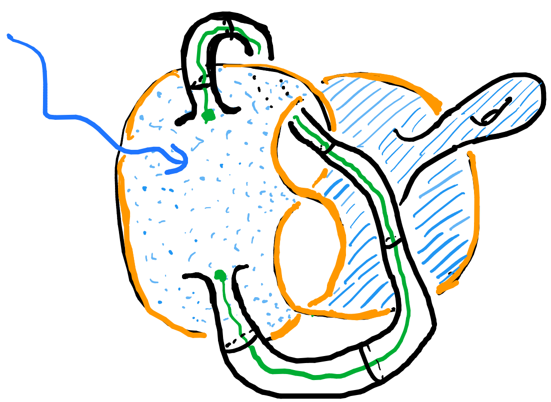
# Unknotting 2-knots with Finger- & Whitney moves

(everything [the manifolds, embeddings, ...] is smooth here)

with Jason Joseph,
Michael Klug & Hannah Schwartz

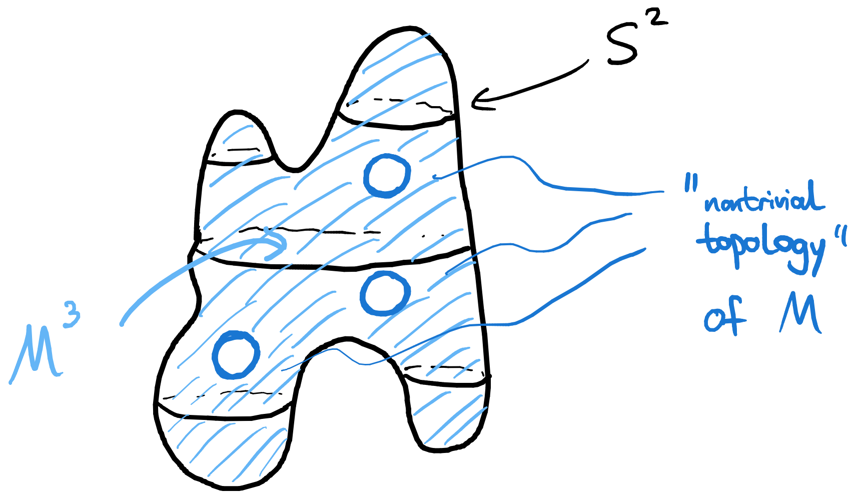
Just as knots $S^1 \hookrightarrow S^3$ bound Seifert surfaces ...



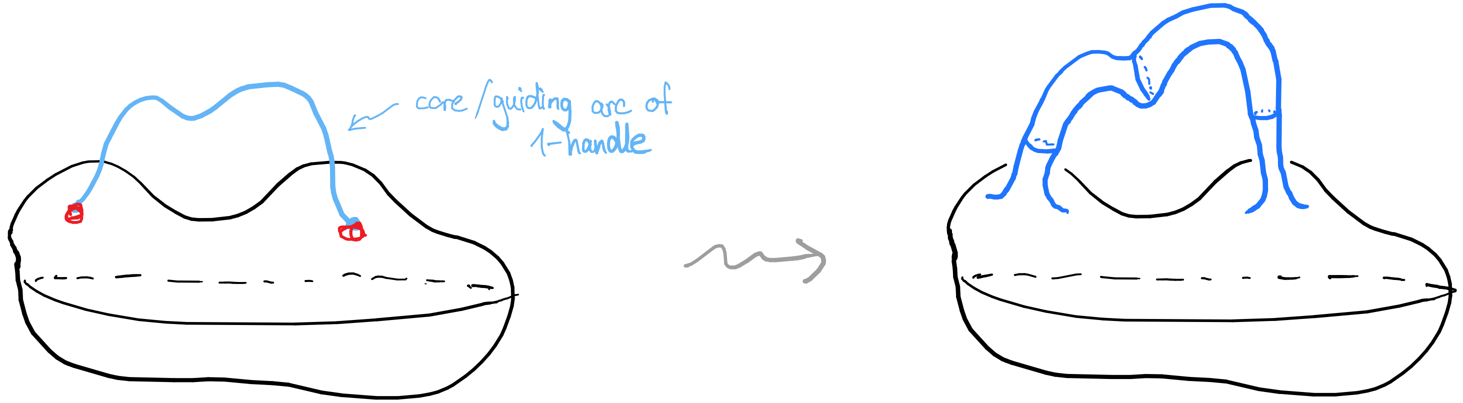
(not unique, you can look up "S-equivalence")

... knotted 2-spheres $S^2 \overset{S}{\hookrightarrow} S^4$ bound Seifert hypersurfaces / Seifert solids

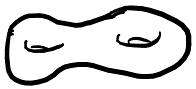
oriented, smooth compact 3-mflds $M^3 \hookrightarrow S^4$ with $\partial M = S$.



# Unknotting by attaching 1-handles:



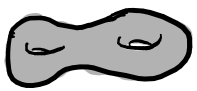

$S \qquad\qquad\qquad S + h^1$

**Claim:** For any (knotted) surface  $\overset{S}{\hookrightarrow} S^4$

you can add a finite number of 1-handles

s.th. $S + h_1 + h_2 + \dots h_k$ is unknotted.

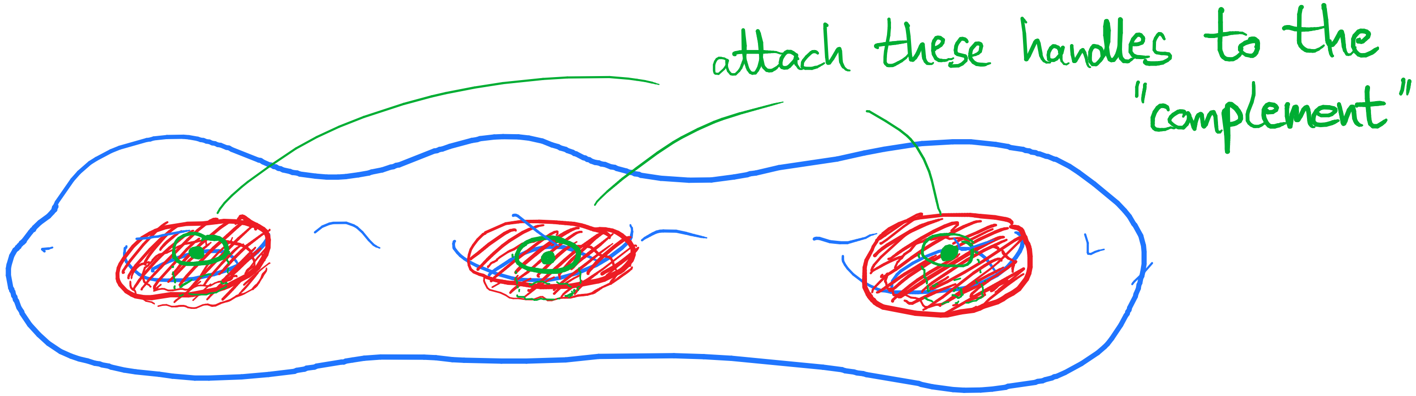
bounds a solid 1-handlebody



Pf.: Drill out cocores of 2-handles of a Seifert solid until you have a handlebody

- each hole you drill corresponds to attaching a 1-handle to the surface

abstractly:




$\square$

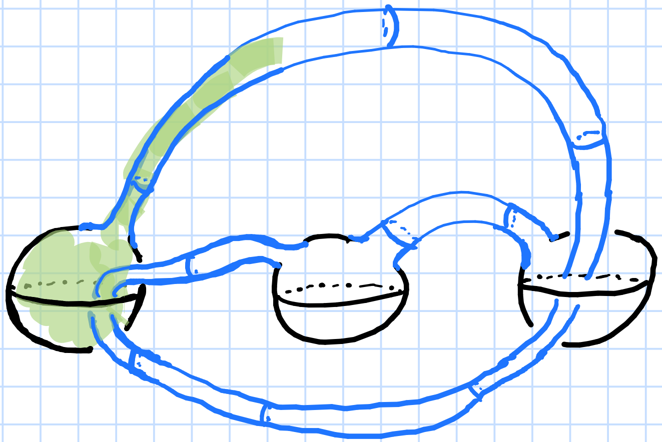
Stabilization #/ 1-handle unknotting # of $S^2 \overset{K}{\hookrightarrow} S^4$:

$u_{1\text{-h.}}(K)$ := min. # of 1-handles that have to be added to $K$ to obtain an unknotted surface

Miyazaki [On the relationship among unknotting number, knotting genus and Alexander invariant for 2-knots, 1985]

has a cool proof that for a ribbon 2-knot $K$

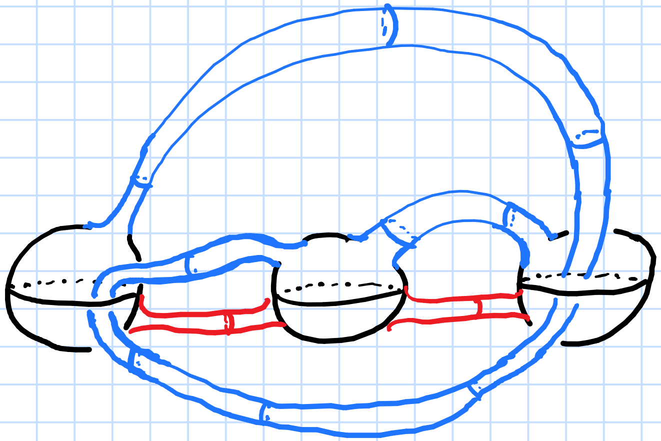
attach fusion 1-handles to minima



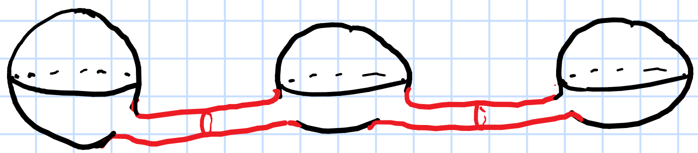
ribbon knot $K$

we have $\boxed{u_{1\text{-h.}}(K) \leq \mathrm{fusion}(K)}$

Pf.:



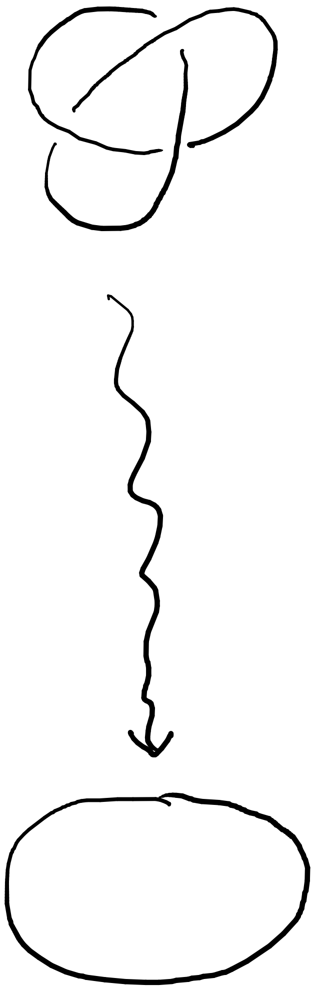
Attach the red ones first
$\rightsquigarrow$ unknotted



Blue are trivial bc. they are attached to an unknot

$\square$

# Unknotting by Finger- & Whitney moves: $S^2 \times [0,1] \to S^4 \times [0,1]$



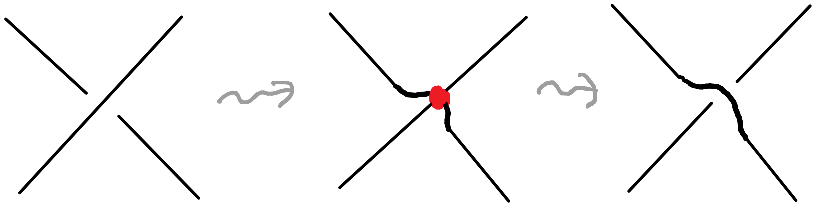
knot $K$ in $S^3$

homotopic to unknot ○

$\pi_1(S^3) = \{1\}$

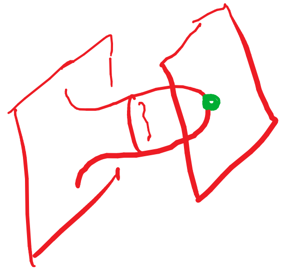
(of course if $K$ non-trivial, not isotopic to unknot)

sequence of isotopies and crossing changes:



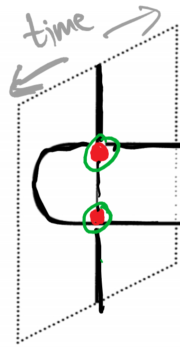
Similarly, any 2-knot $S^2 \overset{S}{\hookrightarrow} S^4$ is (regularly) homotopic to unknot

$\pi_2(S^4) = \{0\}$



2-knot $S$

Finger moves




[from Scorpan: The wild world of 4-manifolds]

immersed middle stage

Whitney moves

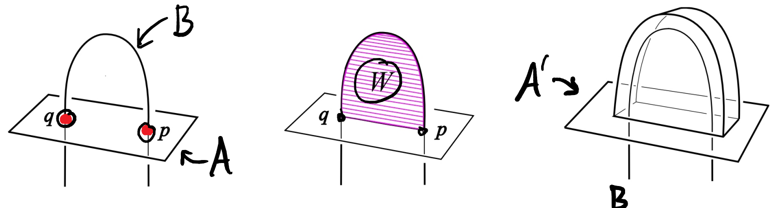
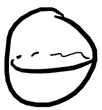



FIGURE 2.3. The pair of intersections $p, q$ (left) admits a purple Whitney disk $W$ (center) which guides a Whitney move eliminating $p, q$ by adding a *Whitney bubble* to the horizontal sheet (right).

[picture borrowed from Schneiderman–Teichner]

unknot

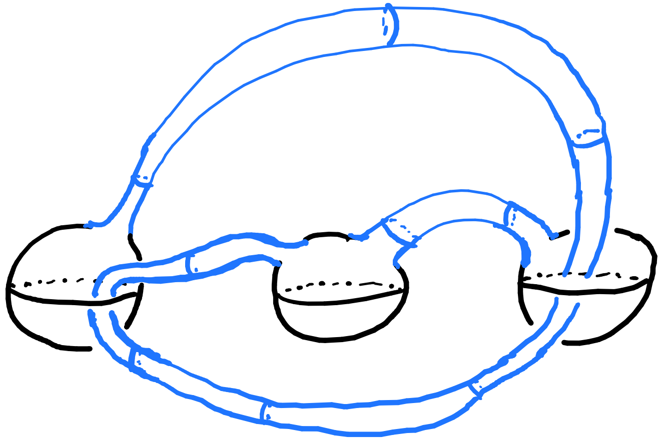


$\subset S^4$

$u_{F.-Wh.}(K)$ := minimal # of Finger moves in a regular homotopy to the unknot

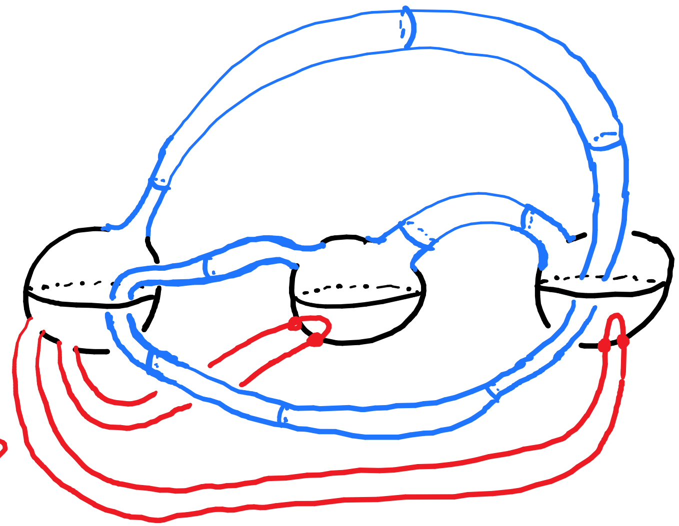
= ——— " ——— Whitney moves ——— " ———

Example: For a ribbon 2-knot



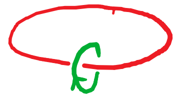
Claim:

After these finger moves can "untangle" the fusion handles ...



... then get rid of the double points via Whitney moves

$$\boxed{K \text{ ribbon 2-knot} \Rightarrow u_{F.-Wh.}(K) \leq \mathrm{fusion}(K)}$$

$\pi_1 \Big( S^3 \setminus$  $\Big) \cong \mathbb{Z}$



Conclusion of Dehn's lemma:

$\pi_1(S^3 \setminus K) \cong \mathbb{Z}$

$\Rightarrow K$ is unknotted

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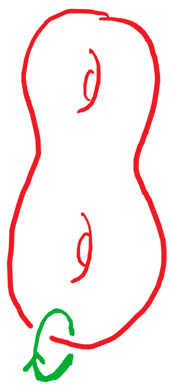
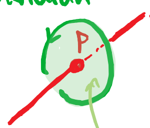
$\pi_1 \Big( S^4 \setminus$  $\Big) \cong \mathbb{Z}$



meridian: boundary of a normal 2-disk of S at point p





BIG open question:

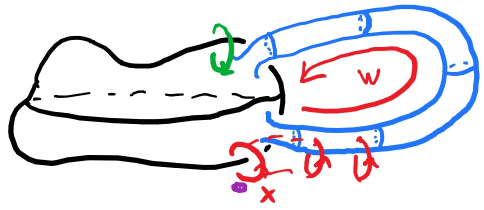
Does $\pi_1$ characterize unknotted surfaces in 4-dim.?

## Effect of 1-handle addition on $\pi_1(\text{complement})$

$$\pi_1(S^4 \setminus (S + h^1)) \cong \frac{\pi_1(S^4 \setminus S)}{\langle\langle [x, w] \rangle\rangle}$$

$x$ = meridian

"guiding arc of 1-handle"



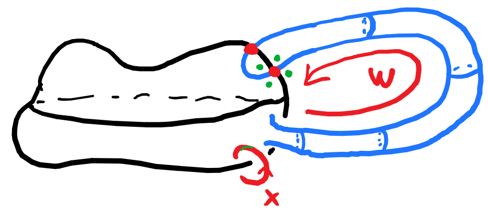

$x = w^{-1} x w$

## Effect of finger move on $\pi_1(\text{complement})$

after finger move on $S$

$$\pi_1(S^4 \setminus S') \cong \frac{\pi_1(S^4 \setminus S)}{\langle\langle [x, \underbrace{w^{-1} x w}_{= x^w}] \rangle\rangle}$$

also a meridian



**Def.:** Weak/algebraic 1-handle unknotting #

$u^{\mathbb{Z}}_{1\text{-h.}}(K)$ := minimal number of elements $g_1, \dots, g_n \in \pi_1(S^4 \setminus K)$

s.th. $$\pi_1(S^4\setminus K) / \langle\langle [g_1, x], \dots, [g_n, x] \rangle\rangle \cong \mathbb{Z}.$$

meridian     normal closure

In words: How many 1-handle relations do we have to add to the group to abelianize it?

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**Def.:** Weak/algebraic Finger-Whitney unknotting #

$u^{\mathbb{Z}}_{\text{F.-Wh.}}$ := minimal number of elements $g_1, \dots, g_n \in \pi_1(S^4 \setminus K)$

s.th. $$\pi_1(S^4\setminus K) / \langle\langle [g_1^{-1} x g_1, x], \dots, [g_n^{-1} x g_n, x] \rangle\rangle \cong \mathbb{Z}.$$

In words: How many finger-move relations do we have to add to the group to abelianize it?

## "Obvious" inequalities:

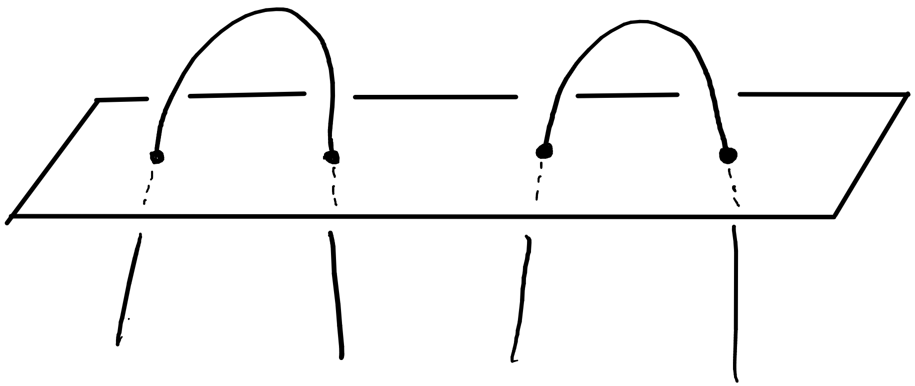
$$\begin{array}{ccc} u^{\mathbb{Z}}_{1-h.} & \leq & u^{\mathbb{Z}}_{F-wh} \\ \rotatebox{-90}{$\leq$} & & \rotatebox{-90}{$\leq$} \\ u_{1-h.} & ??? & u_{F-wh.} \end{array}$$

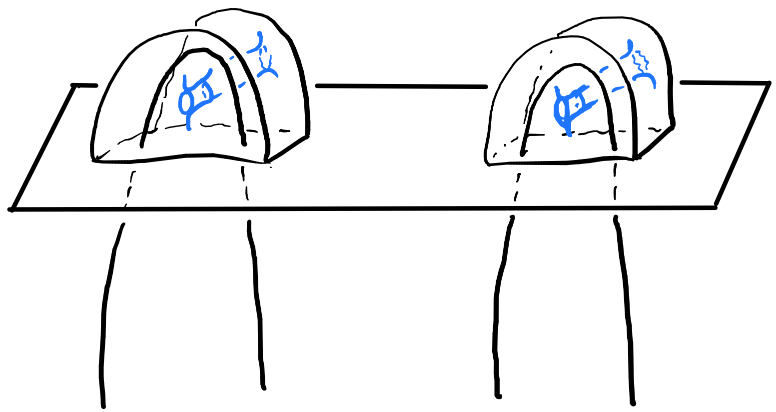
Conjecture: $\leq$ ??

## After Finger moves on $K$:

This is a stabilization of $K$!

(Do you see why?)





We conjecture that this is an unknotted surface

## What we know:

$$\boxed{\begin{array}{ccc} u^{\mathbb{Z}}_{1\text{-}h.} & \leq & u^{\mathbb{Z}}_{F.\text{-}Wh.} \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ u_{1\text{-}h.} & & u_{F.\text{-}Wh.} \end{array}}$$

•) min. # of gen. of Alexander Module of $K \leq u^{\mathbb{Z}}_{F.\text{-}Wh.}(K)$

(Nakanishi index)

$\rightsquigarrow$ $u_{F.\text{-}Wh.}$ can be arbitrarily large

•) We have an example where $1 = u^{\mathbb{Z}}_{1\text{-}h.}(K) \neq u^{\mathbb{Z}}_{F.\text{-}Wh.}(K) = 2$

[Jason]

and the unknotting can be realized geometrically $\rightsquigarrow$ $u_{1\text{-}h.} \neq u_{F.\text{-}Wh.}$

Would like to have examples where the difference is arbitrarily large

•) $u_{F.\text{-}Wh.}\big(\mathrm{Spin}_n(k)\big) \leq u(k)$

later

classical unknotting # of $k: S^1 \hookrightarrow S^3$

•) more upper bounds ...

# Non-additivity

Of course $u_{1\text{-h.}}(K_1 \# K_2) \leq u_{1\text{-h.}}(K_1) + u_{1\text{-h.}}(K_2)$

## But it can fail to be additive:

Miyazaki has an example of ribbon 2-knots $K_1, K_2$ with

$$u_{1\text{-h.}}(K_i) = 1,$$

but there is a new "magical" 1-handle which transforms $K_1 \# K_2$ into an unknotted torus.

MIYAZAKI, K.
KOBE J. MATH.,
3 (1986), 77–85

**ON THE RELATIONSHIP AMONG UNKNOTTING NUMBER, KNOTTING GENUS AND ALEXANDER INVARIANT FOR 2-KNOTS**

By Katura MIYAZAKI

(Received May 24, 1985)

He explicitely draws the isotopy of the fusion bands which shows that the torus is unknotted

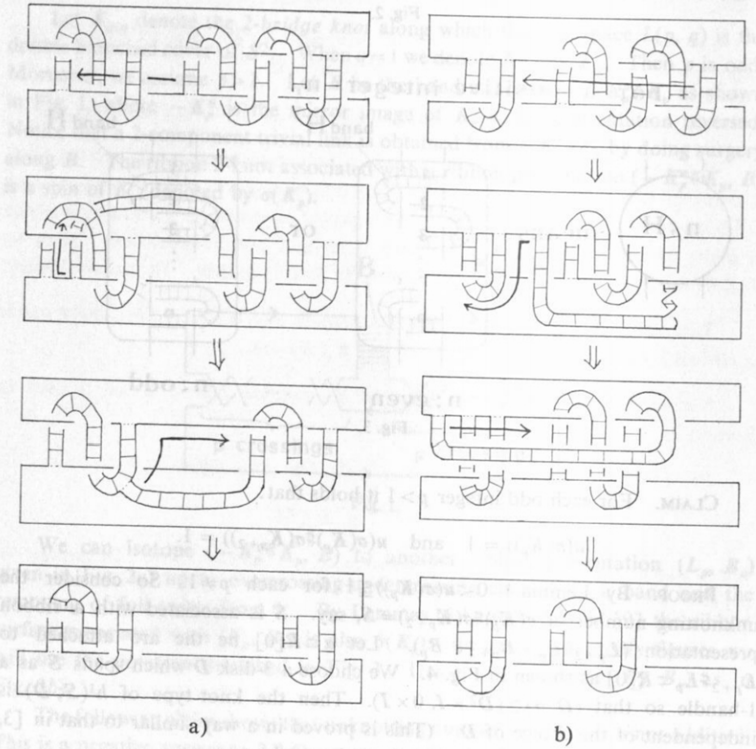



Fig. 6.

Journal of Knot Theory and Its Ramifications, Vol. 5, No. 2 (1996) 161–166


[pictures from Friedman: Knot Spinning]

**WEAK UNKNOTTING NUMBER OF A COMPOSITE 2-KNOT**

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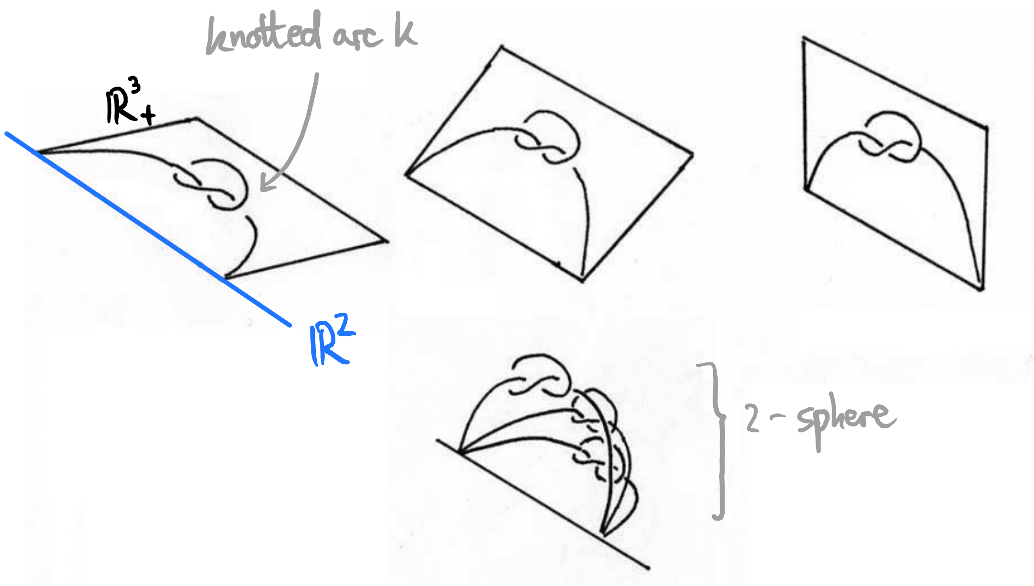
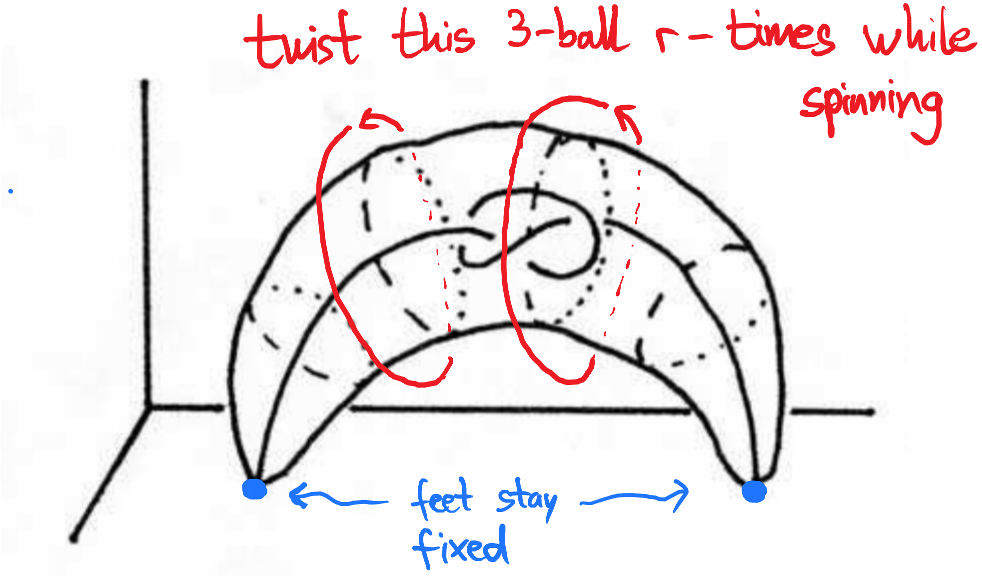
Kanenobu has an example where $u^{\mathbb{Z}}_{1\text{-}h.}$ is as non-additive as it could be

**Recall:** $\mathrm{Spin}_r(k)$ = r-twist spin of a (non-trivial) knot $k$



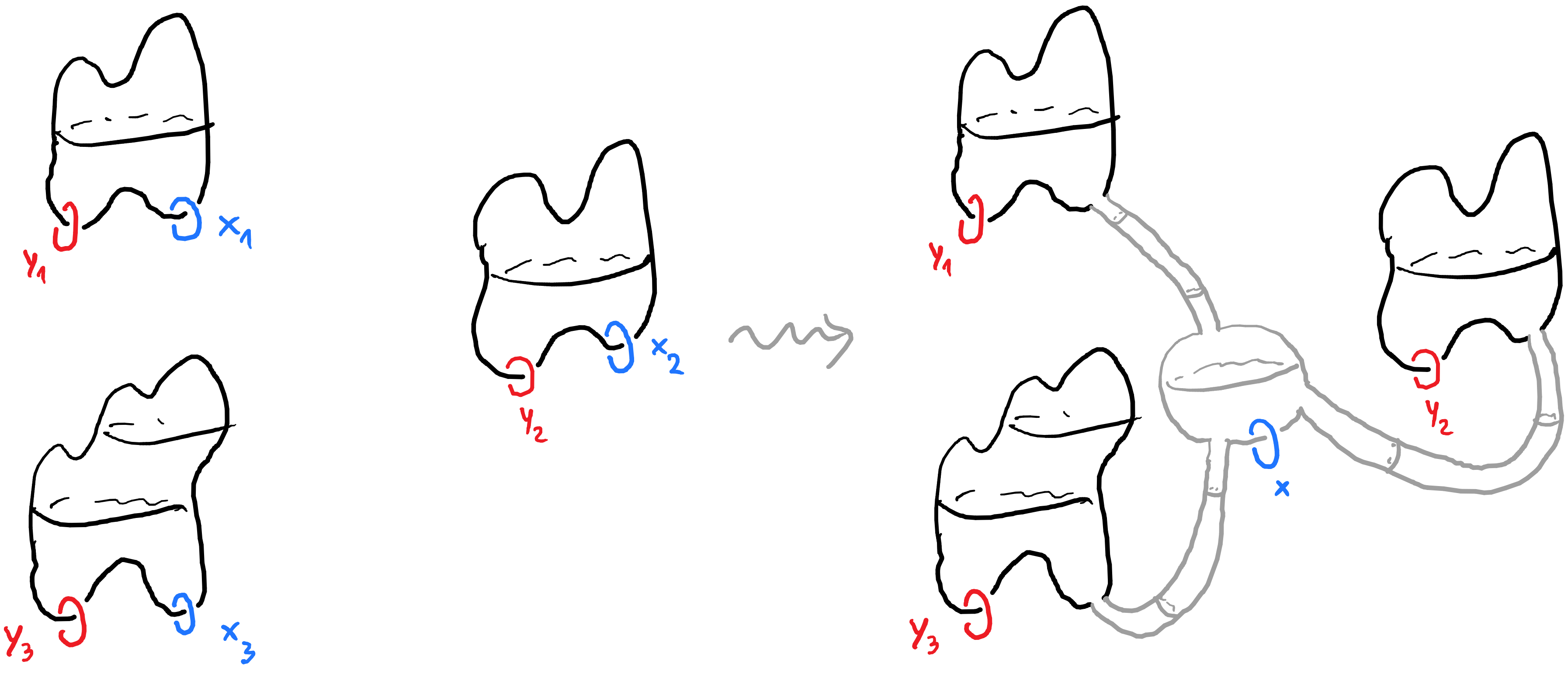
For $k_i$ non-tr. 2-bridge & for natural number $r_i \geq 2$:

$$\boxed{u^{\mathbb{Z}}_{1\text{-}h.}\left(\mathrm{Spin}_{r_i}(k_i)\right) = 1}$$

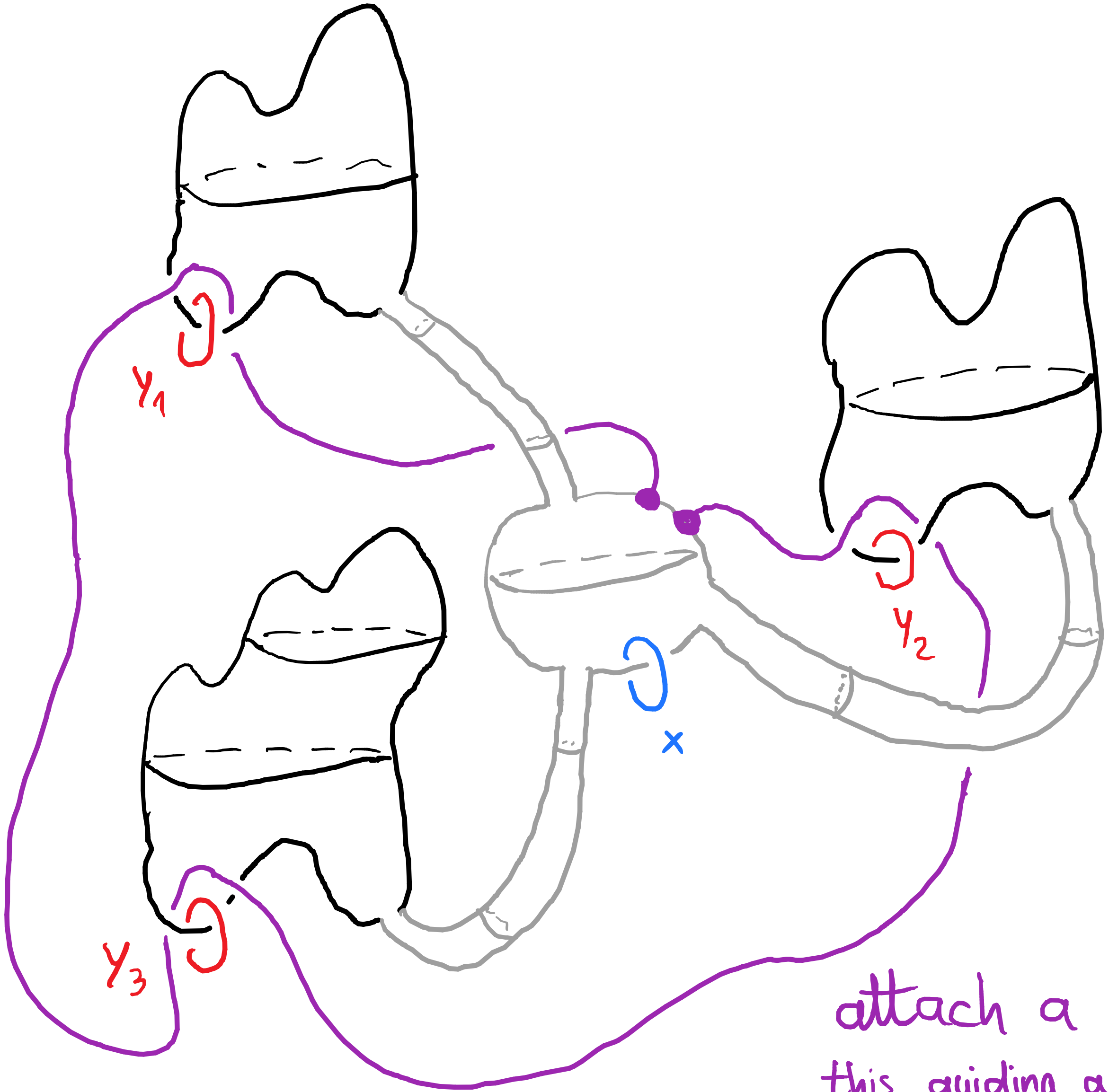
**Lemma:** For $r_1, \ldots, r_n \geq 2$ coprime integers, $k_1, \ldots, k_r$ 2-bridge

$$u^{\mathbb{Z}}_{1\text{-}h.}\left(\mathrm{Spin}_{r_1}(k_1) \# \ldots \# \mathrm{Spin}_{r_n}(k_n)\right) = 1$$

**Kanenobu's example:**



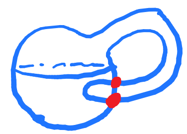
**Claim:** Adding the relation $[x, y_1 \cdot y_2 \cdot \ldots \cdot y_n]$ abelianizes the group




Similar $\pi_1$-calculation works for result of Finger move along this arc

(the immersion complement has $\pi_1 \cong \mathbb{Z}$)

Can we put it in the "standard position?"



attach a 1-handle along this guiding arc to get a torus with $\pi_1(\text{compl.}) \cong \mathbb{Z}$

– is it unknotted?

TODO: Look at "Pulling apart 2-spheres in $S^4$"

for pairing of double points