Effect of 1-handle addition on $\pi_1(\text{complement})$

$$\pi_1\left(\partial^4 \setminus (S + h^1)\right) \cong \frac{\pi_1\left(\partial^4 \setminus S\right)}{[x, w]}$$

$x =$ meridian  
"guiding arc of 1-handle"

$x = w \times w^{-1}$

Effect of finger move on $\pi_1(\text{complement})$

$$\pi_1\left(\partial^4 \setminus S'\right) \cong \frac{\pi_1\left(\partial^4 \setminus S\right)}{[x, w^{-1}xw]}$$

after finger move on $S$

$x = x^w$

also a meridian
Lemma: Let $L: S_a \sqcup S_b \to M^4$ be a link of surfaces in a 4-manifold.

\[ \tau_q(M-L) = \frac{\tau_q(M-L')}{{\langle g^ag, b \rangle}} \]

after finger move

Cartoon: Removing a slice disk from $M^4$
Lemma: Let $L: S_a^2 \sqcup S_b^2 \to M^4$ be a link of surfaces in a 4-manifold (can be immersed).

\[ \mathfrak{x}_1 (M - L') = \frac{\mathfrak{x}_1 (M - L)}{\langle a = g \cdot b \cdot g^{-1} \rangle} \]

link after 1-handle attachment

Have to understand how putting disk back changes $\mathfrak{x}_1$ of complement:

At the attaching regions of the 1-handle:

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\[\text{attaching circle of 2-handle which puts the disk inside}\]
On connected surfaces:

**Finger moves:** \{meridian, meridian\}

1-handle attachments: \{meridian, anything\}
$u_{F-\text{wh.}}(K) \leq \text{ fus}(K)$ for a ribbon 2-knot $K$:

Group after "extending fingers" from the last minimum to all the others:

\[
\langle a \rangle * \langle b, c, d \mid b = w_1 c w_1^{-1}, \; c = w_2 d w_2^{-1} \rangle \quad \overset{[d, G]}{\rightarrow} \quad \mathbb{Z}_a \oplus \mathbb{Z}_{b \circ c \circ d}
\]

"complicated" fusion tube

"twisting $h$ about $a \& b" \Rightarrow [h] = [t] \in \pi_1 \cong \mathbb{Z} \oplus \mathbb{Z}

i.e. in the complement of the immersion, the "complicated" fusion tube is isotopic to this "easy" tube
1-handle $\leftrightarrow$ tunnel #

Finger-wh. $\leftrightarrow$ unknotting #

**Question:** Is $u_{1-h.}(K) \leq u_{F-wh}(K)$?

Interesting Whitney disks taking us to $K$

$$\frac{\pi_1(S^4 - K)}{\langle [m, g_1^{-1}m_1^{-1}], [m, g_2^{-1}m_2^{-1}] \rangle} \cong \mathbb{Z}$$

Pull "fingernails" down

expand tubes

"retract fingers"
Example of a ribbon 2-knot via one finger & one Whitney move on unknot

Preimage in $\mathbb{S}^2$:

\[ p^+ \quad p^- \]
\[ q^+ \quad q^- \]

"fingernail" disk back to unknot

Whitney disk to nontrivial ribbon knot

= Double of ribbon disk for stevedore
For $k_i$ non-tr. 2-bridge & for natural number $r_i \geq 2$:

\[ \mathbb{Z} \cup_{4 \cdot \text{h.}} (\text{Spin}_{r_i}(k_i)) = 1 \]

**Lemma:** For $r_1, \ldots, r_n \geq 2$ coprime integers, $k_1, \ldots, k_r$ 2-bridge

\[ \cup_{4 \cdot \text{h.}} (\text{Spin}_{r_i}(k_i) \# \ldots \# \text{Spin}_{r_n}(k_n)) = 1 \]

**Kanenobu's example:**

\[ y_1 \]
\[ x_1 \]
\[ y_2 \]
\[ x_2 \]
\[ y_3 \]
\[ x_3 \]
\[ y_4 \]
\[ x_4 \]
\[ y_5 \]
\[ x_5 \]

\[ \rightsquigarrow \]

\[ y_1 \]
\[ x_1 \]
\[ y_2 \]
\[ x_2 \]
\[ y_3 \]
\[ x_3 \]
\[ y_4 \]
\[ x_4 \]
\[ y_5 \]
\[ x_5 \]
Claim: Adding the relation \([x, y_1 y_2 \cdots y_n]\) abelianizes the group

Similar \(\pi_1\)-calculation works for result of Finger move along this arc
(the immersion complement was \(\pi_1 \cong \mathbb{Z}\))

Can we put it in the "standard position?"
(finger move an unknot)

attach a 1-handle along this guiding arc to get a torus with \(\pi_1(\text{compl.}) \cong \mathbb{Z}\)

- is it unknotted?