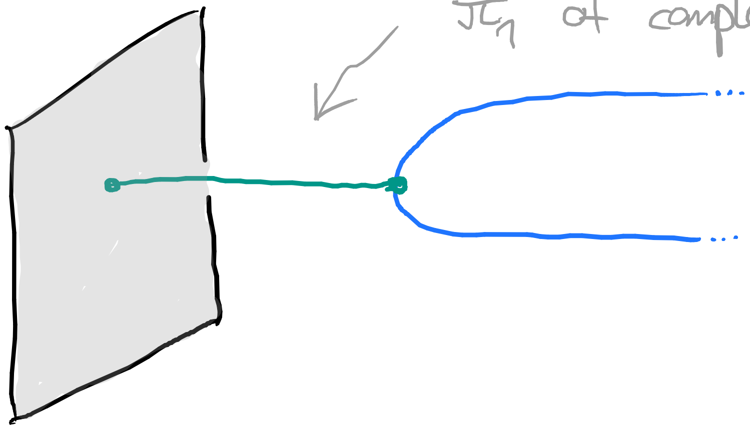
Immersion after finger move on S

A finger move can make a pair of meridians commute

Right before finger move:



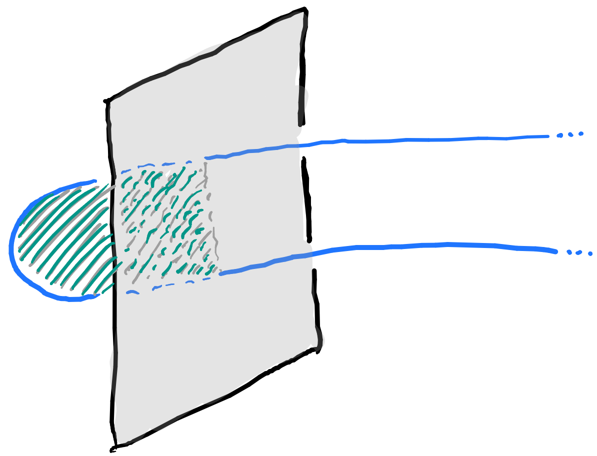
additionally removing this arc doesn't change π_1 of complement



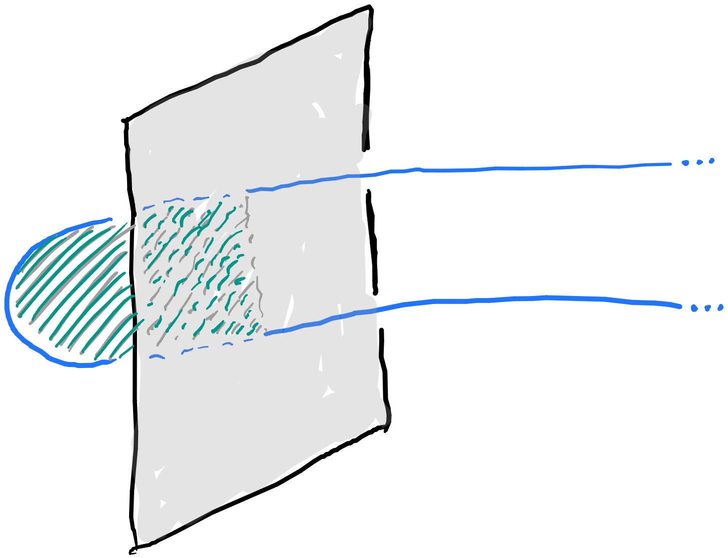
$\simeq$

homotopy
equiv. complements

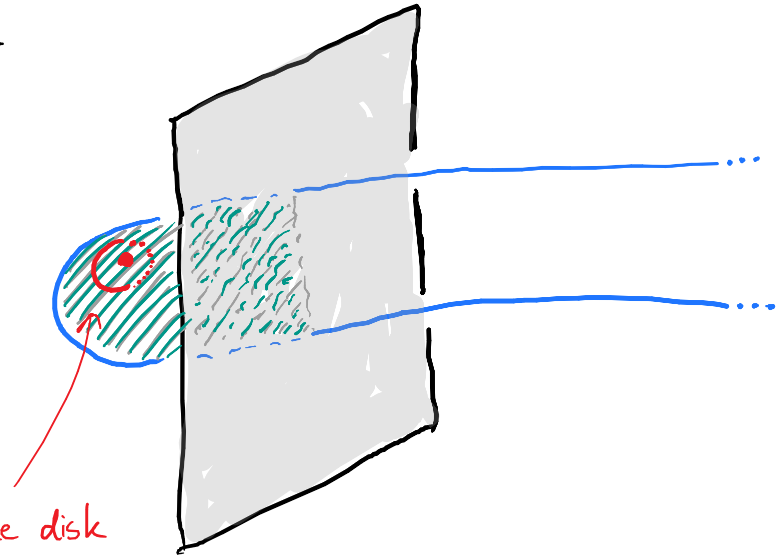
remove black, blue & green disk



remove black, blue & green disk

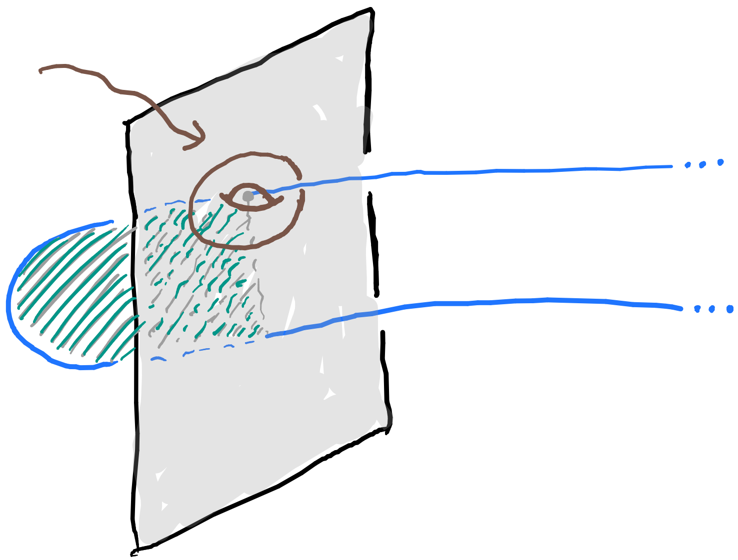


To get the complement after finger move, have to put green disk back



glue disk to this red curve

Clifford torus around double point



Algebraic versions of the unknotting #s:

Finger move: $\pi_1(S^4 - S^{\text{fing.}}) \cong \pi_1(S^4 - S) / \langle\langle [w^{-1}aw, a] \rangle\rangle$

Stabilization: $\pi_1(S^4 - S^{\text{stab}}) \cong \pi_1(S^4 - S) / \langle\langle w^{-1}aw = a \rangle\rangle$

meridians

$a_{cw}(K) :=$ min. # of Finger move relations $[w_i^{-1}aw_i, a_i]$

such that $\pi_1(S^4 - K) / \langle\langle [w_1^{-1}a_1w_1, a_1], [w_2^{-1}a_2w_2, a_2], \ldots, [w_k^{-1}a_kw_k, a_k] \rangle\rangle$

is abelian $(\Leftrightarrow \cong \mathbb{Z})$

$a_{stab}(K) :=$ min. # of 1-handle relations $a_i = w_i^{-1}aw_i$

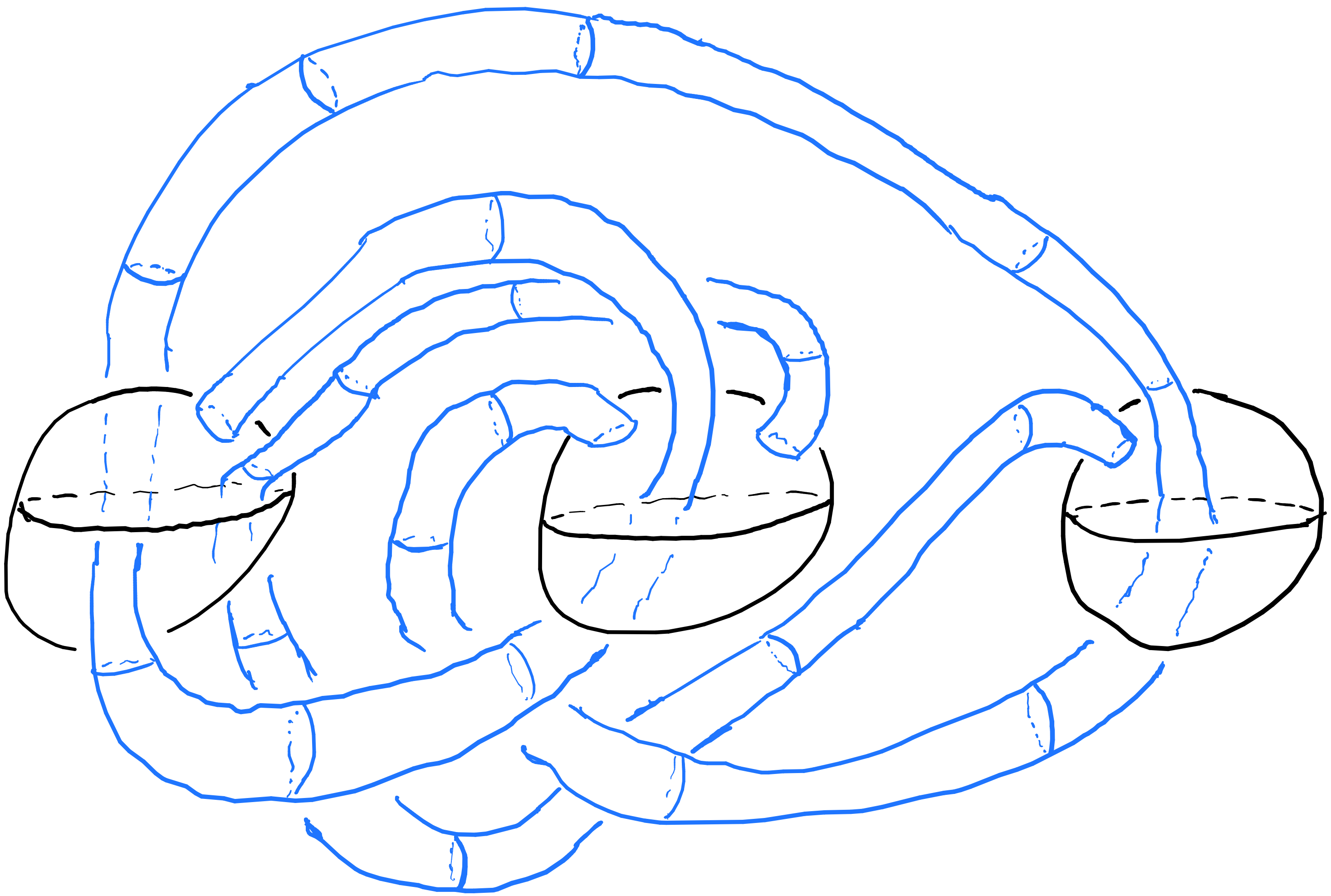
such that ______

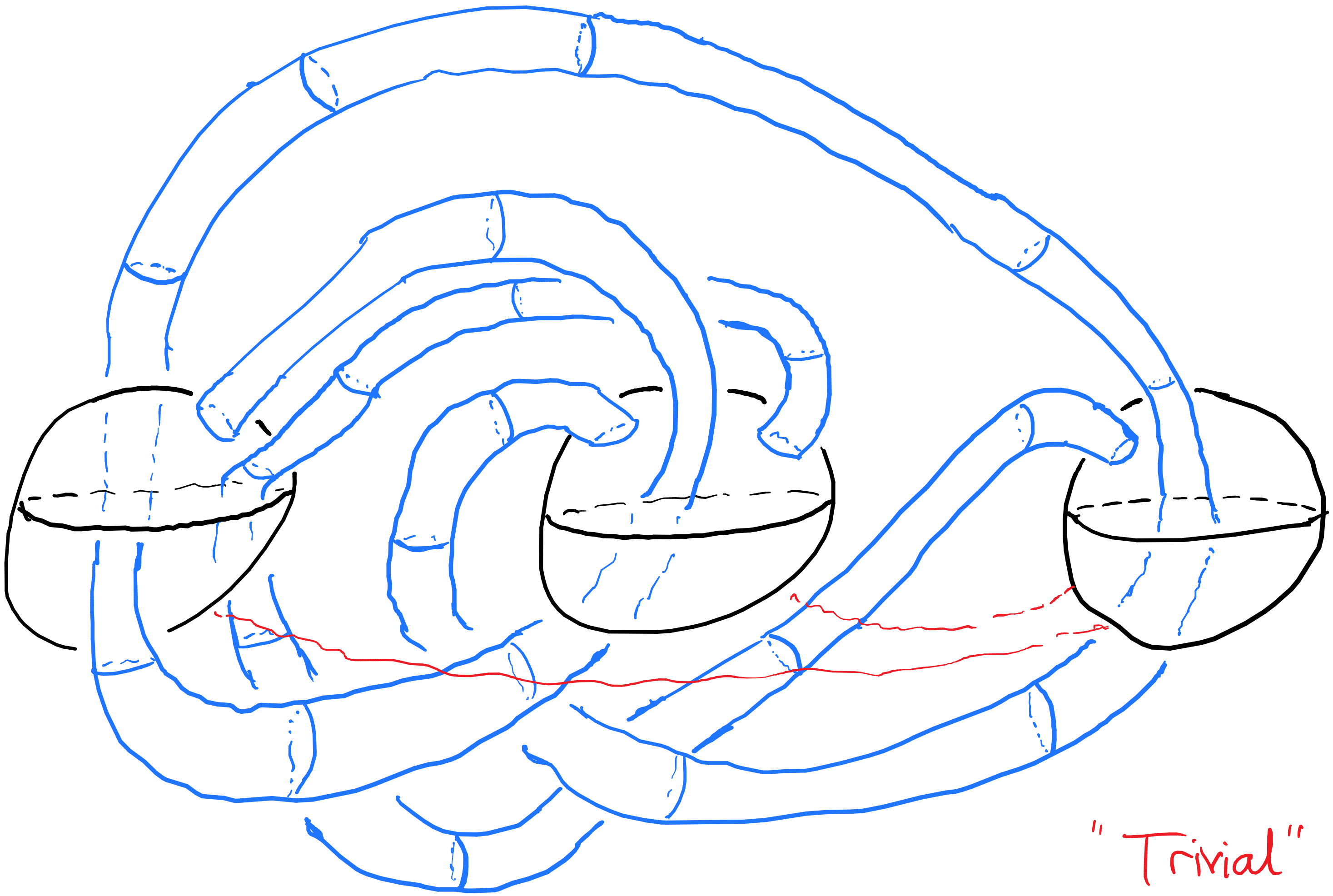
is abelian

Thm: For a ribbon 2-knot K

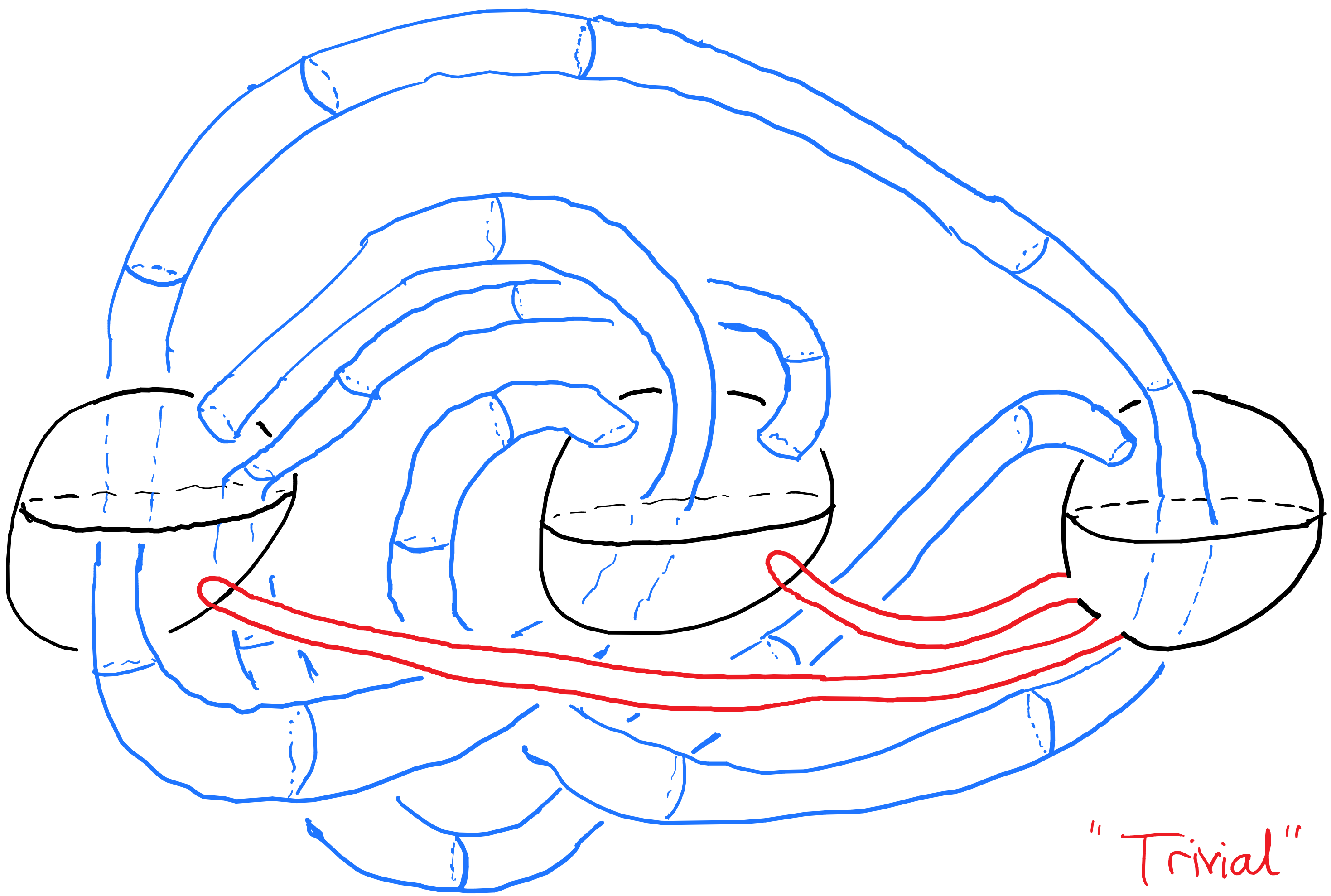
$$u_{CW}(K) \leq \mathrm{fus}(K)$$

The regular homotopy for ribbon 2-knots:





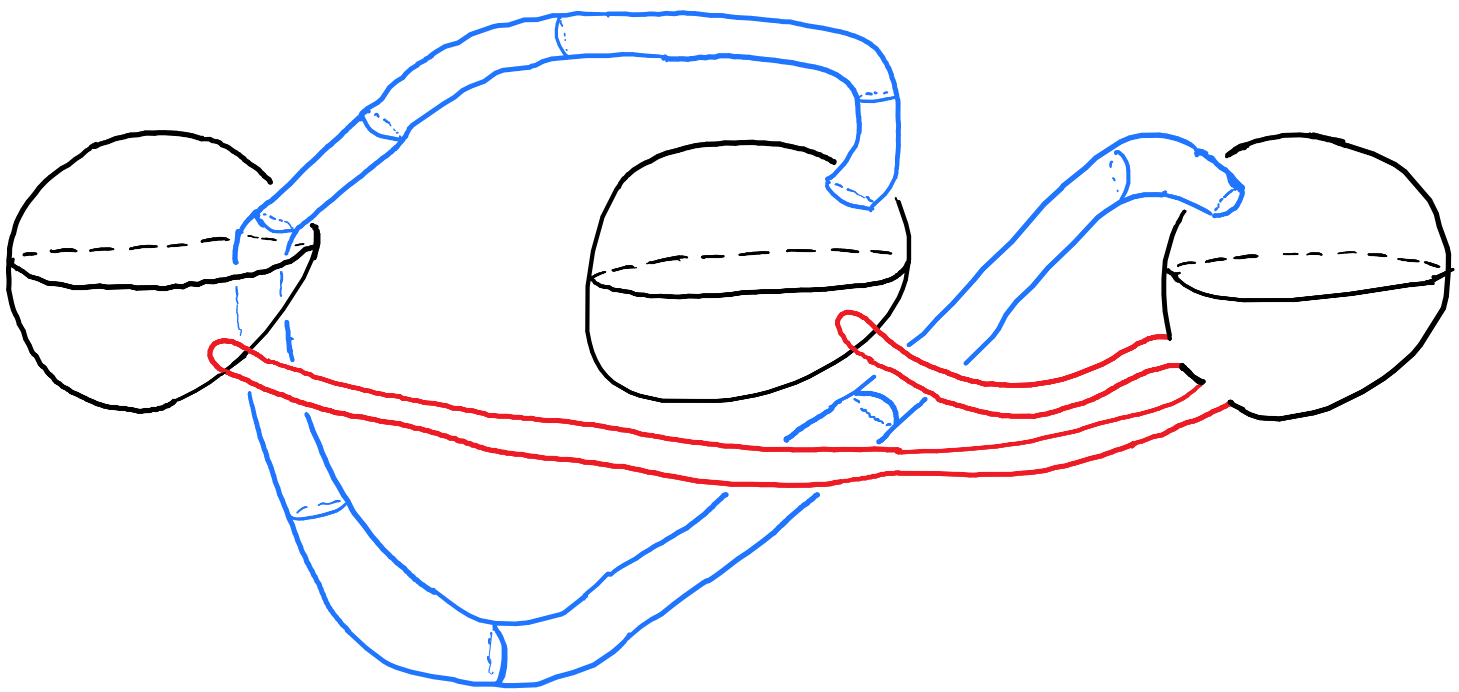
"Trivial" finger moves from the last minimum to all the others

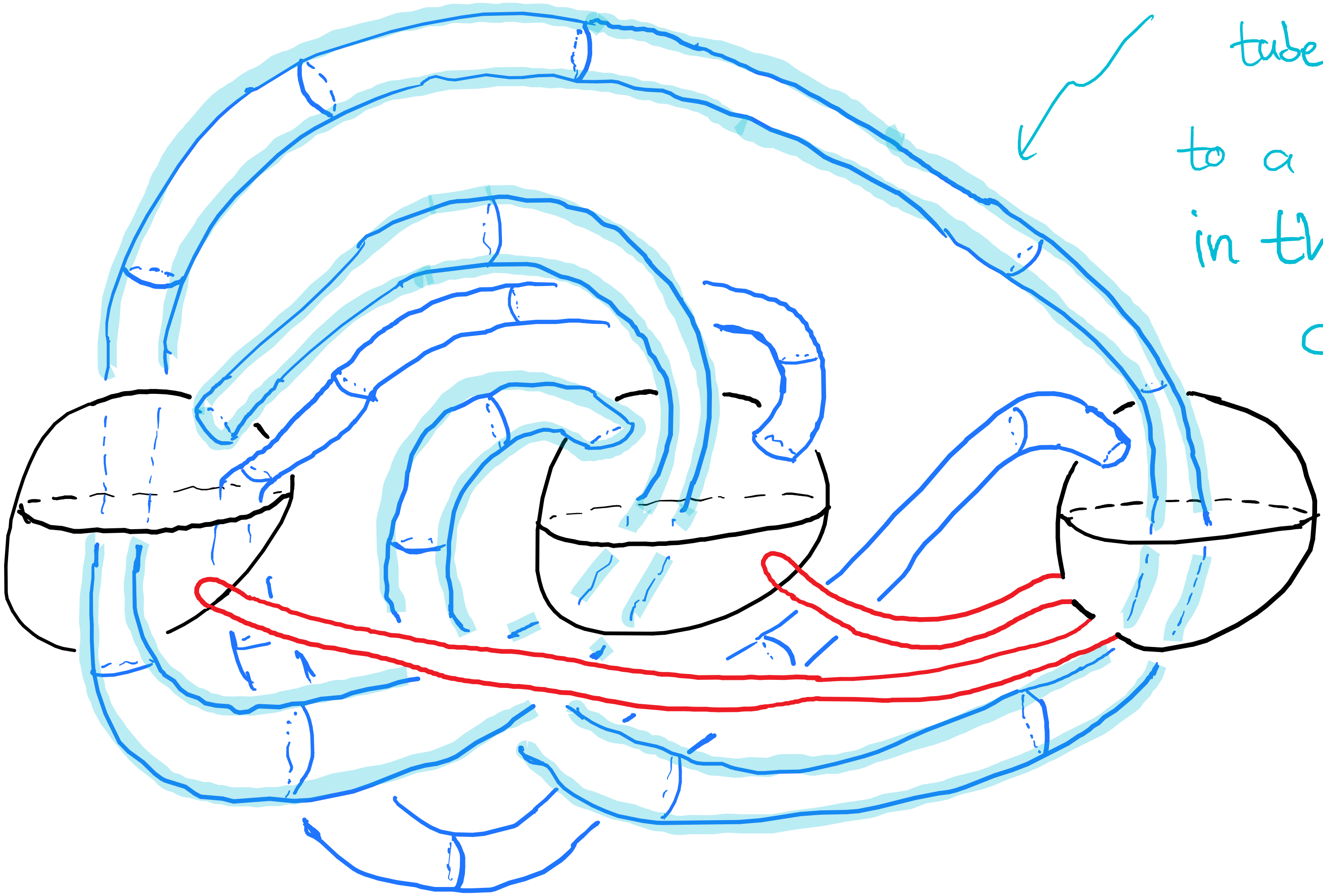


"Trivial" finger moves from the last minimum to all the others

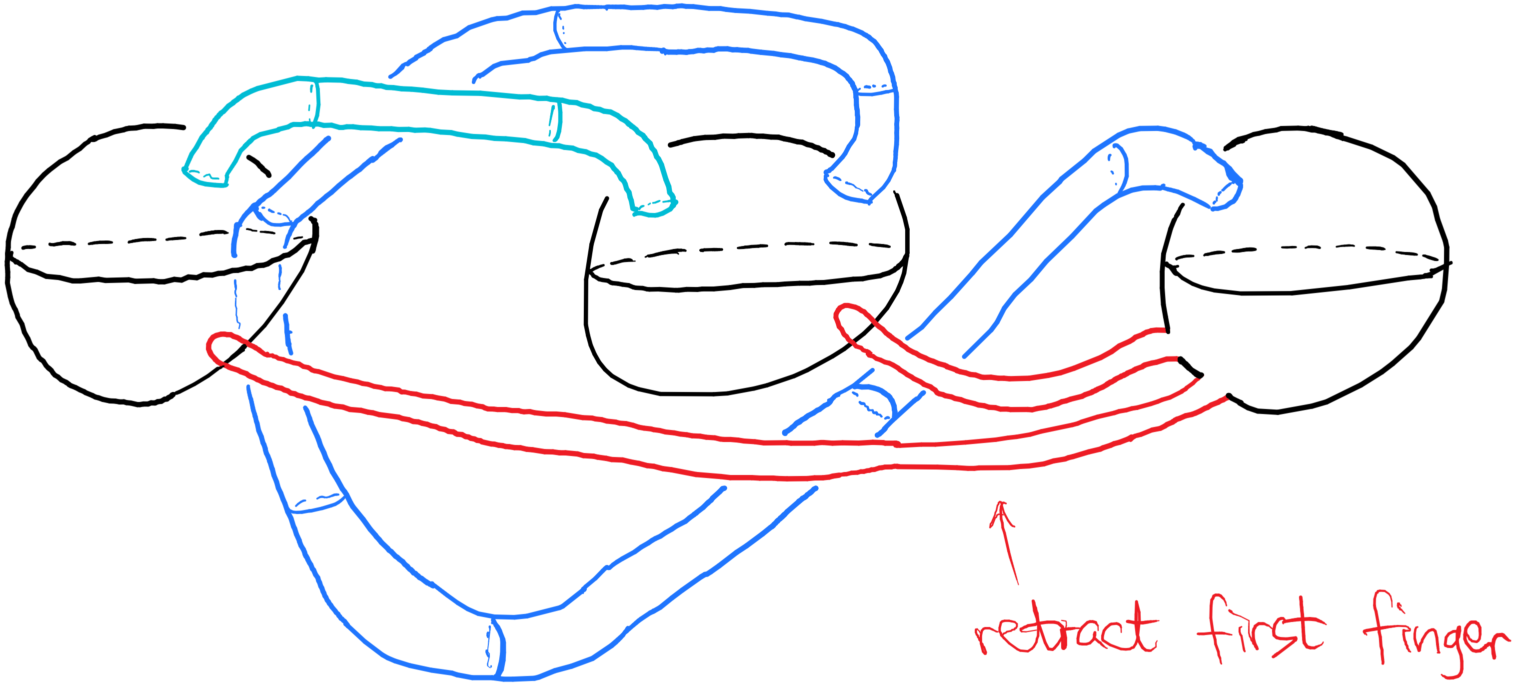
Forget the first fusion tube

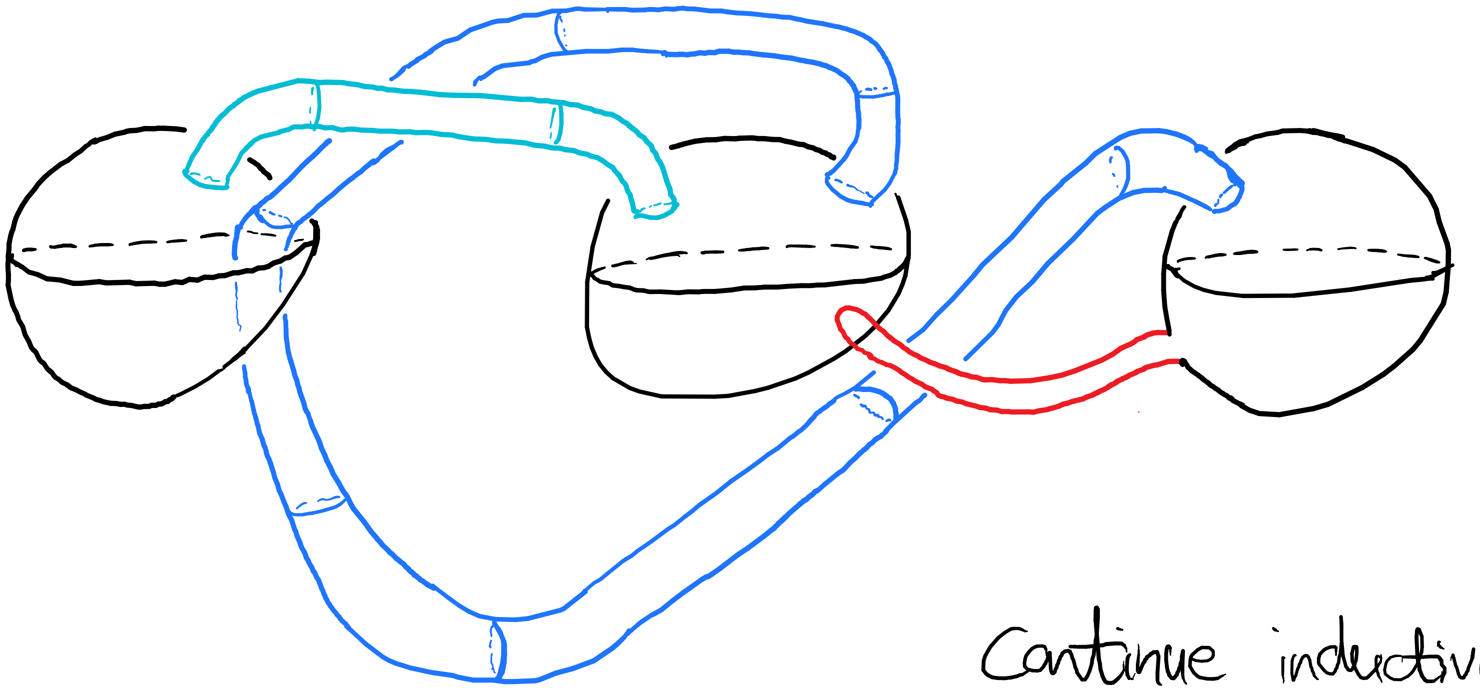
The group of this 2-component link is $\mathbb{Z} \oplus \mathbb{Z}$



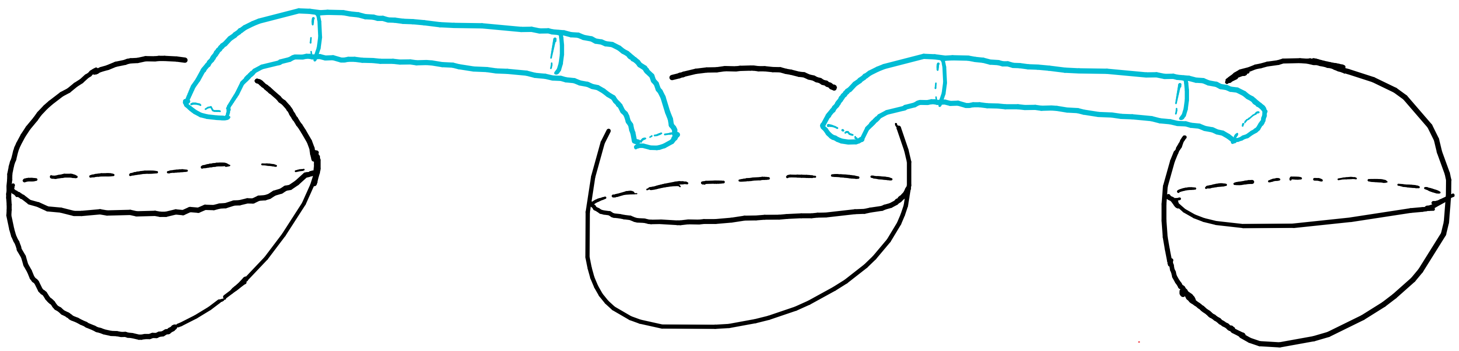


So this complicated tube is isotopic to a trivial fusion in the immersion complement


retract first finger



Continue inductively ...



... Until you have the unknot

□

Some bounds:

this is the best lower bound for the Casson-Whitney number we know of

$$a_{CW}(K) \leq u_{CW}(K)$$

$$\rotatebox{90}{$\leq$}$$

$$a_{stab}(K) \leq u_{stab}(K)$$

$$\rotatebox{90}{$\leq$}$$

minimal size of generating set of Alexander module of K (Nakanishi index)

Some bounds:

$$
\begin{array}{ccc}
a_{cw}(K) & \leq & u_{cw}(K) \\
\rotatebox{90}{$\leq$} & & ?? \\
a_{stab}(K) & \leq & u_{stab}(K)
\end{array}
$$

Oliver Singh's paper was very inspirational

DISTANCES BETWEEN SURFACES IN 4-MANIFOLDS

OLIVER SINGH

Abstract. If Σ and Σ' are homotopic embedded surfaces in a 4-manifold then they may be related by a regular homotopy (at the expense of introducing double points) or by a sequence of stabilisations and destabilisations (at the expense of adding genus). This naturally gives rise to two integer-valued notions of distance between the embeddings: the singularity distance $d_{\mathrm{sing}}(\Sigma, \Sigma')$ and the stabilisation distance $d_{\mathrm{st}}(\Sigma, \Sigma')$. Using techniques similar to those used by Gabai in his proof of the 4-dimensional light-bulb theorem, we prove that $d_{\mathrm{st}}(\Sigma, \Sigma') \leq d_{\mathrm{sing}}(\Sigma, \Sigma') + 1$.

1. Introduction

Let X be a smooth, compact, orientable 4-manifold, possibly with boundary. Let Σ, Σ' be connected, oriented, compact, smooth, properly embedded surfaces in X. We say that Σ' is a *stabilisation* of Σ if there is an embedded solid tube $D^1 \times D^2 \subset X$ such that $\Sigma \cap (D^1 \times D^2) = \{0, 1\} \times D^2$, and Σ' is obtained from Σ by removing these two discs and replacing them with $D^1 \times S^1$, as in Figure 1, and then smoothing corners. In this situation we say that Σ is a *destabilisation* of Σ'.

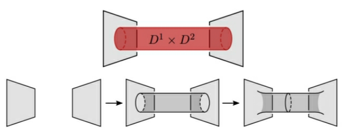


Figure 1. A stabilisation. Given $D^1 \times D^2 \subset X$ which intersects Σ on $S^0 \times D^2$, we remove the two discs $S^0 \times D^2$, add the tube $D^1 \times S^1$, then smooth corners.

Definition 1.1. Given Σ, Σ' as above, both of genus g, define the *stabilisation distance* between Σ and Σ' to be

$$d_{\mathrm{st}}(\Sigma, \Sigma') = \min_{\mathbb{S}} |\max\{g(P_1), \ldots, g(P_k)\} - g|,$$

where $\mathbb{S}$ is the set of sequences $P_1, \ldots, P_k$ of connected, oriented, embedded surfaces where $\Sigma = P_1$, $\Sigma' = P_k$ and P_{i+1} differs from P_i by one of, i) stabilisation, ii) destabilisation, or iii) ambient isotopy. If no such sequence exists we declare $d_{\mathrm{st}}(\Sigma, \Sigma') = \infty$.

1

By carefully manipulating the regular homotopies to the unknot, we can show

$$\boxed{u_{stab}(K) \leq u_{cw}(K) + 1}$$

[the smooth unknotting conjecture would imply that the +1 is not necessary]

and

$$\boxed{u_{cw}(K) = 1 \Rightarrow u_{stab}(K) = 1}$$

Have examples with $\underset{=1}{u_{stab}(K)} \neq \underset{=2}{u_{cw}(K)}$

Used $a_{cw}(K)$ to find the Lower bound by showing that one finger move relation is not enough to abelianize the group:

positive generator of the evaluation of the Alexander ideal at $t=-1$

Thm.: For K_1, K_2 2-knots with determinants $\Delta(K_i)|_{-1} \neq 1$ have $u_{cw}(K_1 \# K_2) \geq 2$

Pf. sketch that $u_{cw}(K_1 \# K_2) \geq 2$:

Will show that a relation of the form $[\text{mer.}, w^{-1}\,\text{mer.}\, w]$ does not abelianize $\pi(K_1 \# K_2)$

- Determinant condition $\rightsquigarrow$ $\pi K_i \twoheadrightarrow \mathrm{Dih}_{p_i} \cong \mathbb{Z}/p_i \rtimes \mathbb{Z}/2$

- Group of connected sum admits surjection $\pi(K_1 \# K_2) \overset{\phi}{\twoheadrightarrow} \underbrace{(\mathbb{Z}/p_1 * \mathbb{Z}/p_2) \rtimes \mathbb{Z}/2}_{=:G}$

- Enough: Induced image $G / \langle\langle \phi([\text{mer.}, w^{-1}\,\text{mer.}\, w]) \rangle\rangle$ not abelian

- Look at commutator subgroup: Want to show $\mathbb{Z}/p_1 * \mathbb{Z}/p_2 / \langle\langle [z, v^{-1} z v] \rangle\rangle$ is not trivial

  $z = \phi(\text{mer.})$ $v = \phi(w)$

- Rewrite $[z, v^{-1} z v] = [z,v]^2$, show this normally gen.

$\rightsquigarrow$ then use a Freiheitssatz of [Fine, Howie, Rosenberger (1988)] to conclude that $\mathbb{Z}/p_1 * \mathbb{Z}/p_2 / \langle\langle g^2 \rangle\rangle$ is nontrivial for any $g \in \mathbb{Z}/p_1 * \mathbb{Z}/p_2$

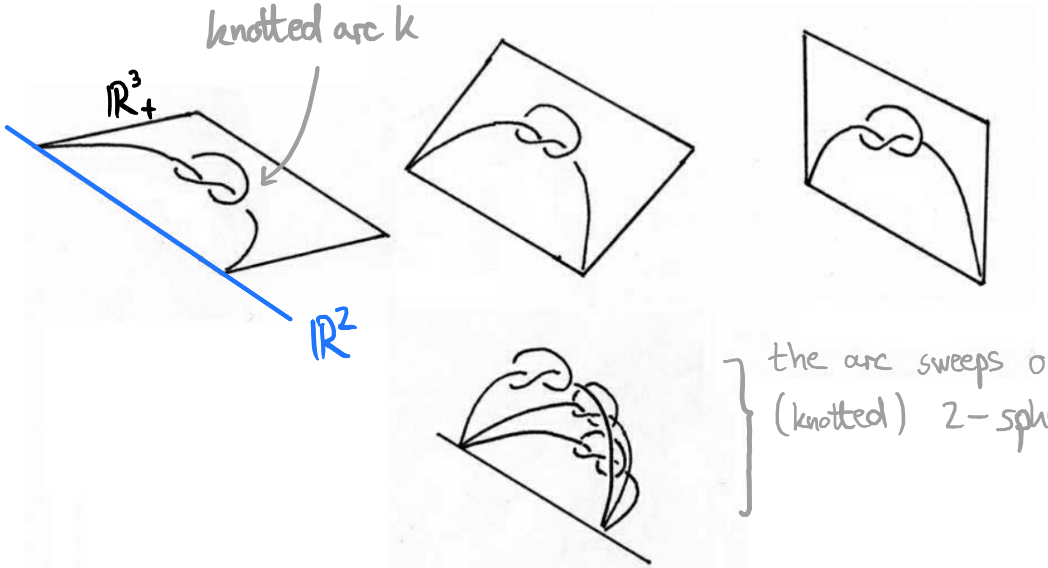
$\square$

Twist-spinning knots: [Artin, Zeeman]

[pictures from Friedman: Knot Spinning]

$\tau^r(k)$ = r-twist spin of a classical knot k

Spinning:



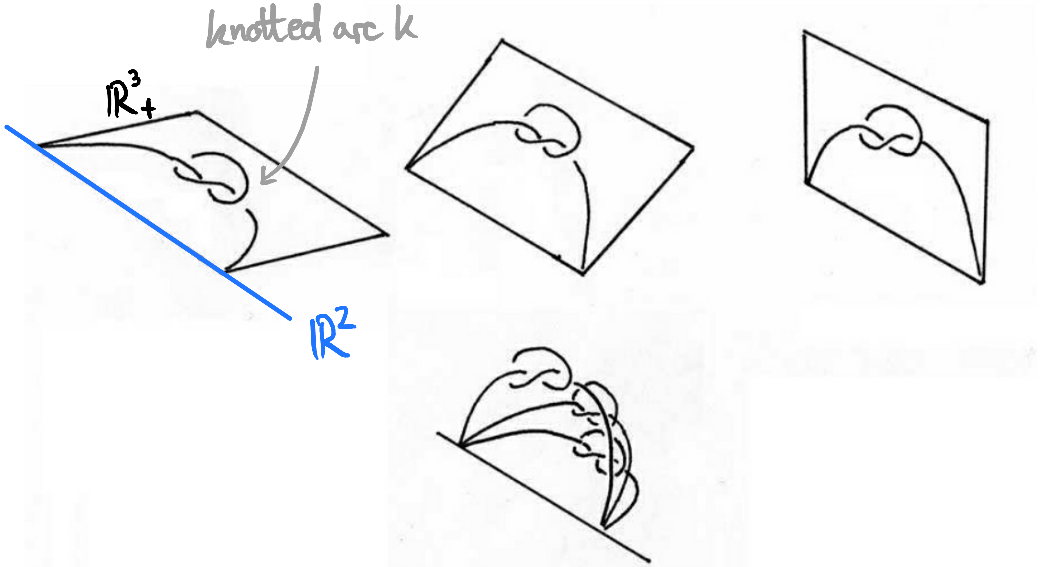
the arc sweeps out a (knotted) 2-sphere in $\mathbb{R}^4 \subset S^4$

Twist-spinning knots: [Artin, Zeeman]

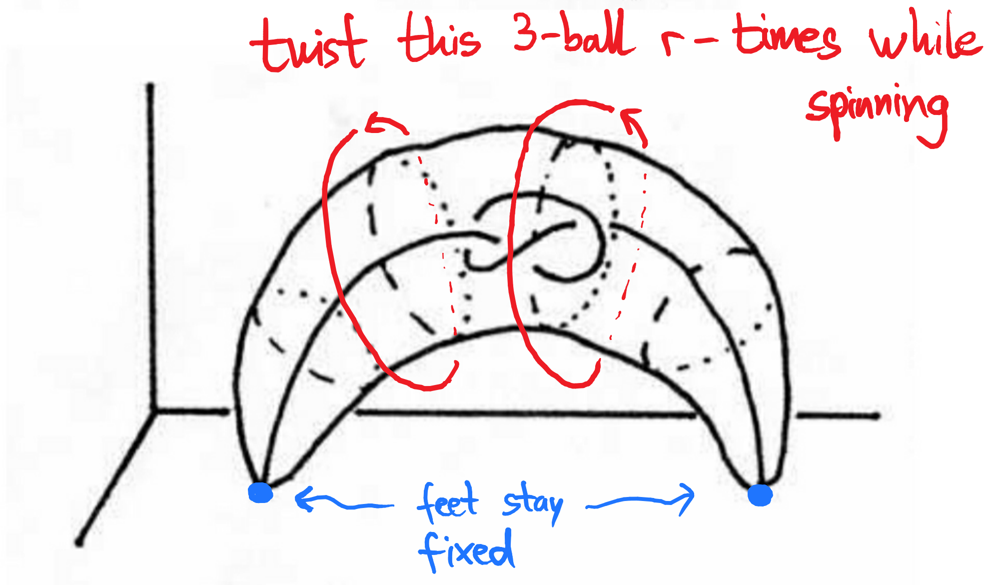
[pictures from Friedman: Knot Spinning]

$$\tau^r(k) = \text{r-twist spin of a classical knot } k$$

Spinning:



Twist - spinning:



Prop.: $$u_{cw}(\tau^n k) \leq u(k)$$

n-twist spin of $k: S^1 \hookrightarrow S^3$

Classical unknotting number of the 1-knot k

Corollary: $a_{cw}(\tau^n k)$ is a lower bound for the classical unknotting number.

Non-additivity

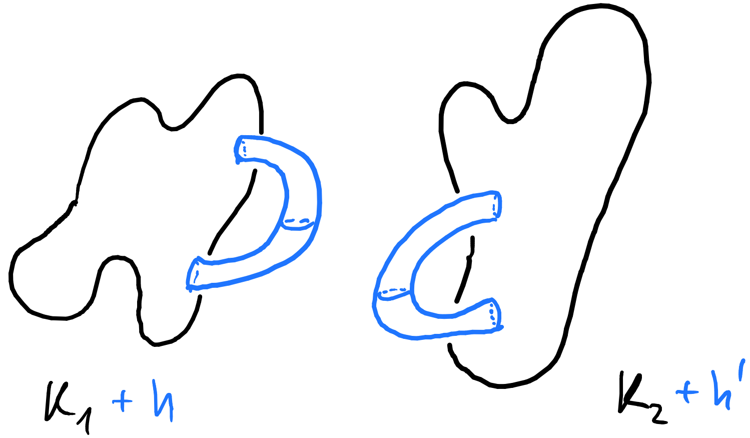
Of course $u_{stab}(K_1 \# K_2) \leq u_{stab}(K_1) + u_{stab}(K_2)$

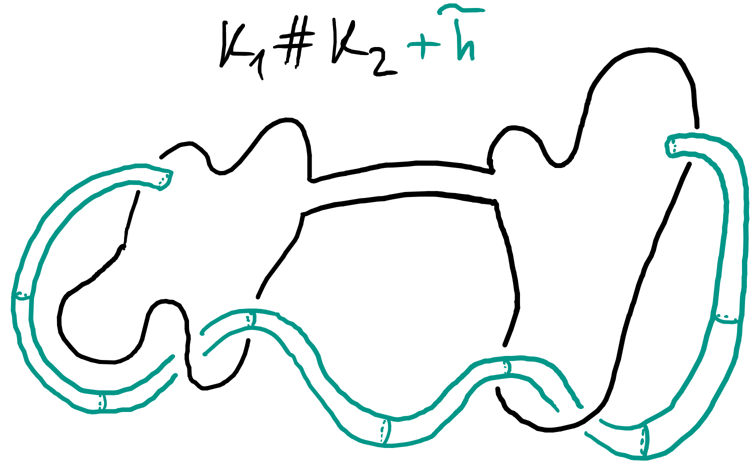
But it can fail to be additive:

Miyazaki has an example of ribbon 2-knots K_1, K_2 with

$$u_{stab}(K_i) = 1,$$

but there is a new 1-handle which transforms $K_1 \# K_2$ into an unknotted torus.





Miyazaki's example:

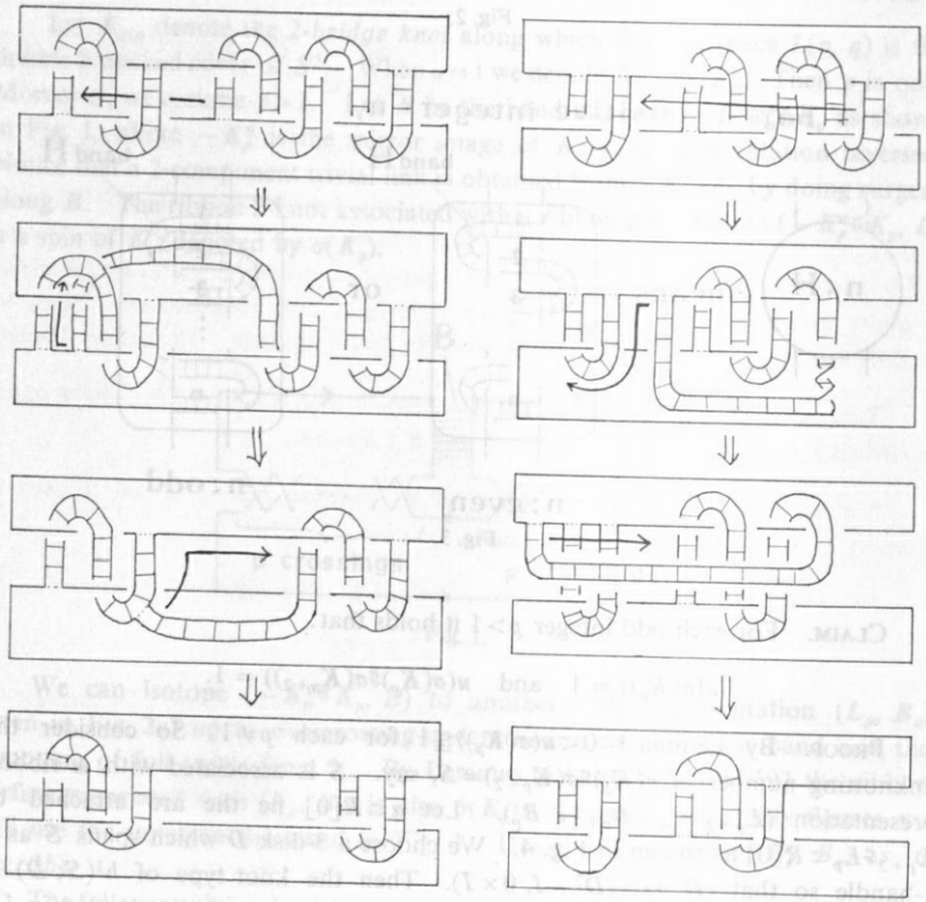
$$\tau(T(2,p) \# T(2,q)) \quad \text{for } \begin{array}{l} q = p+2 \\ q = p+4 \\ q = p+6 \ \text{ if } \gcd(p,p+6)=1 \end{array}$$

MIYAZAKI, K.
KOBE J. MATH.,
3 (1986), 77–85

ON THE RELATIONSHIP AMONG UNKNOTTING NUMBER, KNOTTING GENUS AND ALEXANDER INVARIANT FOR 2-KNOTS

By Katura MIYAZAKI

(Received May 24, 1985)



a) b)

Fig. 6.

He explicitely draws the isotopy of the fusion bands which shows that the torus you get after one stabilization is unknotted.

Miyazaki's example:

$$\tau(T(2,p) \# T(2,q)) \quad \text{for} \quad \begin{array}{l} q = p+2 \\ q = p+4 \\ q = p+6 \quad \text{if } \gcd(p, p+6) = 1 \end{array}$$

$$u_{stab.}\left(\tau(T(2,p) \# T(2,q))\right) = 1$$

From our Lower bound:

$$a_{cw}\left(\tau(T(2,p) \# T(2,q))\right) = 2$$

$$u_{cw}\left(\tau(T(2,p) \# T(2,q))\right) = 2$$

For this example: u_{stab} non-additive but u_{cw} additive

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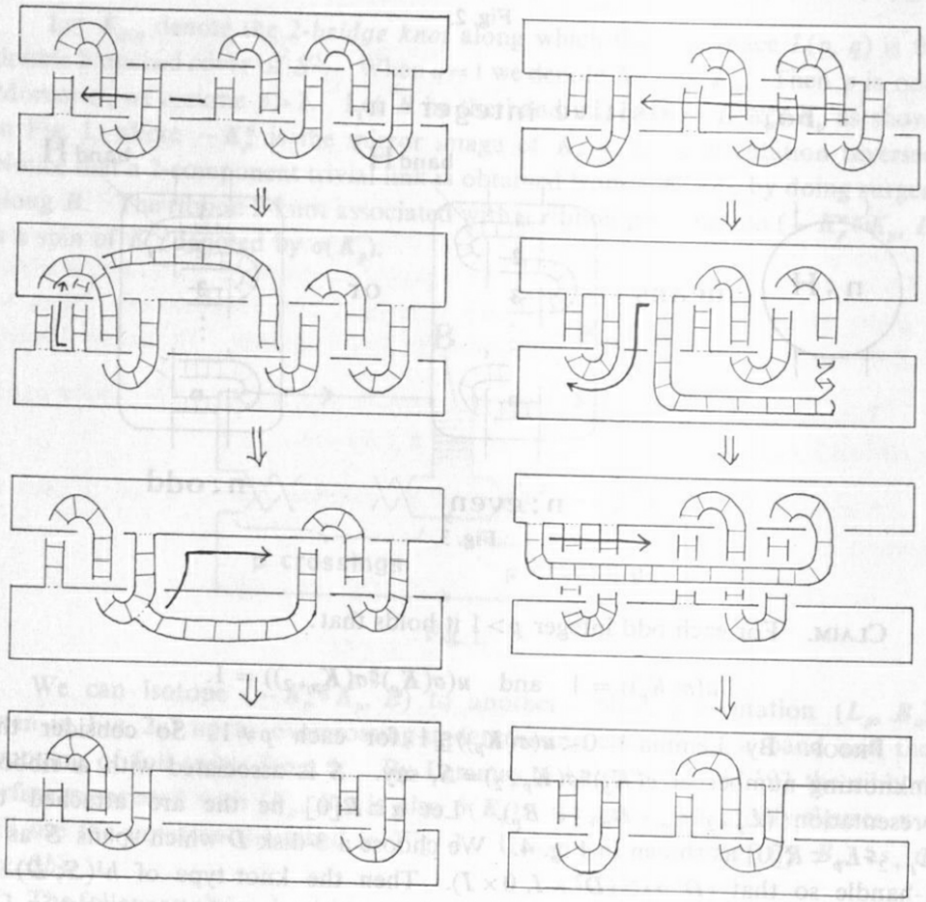


a) b)

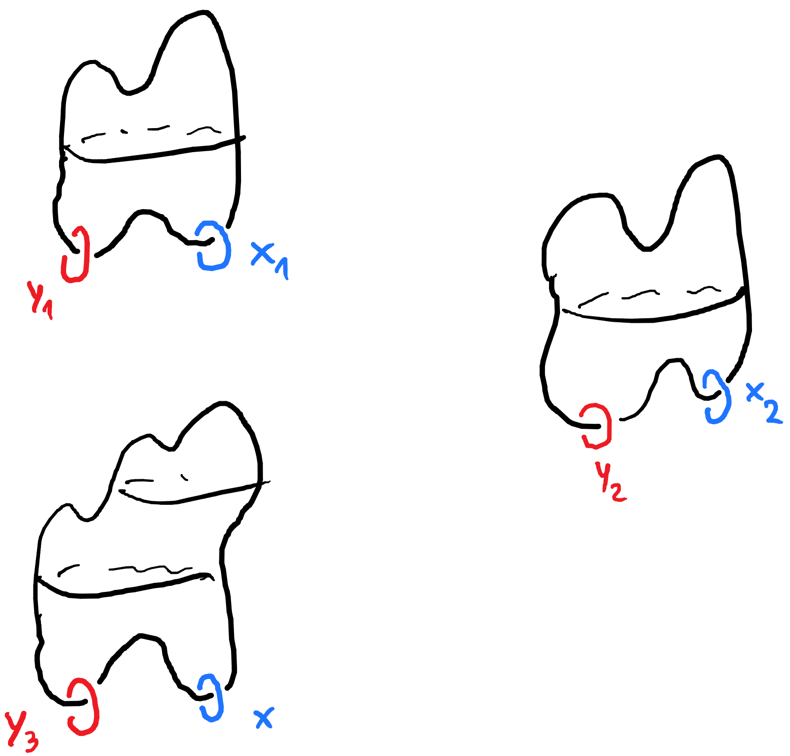
Fig. 6.

"Strong" non-additivity

For K_i non-tr. 2-bridge & for natural number $r_i \geq 2$:

$$\boxed{a_{stab}\left(\tau^{r_i}(K_i) \right) = 1}$$

Kanenobu's example:



"Strong" non-additivity

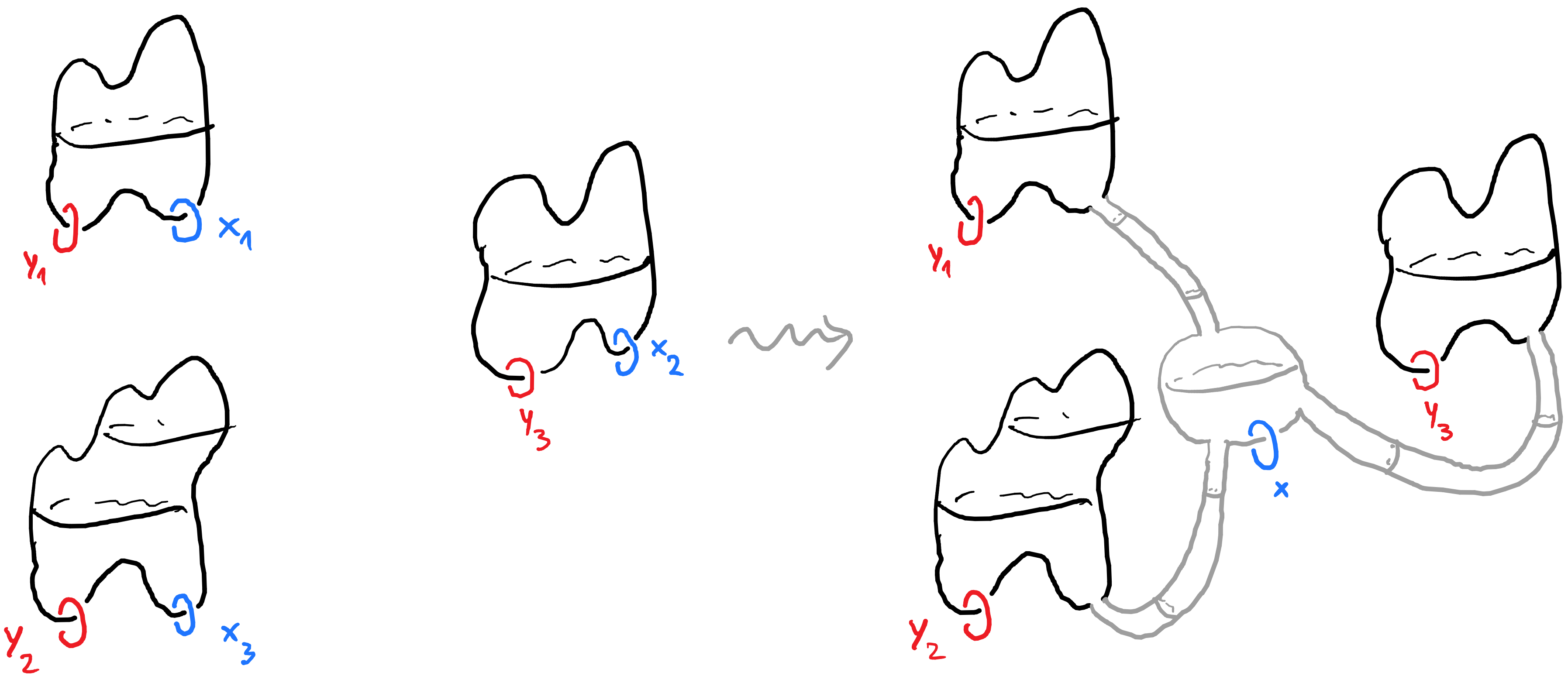
For k_i non-tr. 2-bridge & for natural number $r_i \geq 2$:

$$\boxed{a_{stab}\left(\tau^{r_i}(k_i)\right) = 1}$$

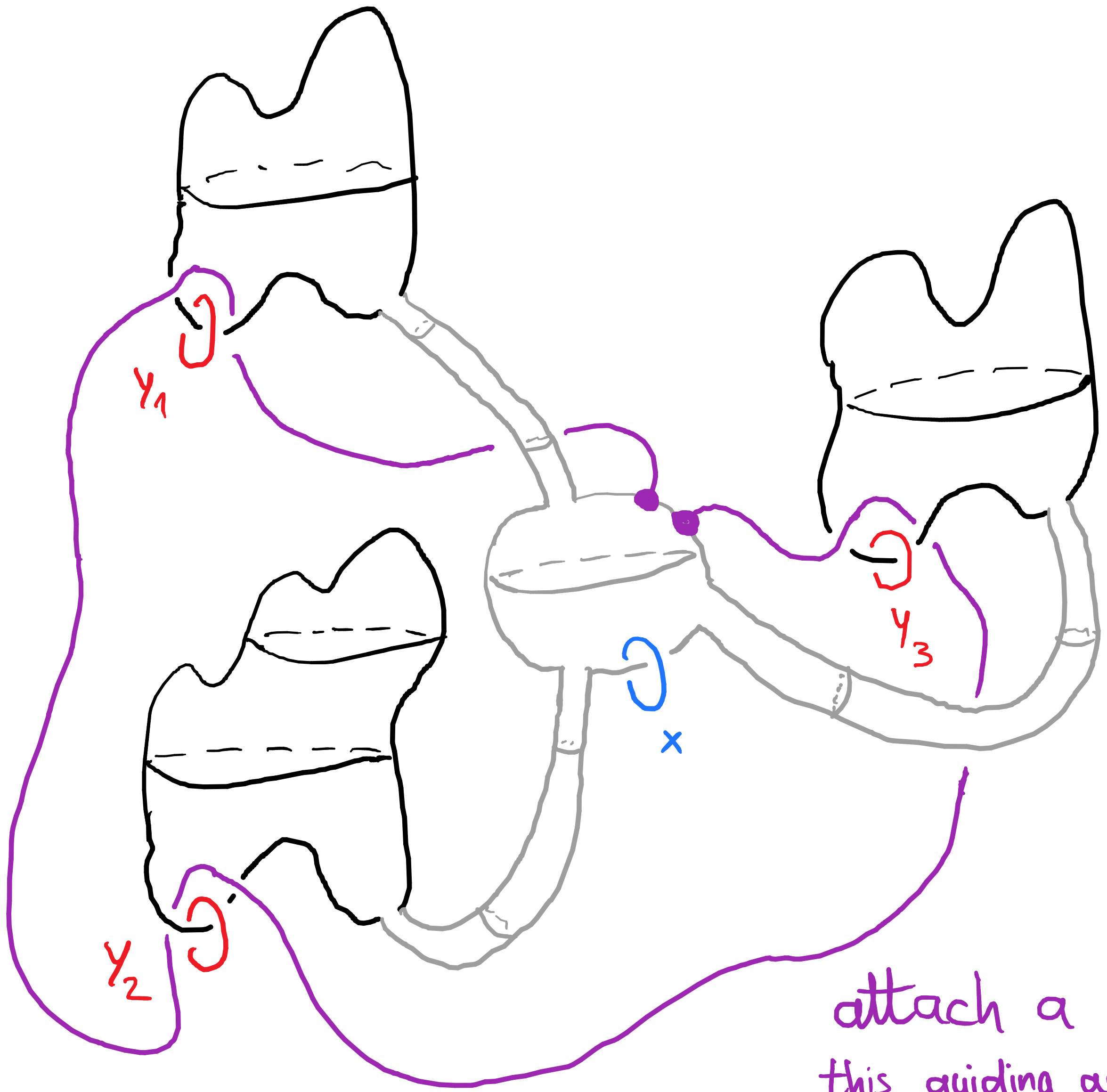
Lemma: [Kanenobu] For $r_1, \dots, r_n \geq 2$ coprime integers, $k_1, \dots, k_r$ 2-bridge

$$a_{stab}\left(\tau^{r_1}(k_1) \# \dots \# \tau^{r_n}(k_n)\right) = 1$$

Kanenobu's example:

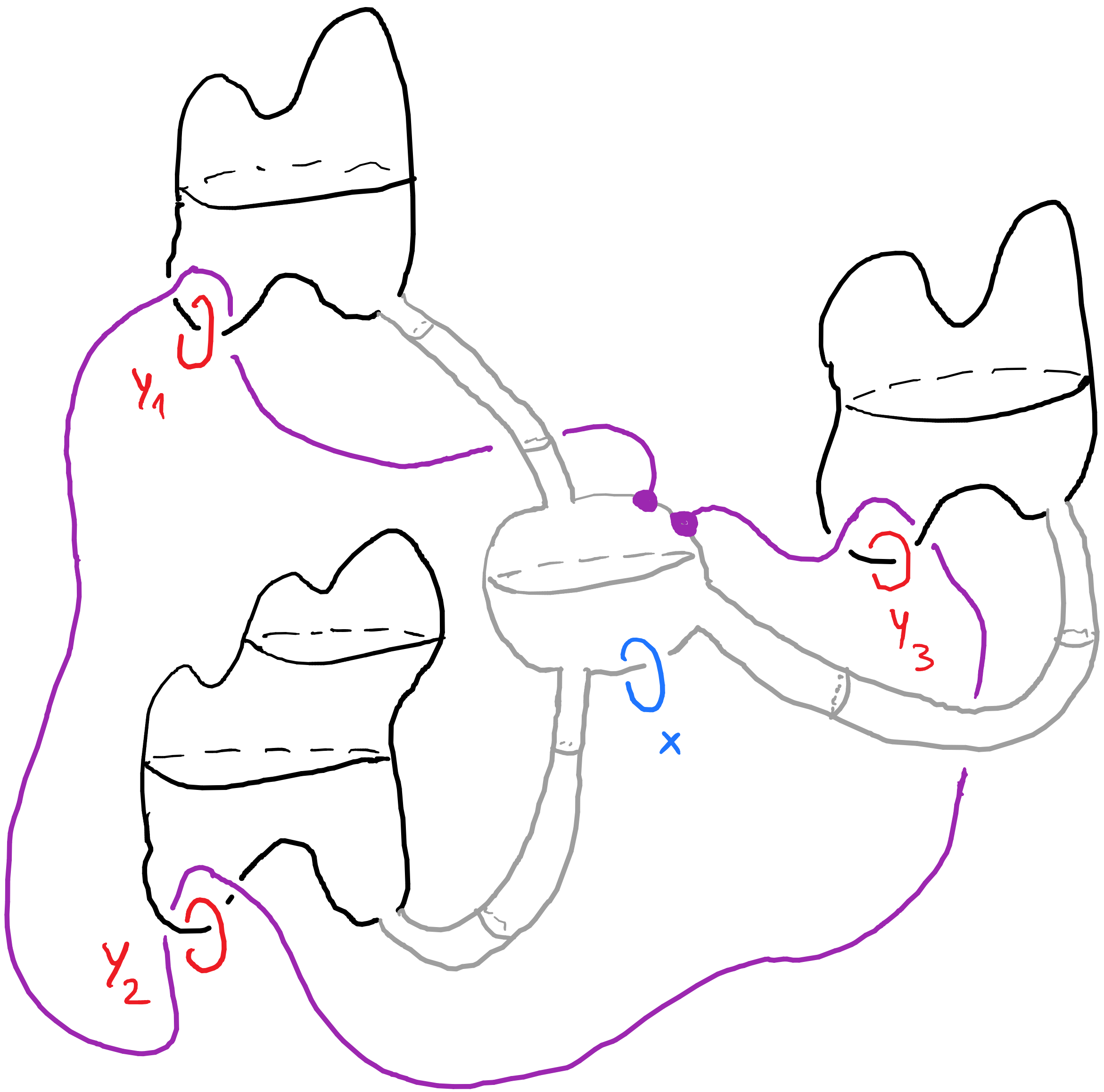


<u>Fact:</u> Adding the relation $[x, y_1 \cdot y_2 \cdot \ldots \cdot y_n]$ abelianizes the group



attach a 1-handle along this guiding arc to get a torus with $\pi_1(\text{compl.}) \cong \mathbb{Z}$

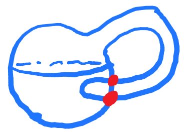
- is it unknotted?



Similar π_1-calculation works for result of Finger move along this arc

(the immersion complement has $\pi_1 \cong \mathbb{Z}$)

Is it isotopic to the "standard position?"



[END]