Rasmussen's s-invariant and the Local Thom conjecture

(Milnor conjecture)

i.e. Rasmussen's combinatorial proof of the Local Thom conjecture using Khovanov homology

[Turner: Five Lectures in Khovanov homology]

[Turner: A Hitchhiker's guide to Khovanov homology]

2020-05-13, IMPRS-seminar @ MPIM Bonn

Plan:

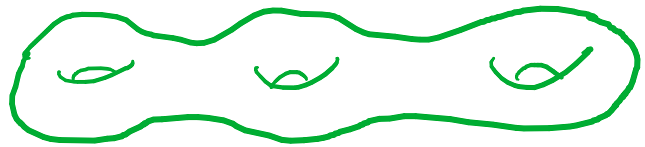
-) Intro: (Local) Thom conjecture
-) Construction of s-invariant from a spectral sequence relating Khovanov homology with Lee homology
-) Applications

Genus of algebraic curves in $\mathbb{CP}^2$:

$$[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$$

fundamental class of a complex line in the projective plane

We would like to "see" these second homology classes as embedded surfaces



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derivatives of F don't vanish all at the same time on the zero-set

Smooth algebraic curve of degree d

zero-set $\{[x:y:z] \mid F(x,y,z)=0\} \subset \mathbb{CP}^2$ of a homogeneous polynomial F of degree d

Ex.: $\{[x:y:z] \mid x^d+y^d+z^d=0\} \subset \mathbb{CP}^2$

Genus - degree formula: C algebr. of degree d, then

[1800s, Riemann-Hurwitz, adjunction formula, ...]

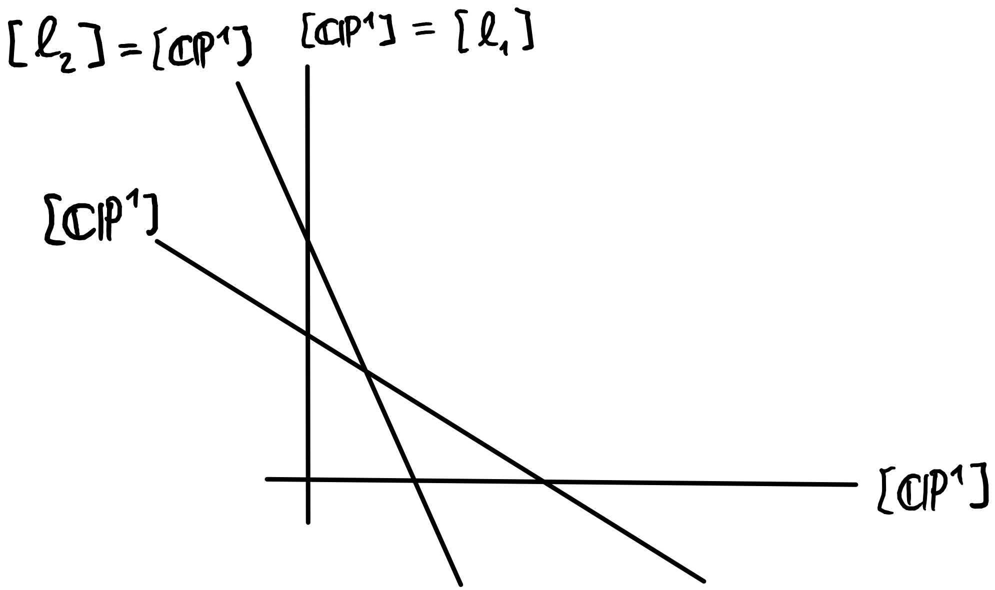
$$\text{genus}(C) = \frac{(d-1)(d-2)}{2}$$

It's enough to show this for our favourite curve of degree d ("space of degree d curves" is path connected, continuity, ...)

Let's start with a singular situation: Generic product of Lines ...

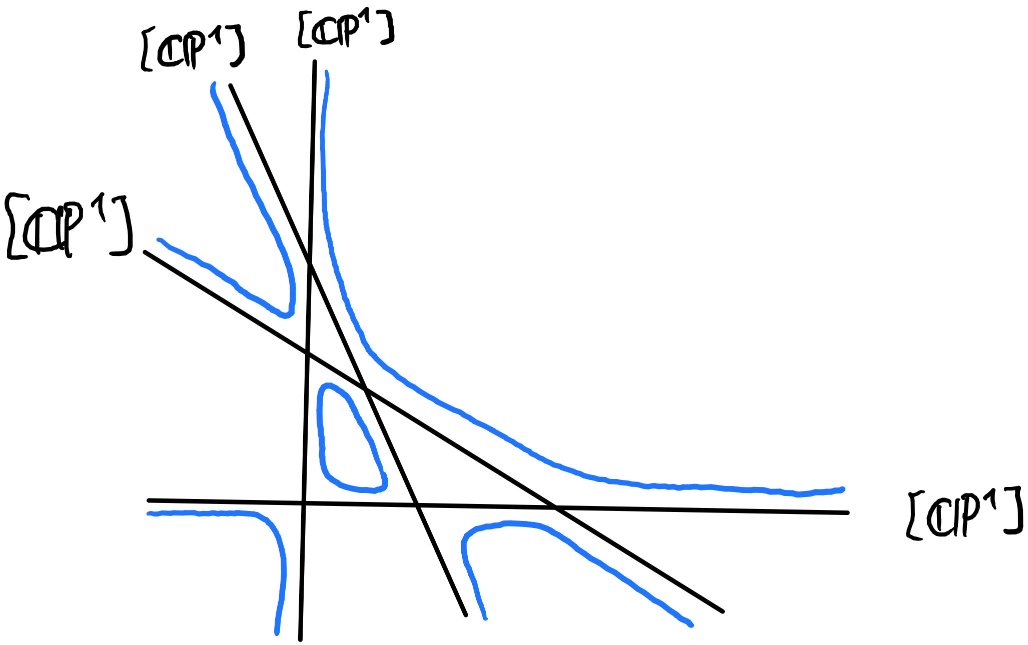
$$(x-\lambda_1 z)\cdot(x-\lambda_2 z)\cdot \ldots \cdot(x-\lambda_d z) = 0 \qquad \lambda_i \in \mathbb{C}$$

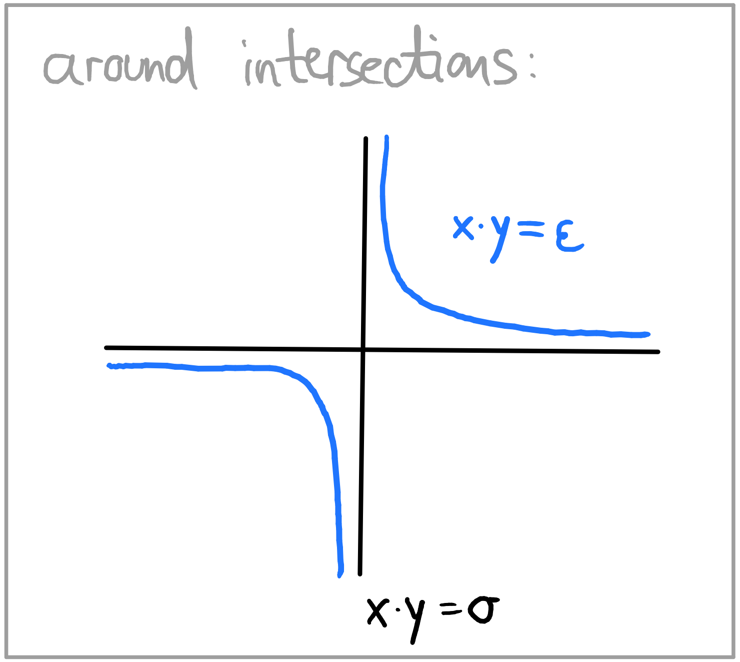
$$\ell_1 \cdot \ell_2 \cdot \ldots \cdot \ell_d = 0 \qquad (x-\lambda_i z) \text{ generic linear forms}$$



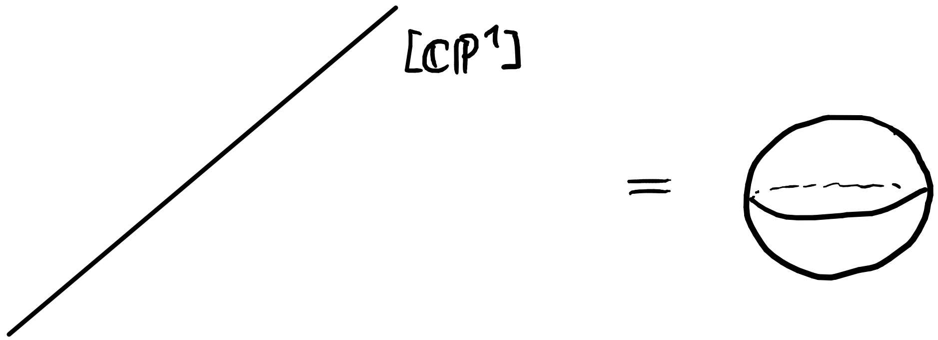
... and perturb this a little to make it smooth

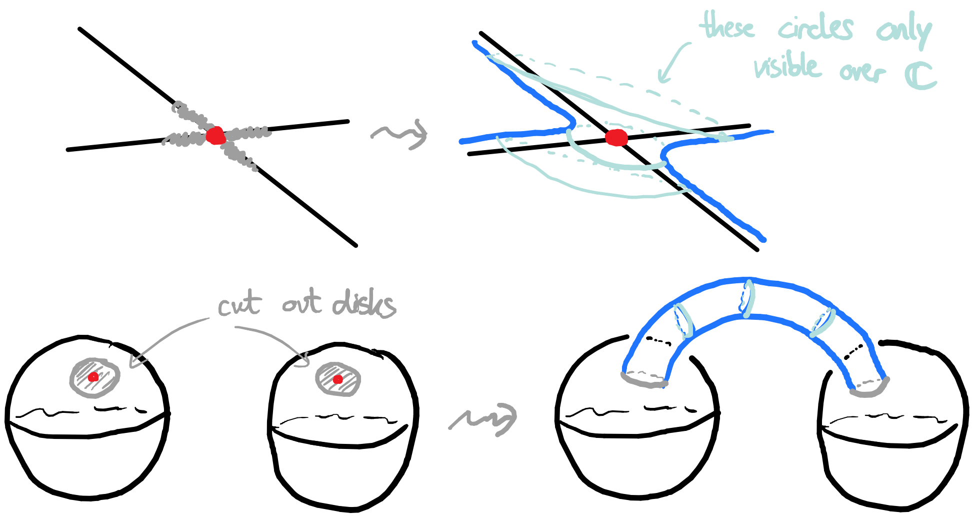
$$(x-\lambda_1 z)\cdot(x-\lambda_2 z)\cdot\ldots\cdot(x-\lambda_d z) \rightsquigarrow (x-\lambda_1 z)\cdot(x-\lambda_2 z)\cdot\ldots\cdot(x-\lambda_d z)+\varepsilon\cdot y^d$$





Dictionary:





Thom conjecture

If we drop the requirement that our surfaces should be algebraic, can we find representing surfaces of lower genus?

$\boxed{\text{NO!}}$

Thm. [Kronheimer, Mrowka (1994)]

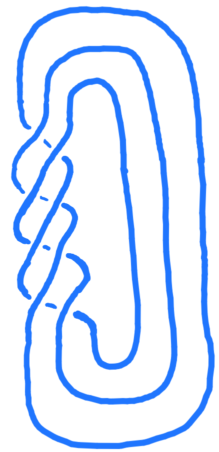
$S \subset \mathbb{CP}^2$ smoothly embedded, oriented, connected surface in $\mathbb{CP}^2$ of positive degree $d = [S] \in H_2(\mathbb{CP}^2) \cong \mathbb{Z}$.

not necess. algebraic

Then

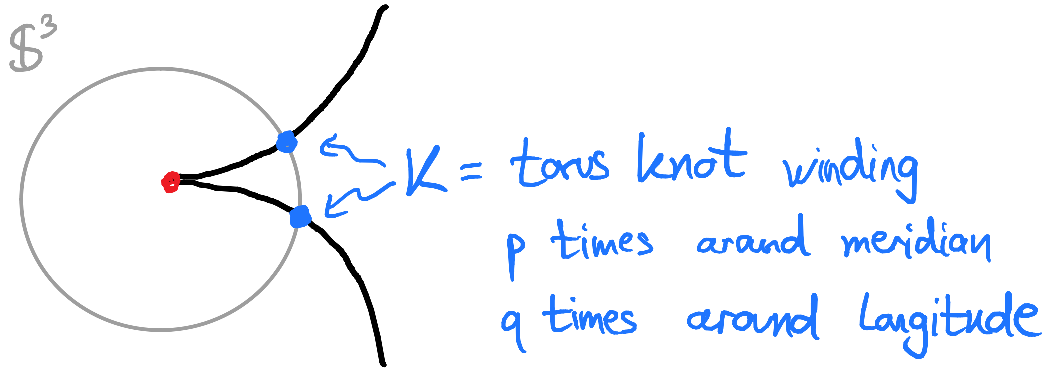
$$\text{genus}(S) \geq \frac{(d-1)(d-2)}{2}$$

Local Thom/Milnor conjecture: The smooth slice genus of the torus knots $T(p,q)$.

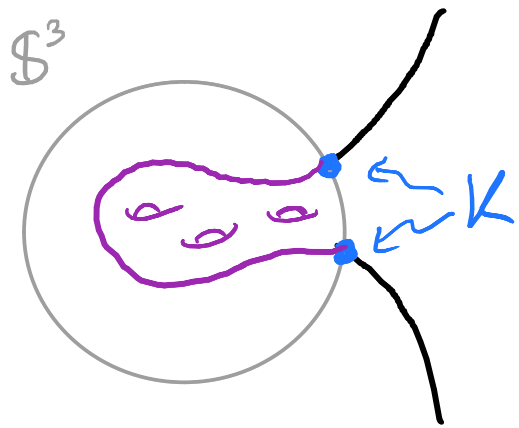


$$V := \{(x,y) \in \mathbb{C}^2 \mid x^p + y^q = 0\} \subset \mathbb{C}^2$$

singularity at (0,0):

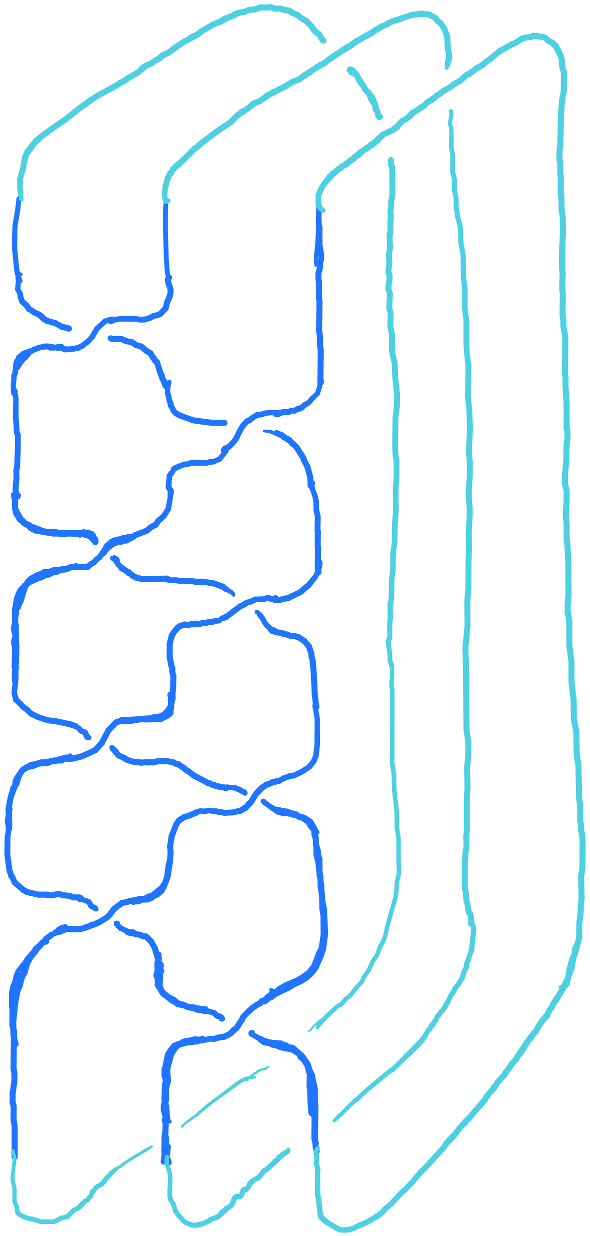


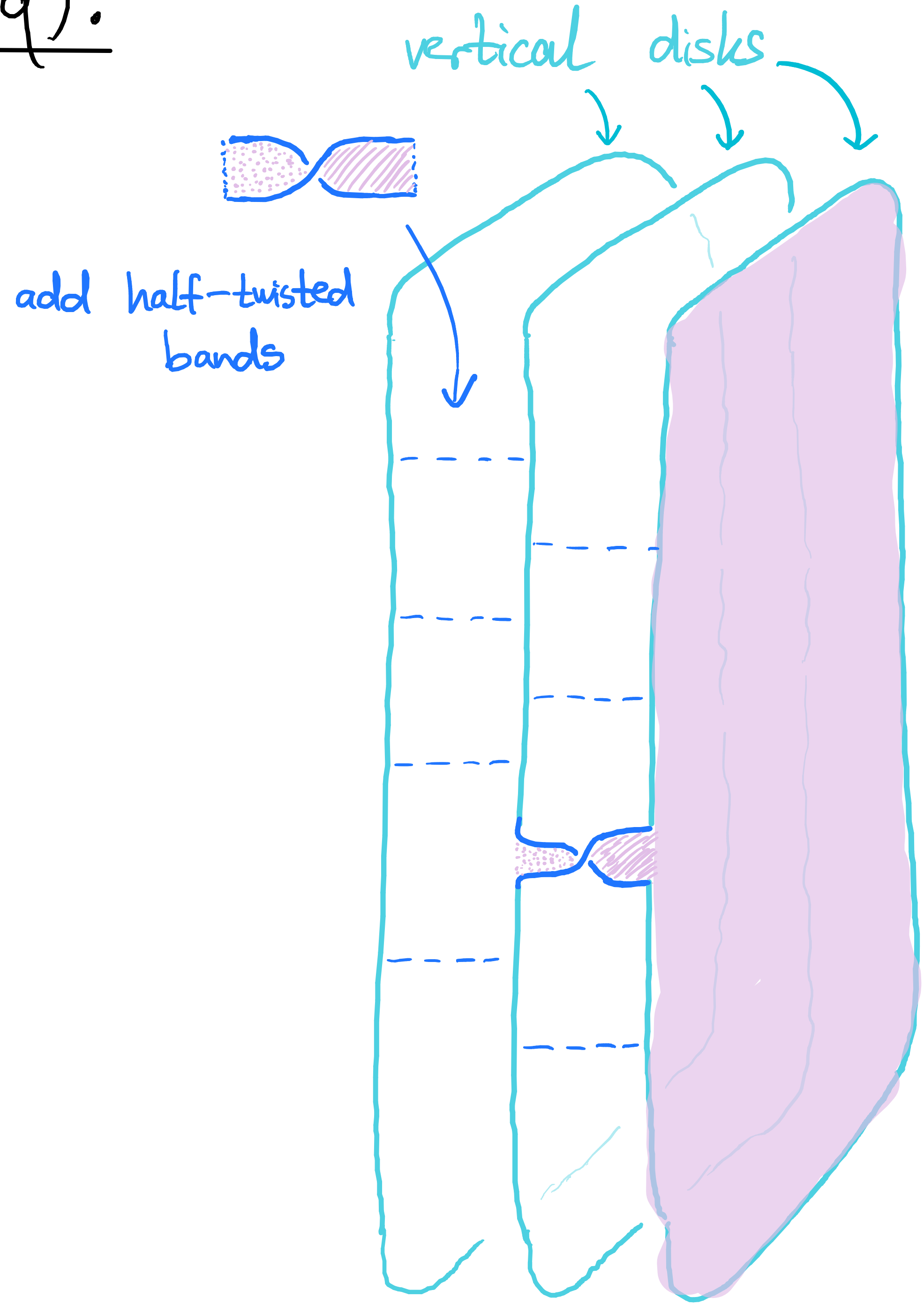
Question: If we want to replace the singularity with a piece of smooth surface, what is the least genus we have to use for this?



Seifert surface for $T(p,q)$:

Lives in 3-space!



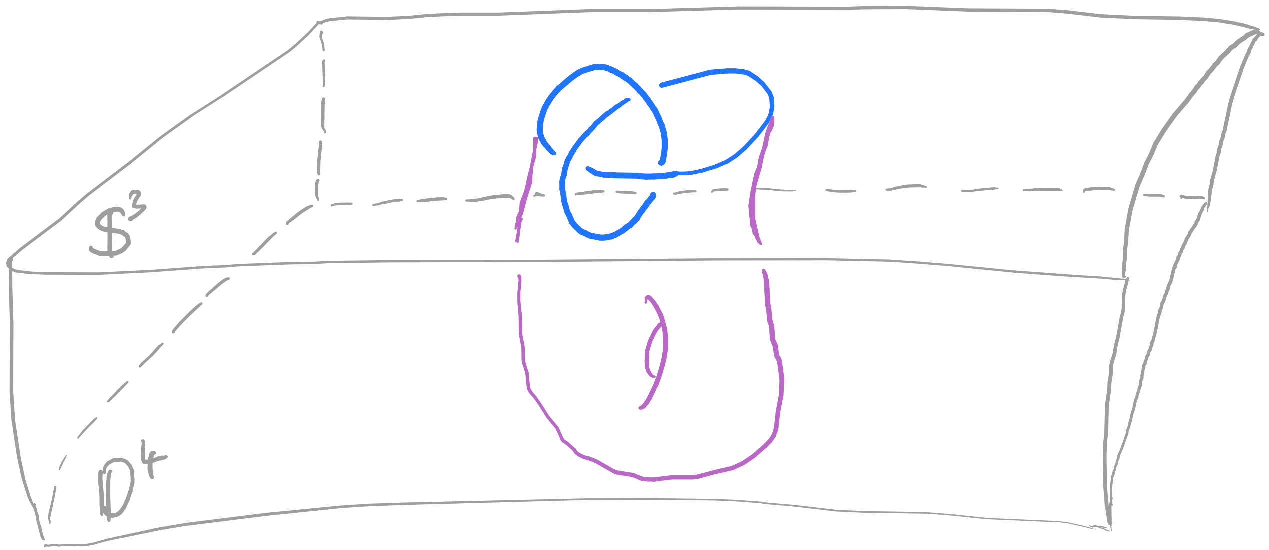


Conjecture [Milnor]

We can't do better than this in the 4-ball, i.e.

$$\text{genus}_4^{\text{sm.}}(T(p,q)) = \frac{(p-1)\cdot(q-1)}{2}$$

proved in 1993 by Kronheimer and Mrowka using gauge theory techniques; reproved by Rasmussen with his s-invariant from Khovanov homology



Shout-out to Peter Lambert-Cole, who showed that the global Thom-conjecture follows from the local Thom conj.

Bridge trisections in $\mathbb{CP}^2$ and the Thom conjecture

Peter Lambert-Cole

In this paper, we develop new techniques for understanding surfaces in $\mathbb{CP}^2$ via bridge trisections. Trisections are a novel approach to smooth 4-manifold topology, introduced by Gay and Kirby, that provide an avenue to apply 3-dimensional tools to 4-dimensional problems. Meier and Zupan subsequently developed the theory of bridge trisections for smoothly embedded surfaces in 4-manifolds. The main application of these techniques is a new proof of the Thom conjecture, which posits that algebraic curves in $\mathbb{CP}^2$ have minimal genus among all smoothly embedded, oriented surfaces in their homology class. This new proof is notable as it completely avoids any gauge theory or pseudoholomorphic curve techniques.

Comments: 33 pages, 18 figures
Subjects: **Geometric Topology (math.GT)**
MSC classes: 57R17, 57R40
Cite as: arXiv:1807.10131 **[math.GT]**
(or arXiv:1807.10131v2 **[math.GT]** for this version)

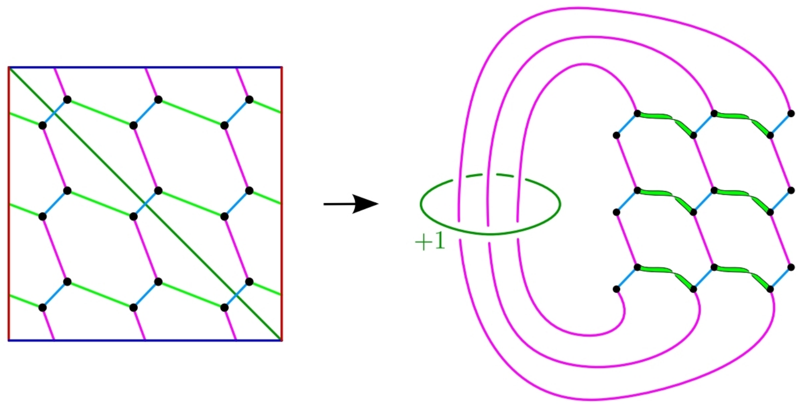


FIGURE 2. (*Left*) A torus diagram for a bridge trisection of a cubic curve in $\mathbb{CP}^2$. (*Right*) A banded link diagram corresponding to the bridge splitting of the cubic.

Definition & Properties

A Quick Reference Guide to Khovanov's Categorification of the Jones Polynomial

borrowed from $\boxed{\text{Dror Bar-Natan}}$, August 17, 2004

The Kauffman Bracket: $\langle\emptyset\rangle = 1$; $\langle \bigcirc L\rangle = (q+q^{-1})\langle L\rangle$; $\langle \times \rangle = \langle \underset{0\text{-smoothing}}{\asymp} \rangle - q\langle \underset{1\text{-smoothing}}{)(} \rangle$.

The Jones Polynomial: $\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-}\langle L\rangle$, where (n_+, n_-) count $(\times, \times)$ crossings.

Khovanov's construction: $[\![L]\!]$ — a chain complex of graded $\mathbb{Z}$-modules;

$$[\![\emptyset]\!] = 0 \to \underset{\text{height }0}{\mathbb{Z}} \to 0; \qquad [\![\bigcirc L]\!] = V \otimes [\![L]\!]; \qquad [\![\times]\!] = \text{Flatten}\left(0 \to \underset{\text{height }0}{[\![\asymp]\!]} \to \underset{\text{height }1}{[\![)(]\!]\{1\}} \to 0\right);$$

$$\mathcal{H}(L) = \mathcal{H}(\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+ - 2n_-\})$$

$$V = \text{span}\langle v_+, v_-\rangle; \qquad \deg v_\pm = \pm 1; \qquad \text{qdim}\, V = q + q^{-1} \quad \text{with} \quad \text{qdim}\,\mathcal{O} := \sum_m q^m \dim \mathcal{O}_m;$$

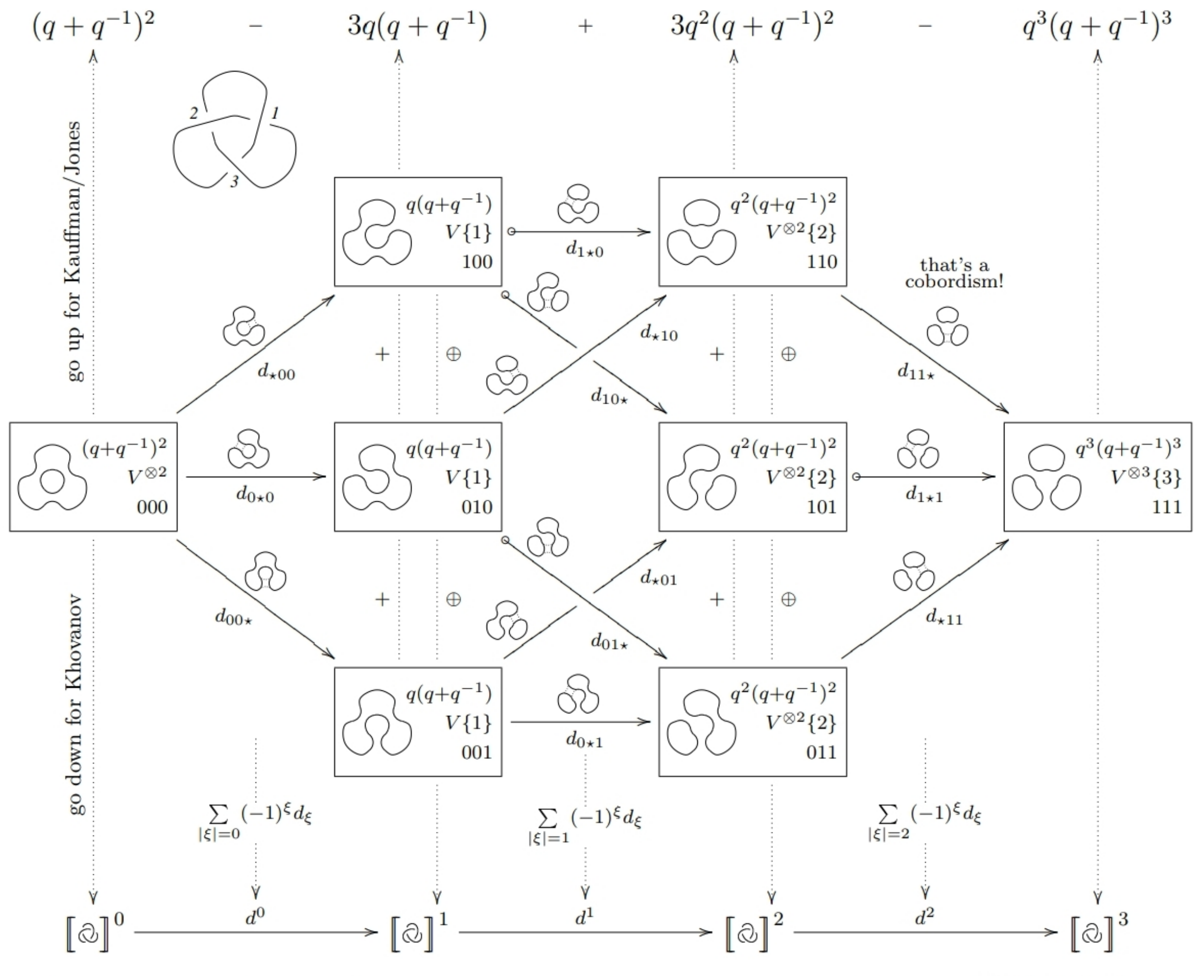
$$\mathcal{O}\{l\}_m := \mathcal{O}_{m-l} \quad \text{so} \quad \text{qdim}\,\mathcal{O}\{l\} = q^l\, \text{qdim}\,\mathcal{O}; \qquad \cdot[s]: \quad \text{height shift by } s;$$


$\longrightarrow (V \otimes V \xrightarrow{m} V)$ $\qquad m: \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$


$\longrightarrow (V \xrightarrow{\Delta} V \otimes V)$ $\qquad \Delta: \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$

That's a Frobenius Algebra! And a (1+1)-dimensional TQFT!

Example: $\qquad q^{-2} + 1 + q^2 - q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q + q^3 + q^5 - q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $\qquad = \quad [\![\text{trefoil}]\!] \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot[-n_-]\{n_+-2n_-\}} \mathcal{C}(\text{trefoil}).$

Theorem 1. The graded Euler characteristic of $\mathcal{C}(L)$ is $\hat{J}(L)$.

Theorem 2. The homology $\mathcal{H}(L)$ is a link invariant and thus so is $Kh_{\mathbb{F}}(L) := \sum_r t^r\, \text{qdim}\, \mathcal{H}^r_{\mathbb{F}}(\mathcal{C}(L))$ over any field $\mathbb{F}$.

Theorem 3. $\mathcal{H}(\mathcal{C}(L))$ is strictly stronger than $\hat{J}(L)$: $\mathcal{H}(\mathcal{C}(\bar{5}_1)) \neq \mathcal{H}(\mathcal{C}(10_{132}))$ whereas $\hat{J}(\bar{5}_1) = \hat{J}(10_{132})$.

Conjecture 1. $Kh_{\mathbb{Q}}(L) = q^{s-1}(1 + q^2 + (1 + tq^4)Kh')$ and $Kh_{\mathbb{F}_2}(L) = q^{s-1}(1+q^2)(1 + (1+tq^2)Kh')$ for even $s = s(L)$ and non-negative-coefficients laurent polynomial $Kh' = Kh'(L)$.

Conjecture 2. For alternating knots s is the signature and Kh' depends only on tq^2.

don't confuse with Rasmussen's s-invariant

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and DBN's http://www.ma.huji.ac.il/~drorbn/papers/Categorification/.

Last week: Khovanov homology is an invariant of knots or links $L: \underset{\text{components}}{\amalg} S^1 \hookrightarrow S^3$

(doubly) graded homology groups $Kh^{*,*}(L)$

($\rightsquigarrow$ graded Euler characteristic is the (unnormalized) Jones polynomial of L)

Now:

γ = # of components of L mod 2

$$E_2^{p,q} = Kh^{p+q,2p+\gamma}(L) \Rightarrow \underbrace{Lee^*(L)}_{\cong \mathbb{Q} \oplus \mathbb{Q}}$$

(i.e. the Lee-Rasmussen spectral sequence leaves only two generators on the E_∞-page)

Rasmussen used this spectral sequence to define a knot invariant $s(K)$

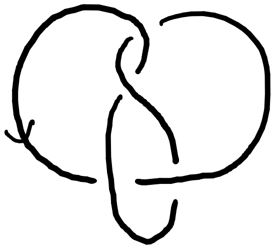
Basic properties of the s-invariant:

$$s : \begin{array}{c}\text{Isotopy classes}\\ \text{of oriented knots}\end{array} \longrightarrow \mathbb{Z}$$

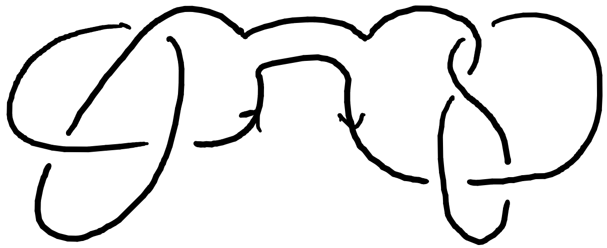
•) $s(\text{unknot}) = 0$

•) $s(K_1 \# K_2) = s(K_1) + s(K_2)$

K_1

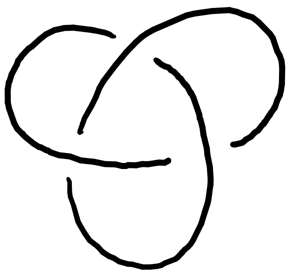


K_2

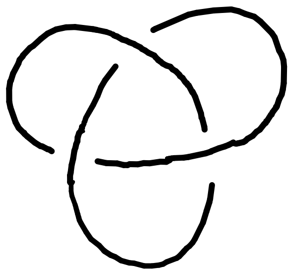


$K_1 \# K_2$

•) $s(\text{mirror}(K)) = -s(K)$



K

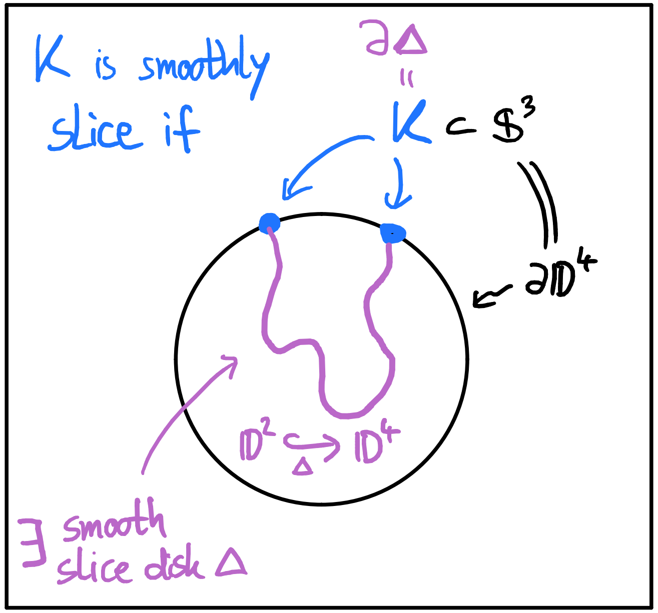


$\text{mirror}(K)$

Cobordism of Links / concordance

From functoriality of Lee homology:

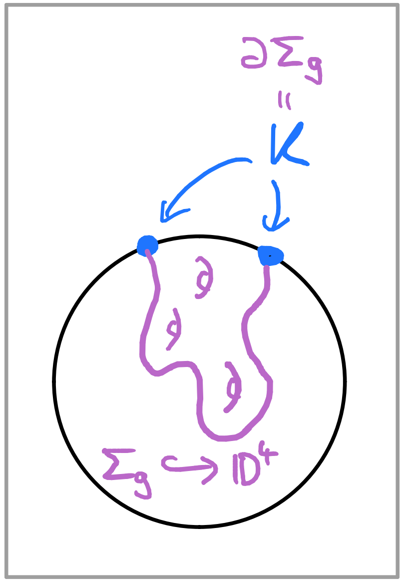
- $s\left(\begin{smallmatrix}\text{smoothly slice}\\ \text{knot}\end{smallmatrix}\right) = 0$



$\rightsquigarrow \mathcal{C}^{\text{smooth}} = \text{knots}/\text{concordance} \xrightarrow{\ s\ } \mathbb{Z}$

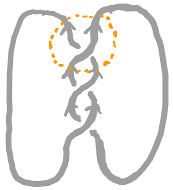
group homomorphism

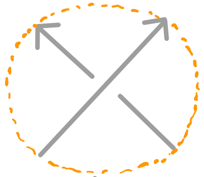
- even better: $|s(K)| \leq \text{genus}_4^{\text{sm.}}(K)$ ← smooth 4-genus of K



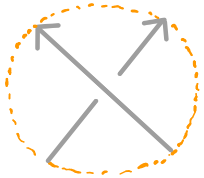
•) K positive knot, then $s(K) = 2 \cdot \text{genus}_4^{\text{sm.}}(K) = 2 \cdot \text{genus}_3(K)$

has a diagram with only positive crossings





positive



negative

Example: $T(p,q)$ **Excercise:** $s(T(p,q)) = (p-1)(q-1)$

and conclude the Milnor conjecture:

The smooth slice genus of the (p,q)-torus knot is $\frac{(p-1)(q-1)}{2}$.

This is amazing: A combinatorially defined invariant can tell us something about smoothness!

[↗ other applications later]

•) K alternating, then $s(K) = \sigma(K)$ ⇜ classical knot signature

if you follow the knot, you see over - under - over - under - ...

- Lee-Rasmussen spectral sequence leaves only 2 generators on E_∞-page

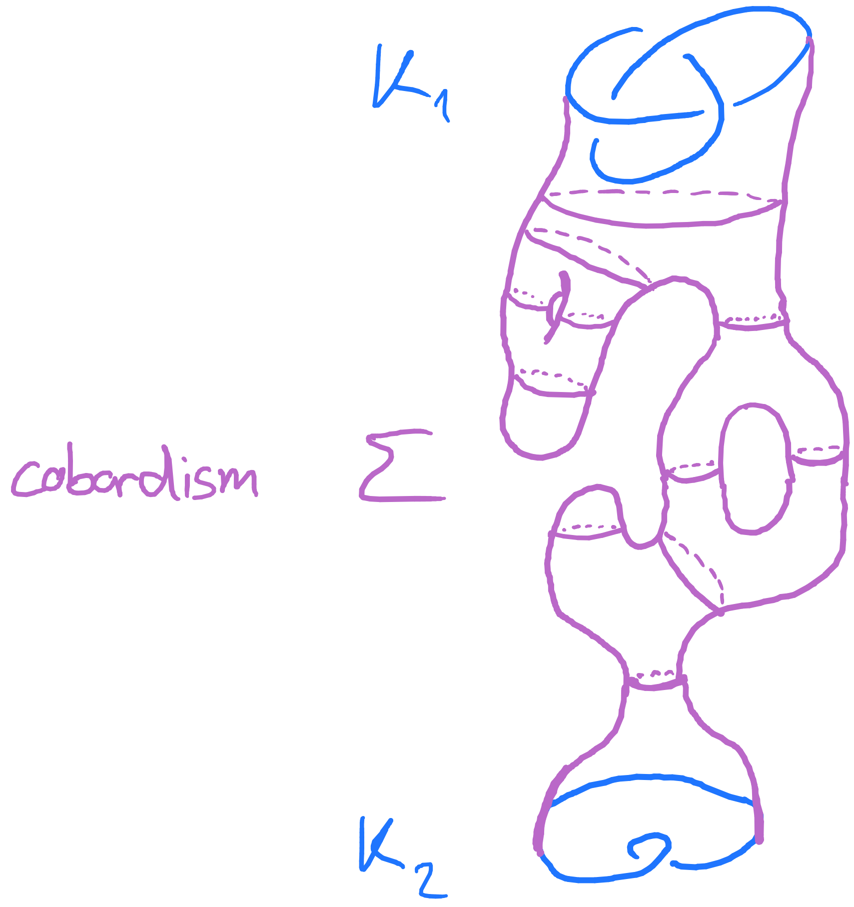
- Convergence of Sp. Seq. means $E_\infty^{i,j} \cong \dfrac{F^j \mathrm{Lee}^i}{F^{j+1}\mathrm{Lee}^i}$ ⇜ $F^*\mathrm{Lee}^i$ induced filtration on Lee theory

- No extension problems $/\mathbb{Q}$ ⇝ $\mathrm{Lee}^i \cong \bigoplus_j E_\infty^{i,j}$

filtration grading is meaningful since $E_2, E_3, \dots$ are knot invariants

Rasmussen s-invariant of K

Prop./Def.: For knot K, there is an even integer $s(K)$ s.th. the two surviving generators in Lee-Rasmussen spectral seq. have filtration degrees $s(K) \pm 1$.



$\rightsquigarrow$ Filtered map

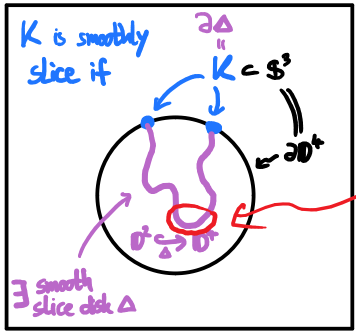
$$\mathrm{Lee}(\Sigma) : \mathrm{Lee}^*(K_1) \longrightarrow \mathrm{Lee}^*(K_2)$$

of filtered degree $\chi(\Sigma)$, i.e.

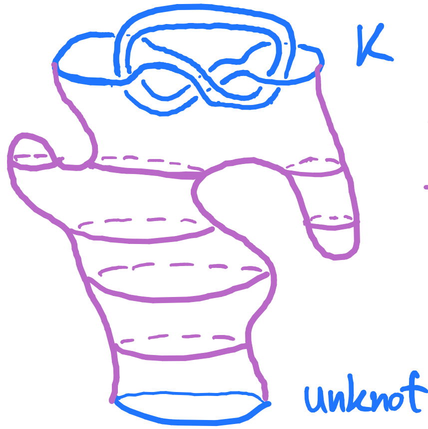
$$\mathrm{Im}\left(F^j \,\mathrm{Lee}^*(K_1) \right) \subseteq F^{j+\chi(\Sigma)} \mathrm{Lee}^*(K_2)$$

<u>Prop. [Rasmussen]:</u> Σ connected $\Rightarrow$ $\mathrm{Lee}(\Sigma)$ isomorphism

Prop.: K smoothly slice $\Rightarrow$ $s(K) = 0$



remove tiny D^2 from slice disk $\rightsquigarrow \Sigma$



(connected) Link cobordism from K to unknot with $\chi(\Sigma) = 0$

$\rightsquigarrow$ filtered iso. of filtered degree zero $\quad \mathrm{Lee}(\Sigma) : \mathrm{Lee}^0(K) \to \mathrm{Lee}^0(\text{unknot})$

two generators in filtration degrees ± 1

$\rightsquigarrow$ Considering gradings forces $s(K) \leq 0$.

Apply same argument to $\mathrm{mirror}(K)$: $\underset{\substack{\| \\ -s(K)}}{s(\mathrm{mirror}(K))} \leq 0$

$\Rightarrow s(K) = 0$

$\square$

Similar proof gives $|s(K)| \leq 2 \cdot \mathrm{genus}_4^{\mathrm{sm.}}(K)$

Applications of the s-invariant

Using the s-invariant to find possibly exotic homotopy 4-balls

Man and machine thinking about the smooth 4-dimensional Poincaré conjecture

Michael Freedman, Robert Gompf, Scott Morrison, Kevin Walker

While topologists have had possession of possible counterexamples to the smooth 4-dimensional Poincaré conjecture (SPC4) for over 30 years, until recently no invariant has existed which could potentially distinguish these examples from the standard 4-sphere. Rasmussen's s-invariant, a slice obstruction within the general framework of Khovanov homology, changes this state of affairs. We studied a class of knots K for which nonzero s(K) would yield a counterexample to SPC4. Computations are extremely costly and we had only completed two tests for those K, with the computations showing that s was 0, when a landmark posting of Akbulut (arXiv:0907.0136) altered the terrain. His posting, appearing only six days after our initial posting, proved that the family of ``Cappell--Shaneson" homotopy spheres that we had geared up to study were in fact all standard. The method we describe remains viable but will have to be applied to other examples. Akbulut's work makes SPC4 seem more plausible, and in another section of this paper we explain that SPC4 is equivalent to an appropriate generalization of Property R (``in S^3, only an unknot can yield S^1 x S^2 under surgery"). We hope that this observation, and the rich relations between Property R and ideas such as taut foliations, contact geometry, and Heegaard Floer homology, will encourage 3-manifold topologists to look at SPC4.

Comments: 37 pages; changes reflecting that the integer family of Cappell-Shaneson spheres are now known to be standard (arXiv:0907.0136)
Subjects: **Geometric Topology (math.GT)**; Quantum Algebra (math.QA)
MSC classes: 57R60, 57N13, 57M25
Journal reference: Quantum Topology, Volume 1, Issue 2 (2010), pp. 171-208
DOI: 10.4171/QT/5
Cite as: arXiv:0906.5177 **[math.GT]**
(or arXiv:0906.5177v2 **[math.GT]** for this version)

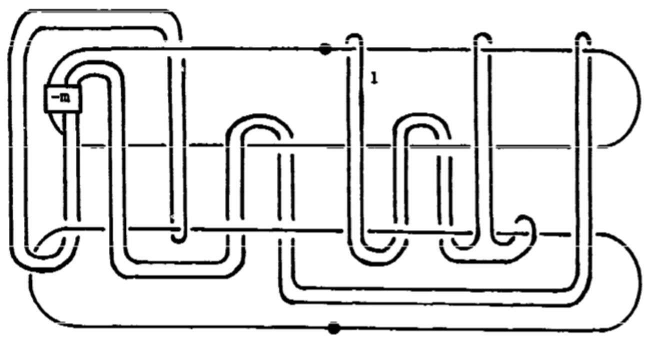


Figure 6: Figure 17 from [16], showing the handle presentation of the Cappell-Shaneson sphere Σ_m.

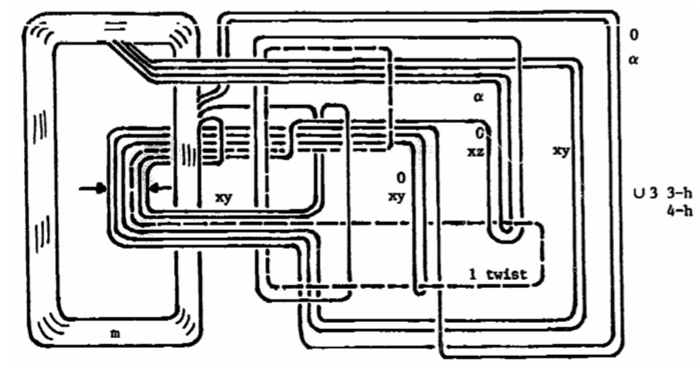


Fig. 9.

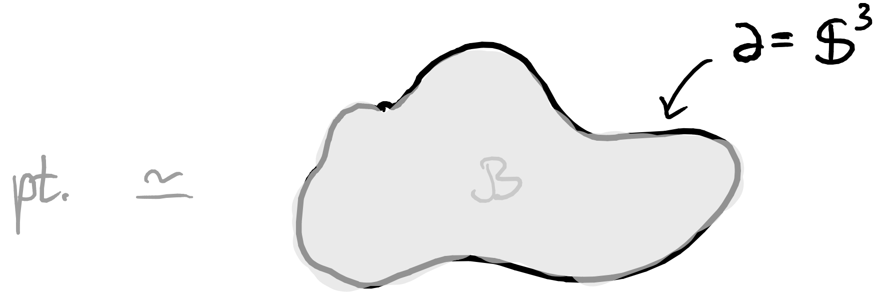
Figure 7: Figure 9 from [16], showing the two component cocore link L_m. What appears to be a third, unknotted, component drawn with a dashed line is actually notation for a full positive twist on the strands passing through it.

Def: (smooth) Homotopy 4-ball

smooth compact 4-manifold B with $\partial B \cong S^3$

$\simeq$ homotopy equiv.

$\mathbb{D}^4$



Smooth Poincaré conjecture in dimension 4 (SPC4) $\Longleftrightarrow$ (not obvious) All homotopy 4-balls are diffeomorphic to $\mathbb{D}^4$

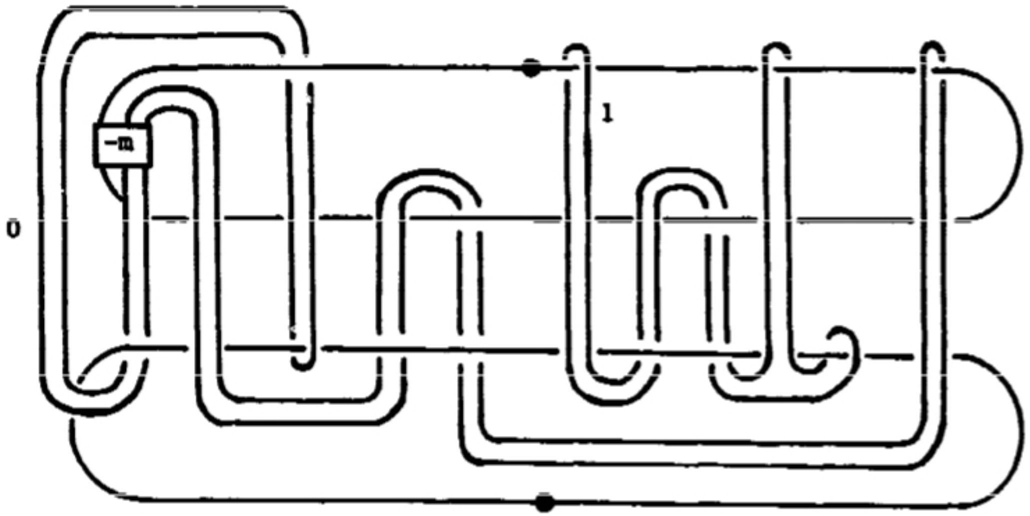


Figure 6: Figure 17 from [16], showing the handle presentation of the Cappell-Shaneson sphere Σ_m.

describes a handle decomposition of a homotopy 4-ball B

(0-handle) ∪ (two 1-handles) ∪ (two 2-handles)

with $\partial \cong S^3$

(so this is also a complicated description of the standard 3-sphere)

this knot $K \subset \partial B = S^3$ is slice in B

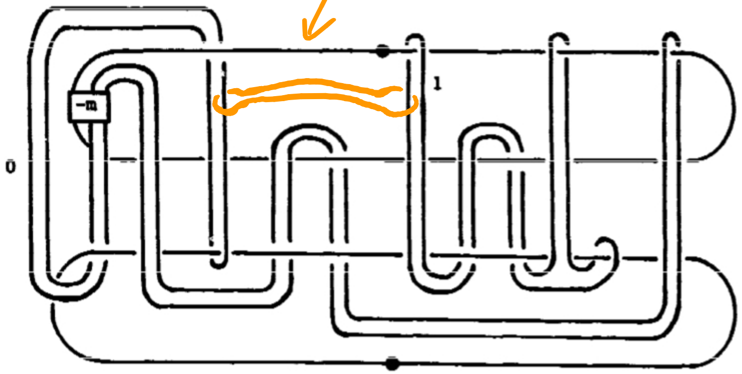


Figure 6: Figure 17 from [16], showing the handle presentation of the Cappell-Shaneson sphere Σ_m.

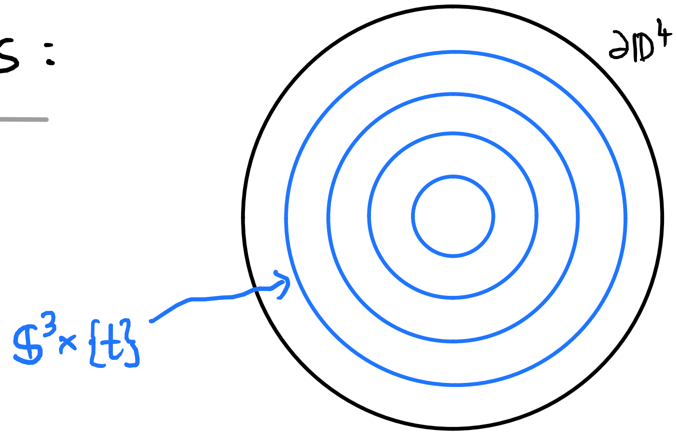
So, if K were non-slice in D^4, B must be exotic!

Maybe the s-invariant can help us ...

Heuristic of this strategy:

The proof of vanishing of the s-invariant for a slice knot in $\mathbb{D}^4$ really depends on diagrams of the knot and uses that

$\mathbb{D}^4$ is build up of $\mathbb{S}^3$-Layers:



What a hypothetical exotic 4-ball won't be!

•) Unfortunately it turned out that the homotopy 4-balls in [FGMW] are actually diffeomorphic to standard B^4:

Cappell-Shaneson homotopy spheres are standard

Selman Akbulut

We show that an infinite sequence of homotopy 4-spheres constructed by Cappell-Shaneson are all diffeomorphic to S^4. This generalizes previous results of Akbulut-Kirby and Gompf.

Comments: 5 pages, 4 figures. A minor correction, a reference and a remark added, to appear in Annals of Mathematics
Subjects: **Geometric Topology (math.GT)**; Algebraic Topology (math.AT)
MSC classes: 58D27, 58A05, 57R65
Cite as: arXiv:0907.0136 **[math.GT]**
(or arXiv:0907.0136v3 **[math.GT]** for this version)

•) It is still open whether the s-invariant automatically vanishes for knots that are slice in some homotopy 4-ball, see the corrigendum to

Gauge theory and Rasmussen's invariant

P. B. Kronheimer, T. S. Mrowka

A previous paper of the authors' contained an error in the proof of a key claim, that Rasmussen's knot-invariant s(K) is equal to its gauge-theory counterpart. The original paper is included here together with a corrigendum, indicating which parts still stand and which do not. In particular, the gauge-theory counterpart of s(K) is not additive for connected sums.

Comments: This version bundles the original submission with a 1-page corrigendum, indicating the error. The new version of the corrigendum points out that the invariant is not additive for connected sums. 23 pages, 3 figures
Subjects: **Geometric Topology (math.GT)**
MSC classes: 57R58, 57R60
DOI: 10.1112/jtopol/jtt008
Cite as: arXiv:1110.1297 **[math.GT]**
(or arXiv:1110.1297v3 **[math.GT]** for this version)

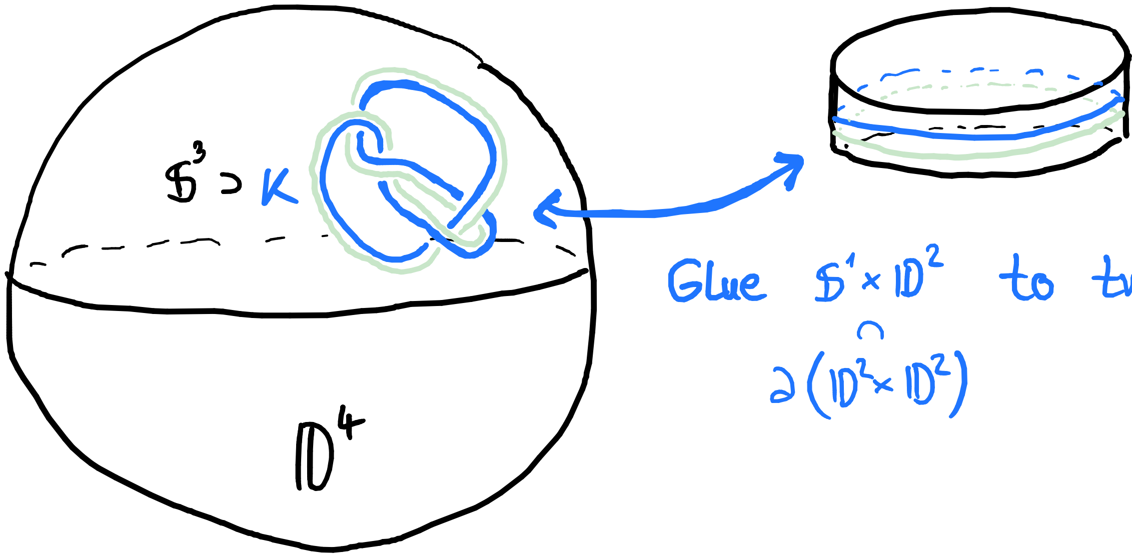
Exotic $\mathbb{R}^4$'s from topologically slice, non smoothly slice knots

[s-invariant not really necessary for this, but it makes it easier to find the knots we will need]

0-trace of a knot $K : S^1 \hookrightarrow S^3$ is

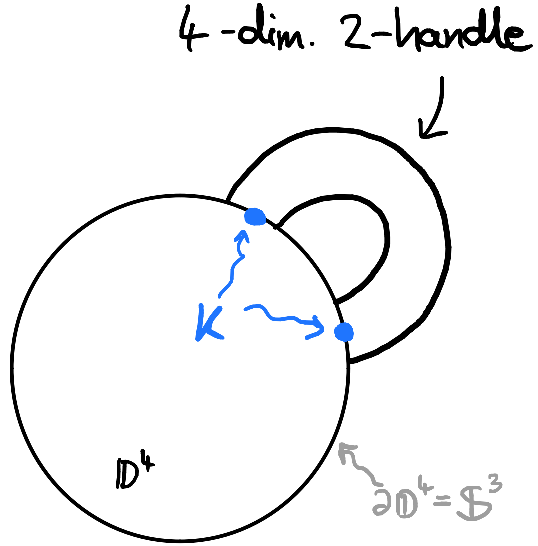
$$\boxed{X_0(K) = D^4 \cup_{\substack{K \times D^2 \\ \cap \\ \partial D^4}} D^2 \times D^2}$$

$D^2 \times D^2$

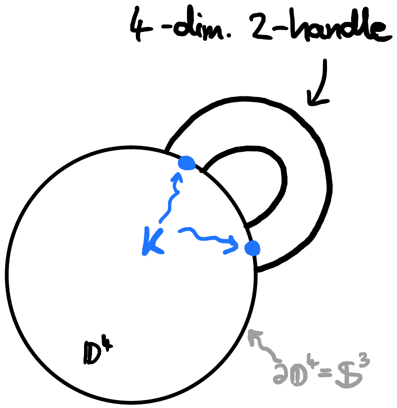


Glue $\underset{\partial (D^2 \times D^2)}{S^1 \times D^2 \atop \cap}$ to tubular neighborhood of K

"Half-dim. schematic":

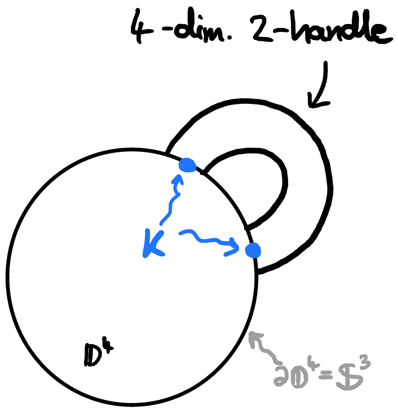


Lemma: K smoothly slice

$\Longleftrightarrow$ 

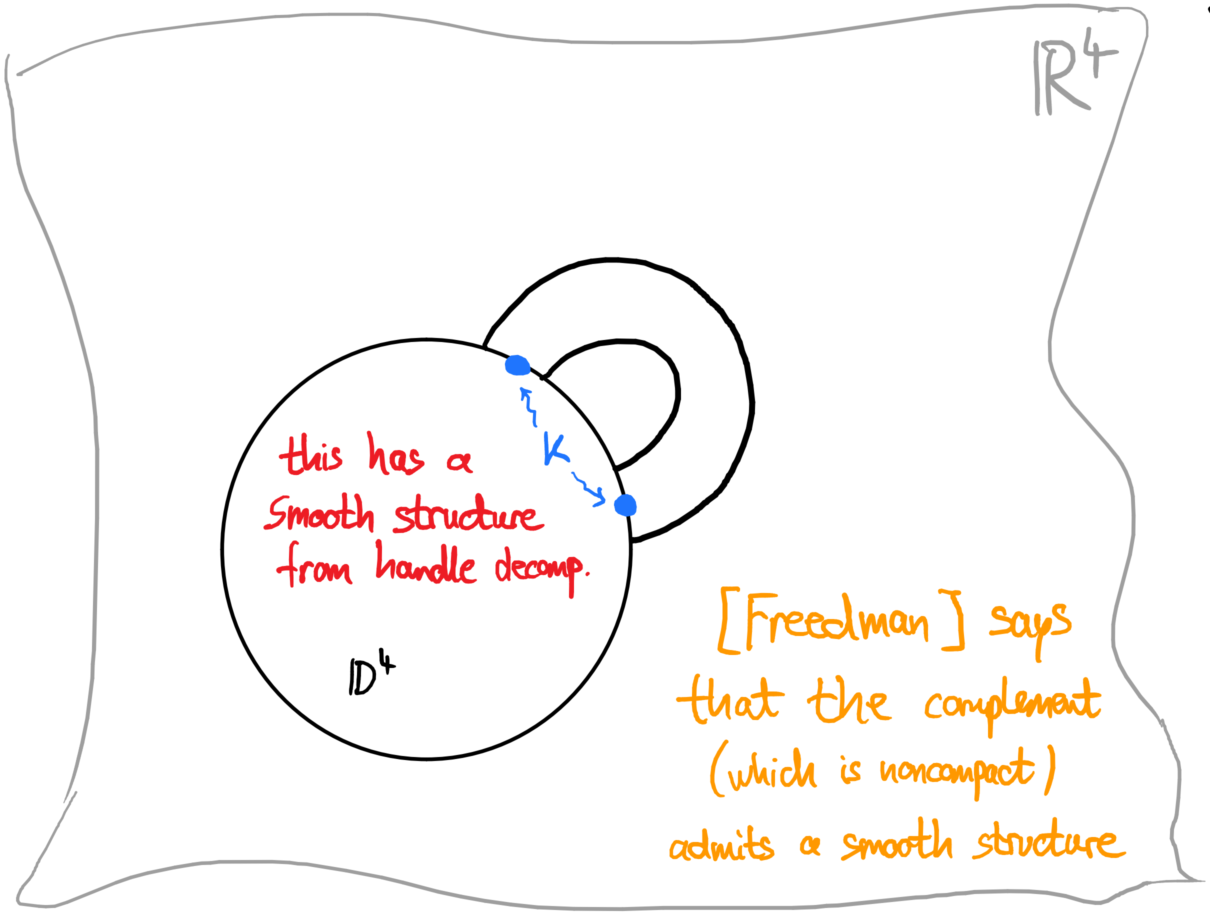
$= X_0(K) \xhookrightarrow{\text{smoothly}} \mathbb{R}^4_{\text{std.}}$

K topologically slice $\rightsquigarrow$ (bounds a topologically locally flat embedded disk in $\mathbb{D}^4$)

$\Longleftrightarrow$ 

$= X_0(K) \xhookrightarrow{\text{topologically}} \mathbb{R}^4_{\text{std.}}$

Construction: Start with topologically slice, non-smoothly slice knot K, and a topological embedding in $\mathbb{R}^4$



Red & Orange together give smooth structure $\mathcal{R}$ on $\mathbb{R}^4$ ...

... which can't be diffeomorphic to $\mathbb{R}^4_{std.}$ because otherwise we would have a smoothly embedded $X_0(K)$

Ex.: Knots with Alexander polynomial $\Delta_K = 1$ are topologically slice [Freedman]

Only have to find one with $s \neq 0$ ($\Rightarrow$ not smoothly slice) and we can build an exotic $\mathbb{R}^4$!

E.g. $(-3, 5, 7)$ pretzel knot