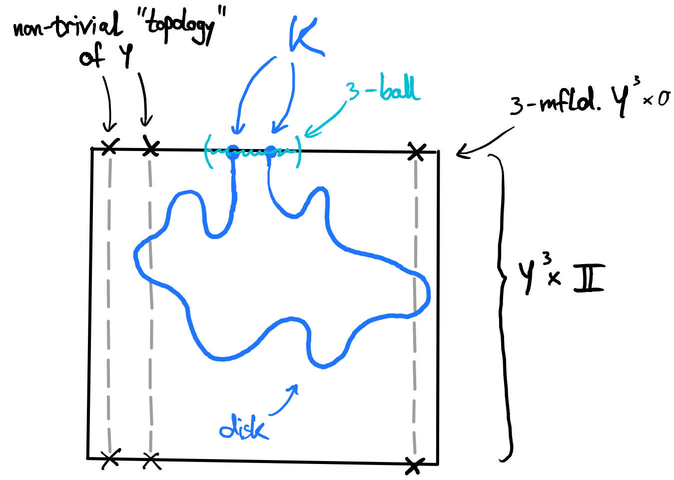
Deeply Slice Knots

2020-04-02

[with Michael Klug]

local knot in 3-ball

Local means that K is contained in a 3-ball: $\underbrace{K \subset D^3 \subset Y^3}$



"Surprising" fun fact:

If a local knot K in a 3-mfld. Y^3 bounds a disk in $Y \times [0,1]$ then K is alredy slice in D^4 !

(in other words, also bounds a disk in $S^3 \times [0,1]$)

- "Surprising", because you might expect that the "extra room" in Y would allow you to construct more disks in $Y \times [0,1]$ than in $S^3 \times [0,1]$, but apparently this is not the case

- Proof uses the nontrivial fact that the universal cover $\widetilde{Y \setminus \operatorname{int} D^3}$ of every (punctured) compact 3-mfld. embedds in S^3.

This was probably known to Perelman and is a corollary of his Geometrization thm., but appears in print later in

Concordance group of virtual knots

Hans U. Boden, Matthias Nagel

(Submitted on 21 Jun 2016)

We study concordance of virtual knots. Our main result is that a classical knot K is virtually slice if and only if it is classically slice. From this we deduce that the concordance group of classical knots embeds into the concordance group of long virtual knots.

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MSC classes: 57M25, 57M27
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Cite as: arXiv:1606.06404 [math.GT]

- Also, compare with

Smooth and topological almost concordance

Matthias Nagel, Patrick Orson, JungHwan Park, Mark Powell

(Submitted on 4 Jul 2017 (v1), last revised 4 Jan 2018 (this version, v2))

We investigate the disparity between smooth and topological almost concordance of knots in general 3-manifolds Y. Almost concordance is defined by considering knots in Y modulo concordance in Yx[0,1] and the action of the concordance group of knots in the 3-sphere that ties in local knots. We prove that the trivial free homotopy class in every 3-manifold other than the 3-sphere contains an infinite family of knots, all topologically concordant, but not smoothly almost concordant to one another. Then, in every lens space and for every free homotopy class, we find a pair of topologically concordant but not smoothly almost concordant knots. Finally, as a topological counterpoint to these results, we show that in every lens space every free homotopy class contains infinitely many topological almost concordance classes.

Comments: 25 pages, 13 figures. To appear in International Mathematics Research Notices
Subjects: Geometric Topology (math.GT)
MSC classes: 57M27, 57N70
Cite as: arXiv:1707.01147 [math.GT]

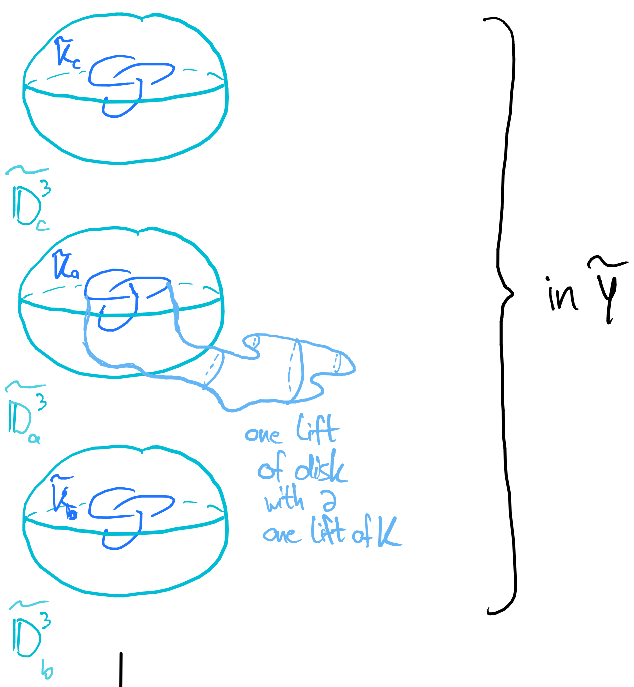
Pf. (of fun fact):

universal cover

$$\begin{tikzcd}
 & \tilde{Y}^3 \times I \arrow[r, hook, "\text{[Perelman, Boden–Nagel]}"] \arrow[d] & S^3 \times I \\
D^2 \arrow[r, hook, "\Delta"'] & Y^3 \times I &
\end{tikzcd}$$

with ∂ = local knot $K \subset Y \times \{0\}$

We can remove $\left(\text{small open 3-ball}\right) \times I$ missing the disk Δ from $Y \times I$ so that we can cite the embedding result for the punctured 3-mfld. $Y \setminus \operatorname{int} D^3$



Lift of disk with ∂ = one of the lifts $\tilde{K}_a$

$$\begin{tikzcd}
 & \tilde{Y}^3 \times I \arrow[r, hook] \arrow[d] & S^3 \times I \\
D^2 \arrow[r, hook, "\Delta"'] \arrow[ur, dashed] & Y^3 \times I &
\end{tikzcd}$$

with ∂ = local knot $K \subset Y \times \{0\}$

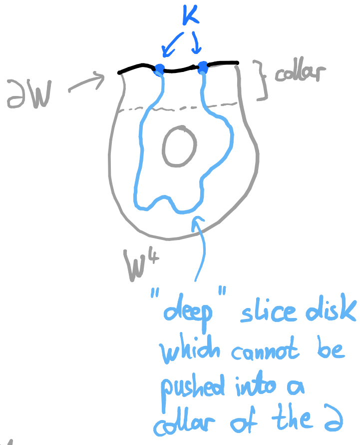
$\square$

Def: $K \subset \partial W^4$ is __deeply slice__ in W^4

if it bounds a properly embedded disk $D^2 \hookrightarrow W$, $\partial D^2 = K$

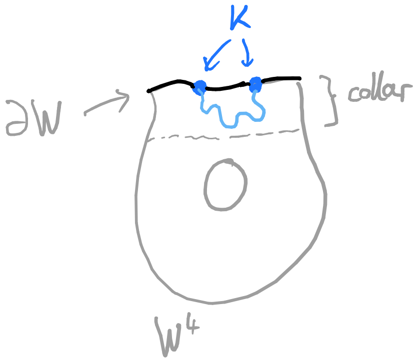
but K is __not__ null-concordant in a collar $\partial W \times I$

of the boundary of W.



All the slice disks of K have to "go deep" into W.

Knots which are null-concordant in a neighborhood of the boundary could be called __shallowly slice__.



__Stupid observation:__

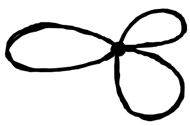
Slice knots in W which are __not even nullhomotopic in ∂W__ will always be deeply slice, since they won't even bound an immersed disk in a collar.

e.g. the $\{pt\} \times S^1 \subset S^2 \times D^2$ is deeply slice

⇝ From now on, only consider nullhomotopic knots $K \subset \partial W$ in the boundary.

Prop: 4-dim. solid 1-handlebodies $\natural^k S^1 \times D^3$ don't contain deeply slice knots.

Pf.: By general position, can make any (2-dim.) disk disjoint from the "core"

 $= \bigvee^k S^1$

then push disk (radially) towards boundary collar

$\square$

Thm.: Any 2-handlebody $(\neq \mathbb{D}^4)$ contains deeply slice knots (null-homotopic) in its boundary.

$\underbrace{\text{2-handlebody}}_{\text{4-dim. } 0\text{-handle} \,\cup\, \text{some number of 2-handles}}$

Pf.: Splits up in two cases $\quad V^4 := \text{2-handlebody} = h^0 \cup \bigcup_\alpha h^2_\alpha$

① $\partial V^4 \neq S^3$ and thus not simply-connected

$\rightsquigarrow$ Wall's self-intersection concordance invariant

$$\mu(K) \in \frac{\mathbb{Z}[\pi_1 \partial V]}{\langle g - g^{-1} \mid g \in \pi_1(\partial V) \rangle \oplus \mathbb{Z}[e]}$$

② $\partial V \cong S^3$, here we consider whether some homology classes can be represented by embedded surfaces and apply one of Rohlin's thms.

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①

Algebraic linking numbers of knots in 3-manifolds

Rob Schneiderman

(Submitted on 4 Feb 2002 (v1), last revised 4 Oct 2003 (this version, v4))

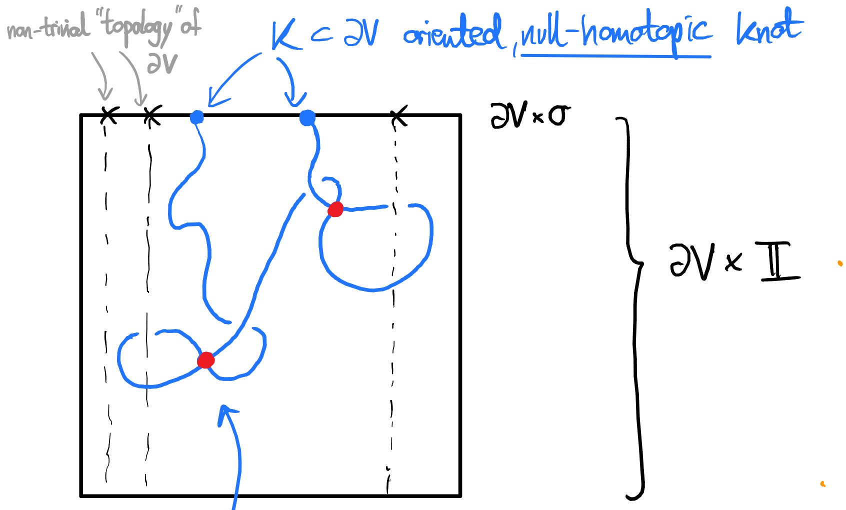
Relative self-linking and linking "numbers" for pairs of knots in oriented 3-manifolds are defined in terms of intersection invariants of immersed surfaces in 4-manifolds. The resulting concordance invariants generalize the usual homological notion of linking by taking into account the fundamental group of the ambient manifold and often map onto infinitely generated groups. The knot invariants generalize the cyclic (type 1) invariants of Kirk and Livingston and when taken with respect to certain preferred knots, called spherical knots, relative self-linking numbers are characterized geometrically as the complete obstruction to the existence of a singular concordance which has all singularities paired by Whitney disks. This geometric equivalence relation, called W-equivalence, is also related finite type-1 equivalence (in the sense of Habiro and Goussarov) via the work of Conant and Teichner and represents a `first order' improvement to an arbitrary singular concordance. For null-homotopic knots, a slightly weaker geometric equivalence relation is shown to admit a group structure.

A Note on Knot Concordance

Eylem Zeliha Yildiz

(Submitted on 6 Jul 2017 (v1), last revised 28 May 2018 (this version, v3))

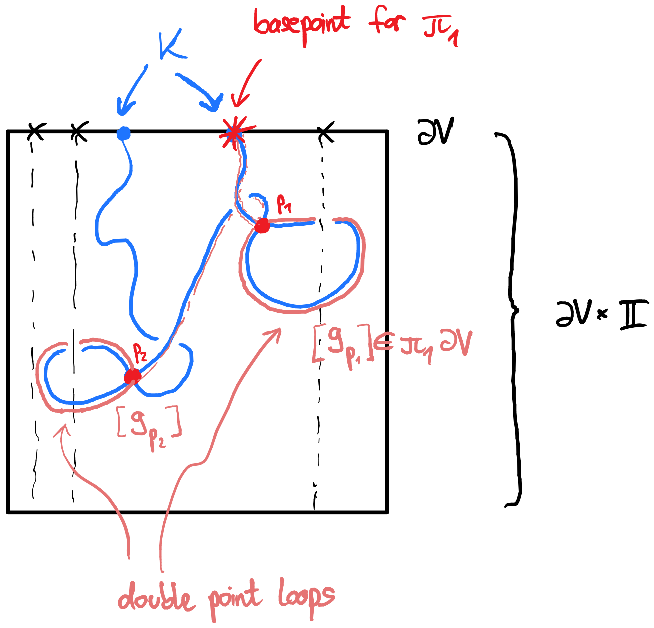
We use classical techniques to answer some questions raised by Daniele Celoria about almost-concordance of knots in arbitrary closed 3-manifolds. We first prove that, given $Y^3 \neq S^3$, for any non-trivial element $g \in \pi_1(Y)$ there are infinitely many distinct smooth almost-concordance classes in the free homotopy class of the unknot. In particular we consider these distinct smooth almost-concordance classes on the boundary of a Mazur manifold and we show none of these distinct classes bounds a PL-disk in the Mazur manifold, but all the representatives we construct are topologically slice. We also prove that all knots in the free homotopy class of $S^1 \times pt$ in $S^1 \times S^2$ are smoothly concordant.



track of nullhomotopy:

immersed $\mathbb{D}^2 \looparrowright \partial V \times I$ with $\partial = K$

and finite number of self intersections



null-homotopic knot

$$\underline{\mu(K)} := \sum_{\substack{p \in \text{double points} \\ \text{in the track} \\ \text{of null-htpy.}}} \underbrace{\text{sign}(p)}_{\in\{\pm 1\}} \cdot \underbrace{g_p}_{\in \pi_1(\partial V)} \in \frac{\mathbb{Z}[\pi_1 \partial V]}{\langle g - g^{-1} \rangle \oplus \mathbb{Z}[e]}$$

(quotient as abelian groups)

changing order of sheets at a double point transforms group element from g_p to g_p^{-1}

cusps introduce intersections with trivial group element

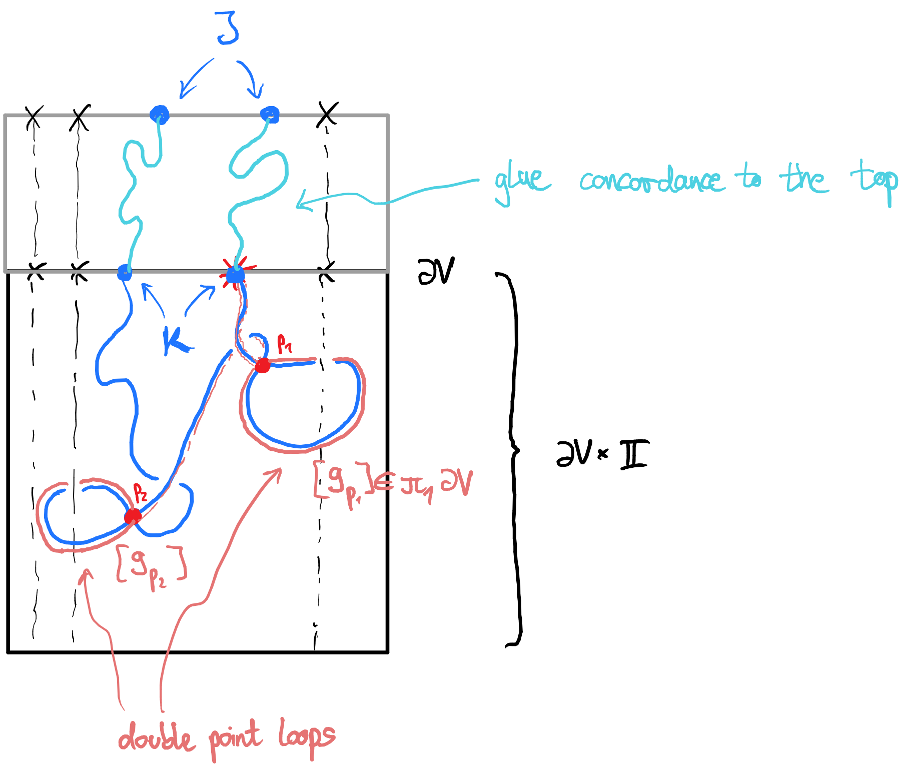
Independent of choice of:

·) nullhomotopy [use Freedman-Quinn to decompose regular homotopy into finger & Whitney moves + statements about embedded spheres representing π_2(3-mfld.]

·) orderings of sheets

·) whiskers on the disk (which is simply-connected)

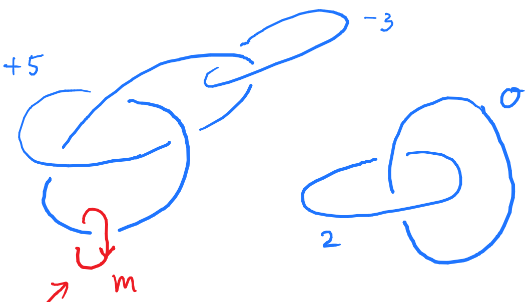
μ is concordance invariant:



Remember we are trying to find a deeply slice knot γ

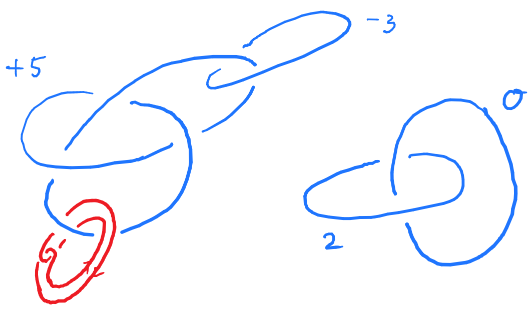
in $\partial V = \partial\left(h^0 \cup \bigcup_{\alpha} h^2_\alpha\right)$ under the assumption that $\pi_1(\partial V) \neq 1$

described by framed Link (attaching circles of the 2-handles)



$\pi_1(\partial V) \neq 1$, so at least one of the meridians is nontrivial in π_1

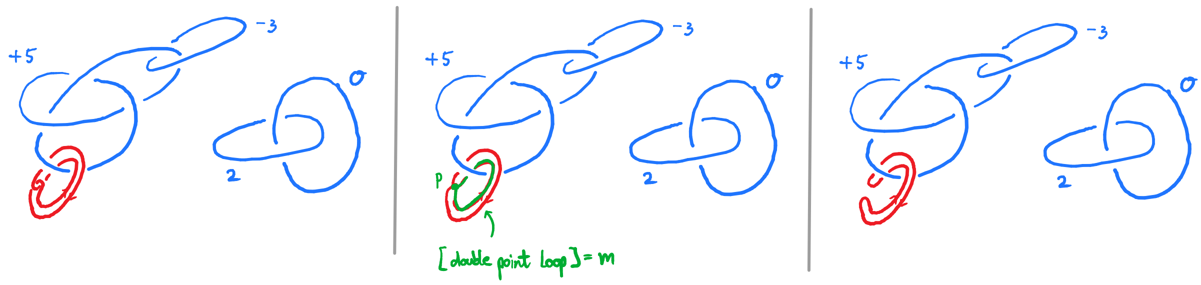
Let's suppose it's this one



Then $\gamma = Wh(m)$ is deeply slice!

The unknotted curve  bounds a disk in the 0-handle

Null-homotopy of γ in ∂V:



$\Rightarrow \mu(\gamma) = m \neq 0 \in \dfrac{\mathbb{Z}[\pi_1 \partial V]}{\langle g - g^{-1} \rangle \oplus \mathbb{Z}[e]}$ and thus γ not null-concordant in $\partial V \times I$

$\square$ 1

② Suppose $\partial V = \partial(h^0 \cup \bigcup h^2_\alpha)$ simply connected

$\Rightarrow \partial V \cong S^3$

[μ is of no use since it takes values in the trivial module]

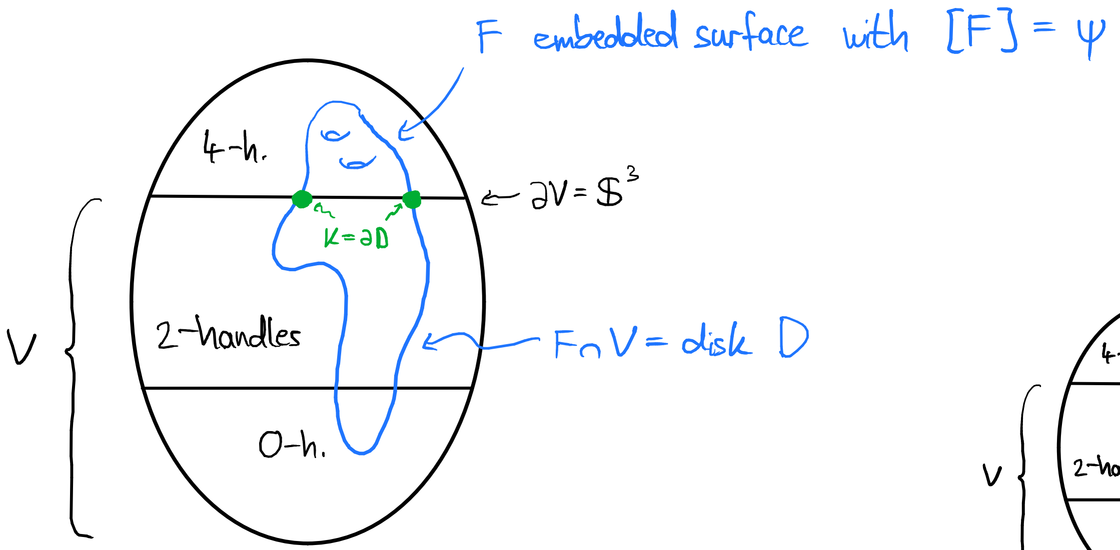
But now we can close V off with a 4-handle: $\hat{V} := V \cup_{\partial = S^3} D^4$

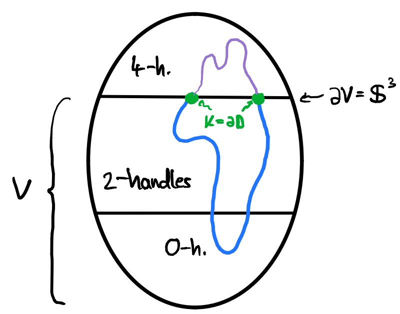
Theorem 3.5 (Rohlin, [Roh71]). *Let X be an oriented closed smooth 4-manifold with $H_1(X;\mathbb{Z}) = 0$. Let $\psi \in H_2(X;\mathbb{Z})$ be an element that is divisible by 2, and let A be a closed oriented surface of genus g smoothly embedded in X that represents ψ. Then*

$$4g \geq |\psi \cdot \psi - 2\sigma(X)| - 2b_2(X)$$

Lemma 3.6. *Let X be a closed smooth 4-manifold with $H_1(X;\mathbb{Z}) = 0$, and $H_2(X;\mathbb{Z}) \neq 0$. Then there exists a homology class $\psi \in H_2(X;\mathbb{Z})$ that cannot be represented by a smoothly embedded sphere.*

We'll use this: Let $\psi \in H_2(\hat{V};\mathbb{Z})$ be a homology class that cannot be represented by an embedded sphere.





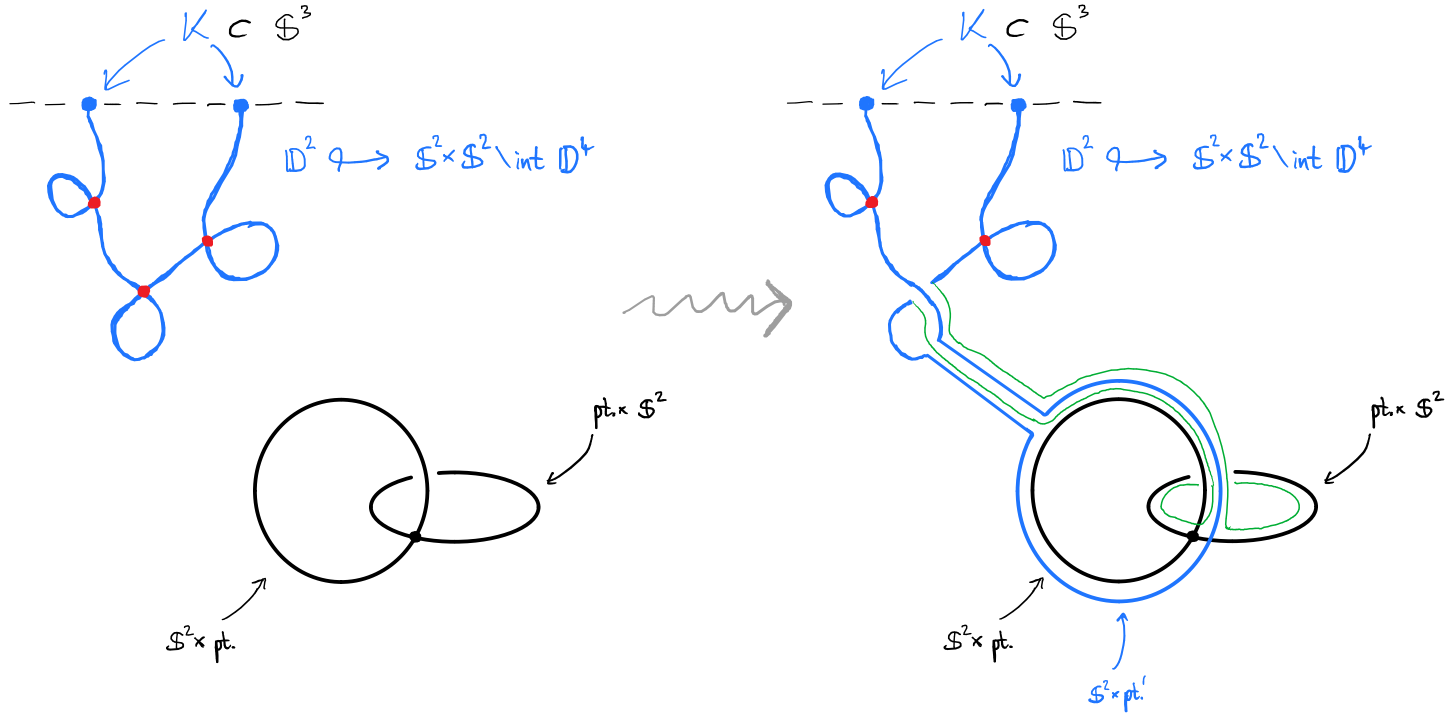
$K = \partial D$ is deeply slice in V, otherwise we could replace the part of F in the 4-handle with an embedded disk and get an (impossible) sphere representative of ψ

□ ②

"Universal slicing" - manifolds

Norman's trick: Any knot $K \subset S^3$ bounds a properly embedded disk in $S^2 \times S^2 \setminus \operatorname{int} D^4$

Pf.: Track of nullhomotopy gives immersed disk, tube into $S^2 \times pt.$, $pt. \times S^2$ to remove double points.



Accurate picture in 2 dim.:

Tubing into the coordinate spheres of $S^2 \times S^2$ changes the homology class, i.e. this process creates embedded disks which are usually not null-homologous

Most people require that a (properly embedded) slice disk

$\Delta : D^2 \hookrightarrow V^4$

satisfies $[\Delta, \partial\Delta] = 0 \in H_2(V^4, \partial V; \mathbb{Z})$

$\Leftrightarrow$ Δ intersects all oriented closed surfaces algebraically zero times

Prop.: K slice (via a null-homologous disk) in a $\left(\#^n S^2 \times S^2\right) \setminus \operatorname{int} D^4$

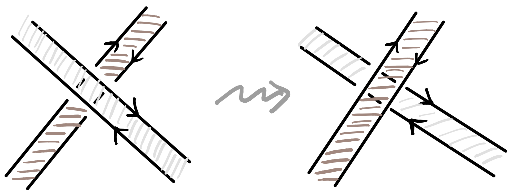
iff. $\operatorname{Arf}(K) = 0$.

STABLY SLICE DISKS OF LINKS

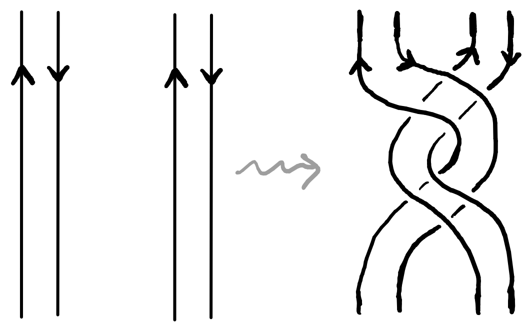
ANTHONY CONWAY AND MATTHIAS NAGEL

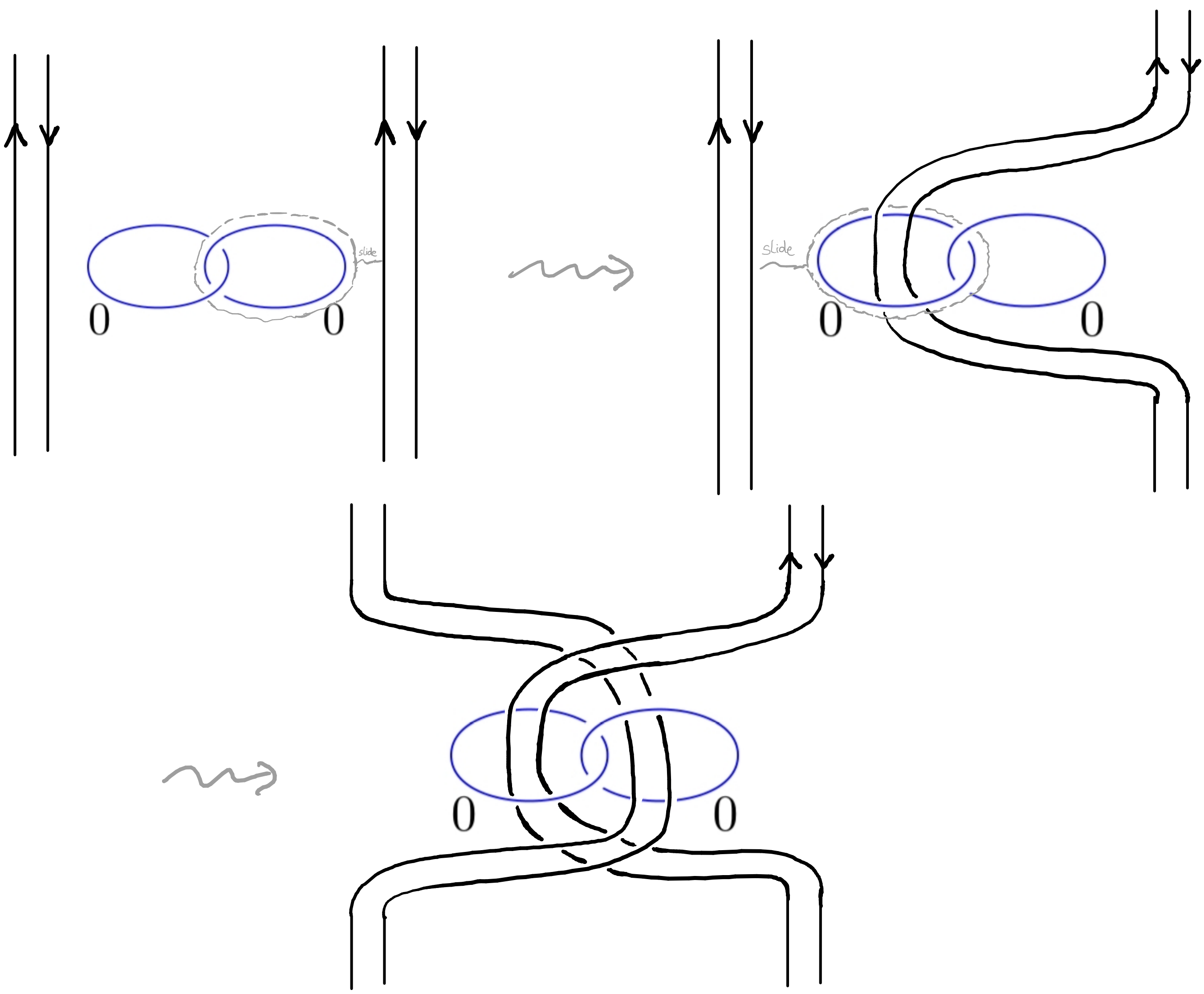
ABSTRACT. We define the stabilizing number $\operatorname{sn}(K)$ of a knot $K \subset S^3$ as the minimal number n of $S^2 \times S^2$ connected summands required for K to bound a nullhomotopic locally flat disc in $D^4 \# nS^2 \times S^2$. This quantity is defined when the Arf invariant of K is zero. We show that $\operatorname{sn}(K)$ is bounded below by signatures and Casson-Gordon invariants and bounded above by the topological 4-genus $g_4^{\mathrm{top}}(K)$. We provide an infinite family of examples with $\operatorname{sn}(K) < g_4^{\mathrm{top}}(K)$.

Pf.: Use the coordinate spheres of $S^2 \times S^2$ to perform band passes



or equivalently





Since we are always sliding pairs of oppositely oriented strands, the track of this movie is a null-homologous cobordism

Prop.: Any knot $K \subset S^3$ is slice (via a null-homologous disk) in some $\left(\#^k \mathbb{CP}^2 \# ^l \overline{\mathbb{CP}^2}\right) \setminus \operatorname{int} D^4$.

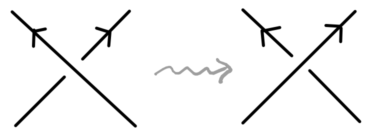
TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 297, Number 1, September 1986

UNKNOTTING INFORMATION FROM 4-MANIFOLDS

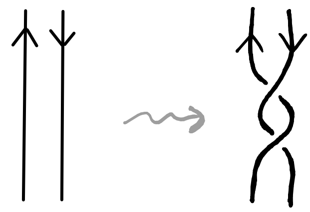
T. D. COCHRAN[1] AND W. B. R. LICKORISH

ABSTRACT. Results of S. K. Donaldson, and others, concerning the intersection forms of smooth 4-manifolds are used to give new information on the unknotting numbers of certain classical knots. This information is particularly sensitive to the signs of the knot crossings changed in an unknotting process.

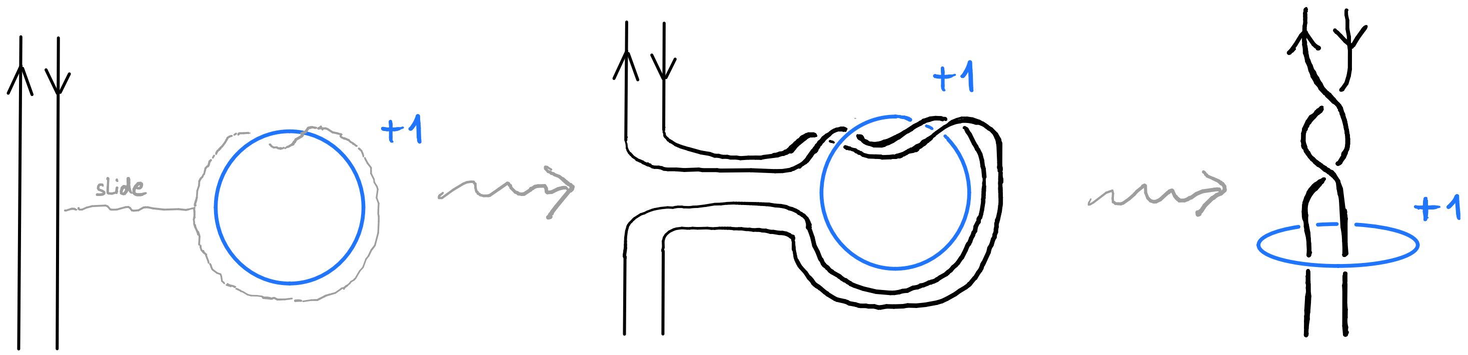
Example of a crossing change:



or equivalently



Pf.:
- Sequence of positive and negative crossing changes leads from K to unknot
- Use the "correct" $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ to remove the double points



Just like before: Since we are always sliding pairs of oppositely oriented strands, the track of this movie is a null-homologous cobordism

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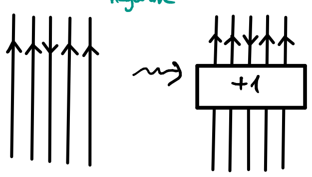
More generally, by sliding over the $(+1)$-2-sphere $\mathbb{CP}^1 \subset \mathbb{CP}^2$,
(-1)-2-sphere $\mathbb{CP}^1 \subset \overline{\mathbb{CP}^2}$,
can add a positive (negative) full twist to a bundle of parallel strands



If the track of this isotopy / movie should give a null-homologous disk, we better only do this to bundles where the orientations algebraically sum up to zero

- There are examples of knots which are slice in $\#^M \mathbb{CP}^2$, but not in $\#^{M-1} \mathbb{CP}^2$

It is probably very interesting to think about a "stabilizing number" for slicing in connected sums of projective planes

Q: Is there a (closed, sm.) 4-mfld. $V^4_{\text{universal}}$

s.th. every knot $K \subset S^3$ is slice in $V_{\text{universal}} \setminus \text{int } D^4$?

(via a nullhomologous disk)

No!

Prop.: Any compact (sm.) 4-mfld W with $\partial W = S^3$ contains a knot in its boundary which is not even topologically slice in W.

Let's construct such a non-slice knot under the additional assumption that $H_1(W) = 0$. We'll use the following generalization of the Murasugi-Tristram inequality:

The next theorem provides an obstruction for two links to cobound a nullhomologous cobordism. [Conway, Nagel] Stably slice disks of links

Theorem 3.8. *Let V be a closed topological 4–manifold with $H_1(V;\mathbb{Z}) = 0$. If $\Sigma \subset (S^3 \times I)\#V$ is a nullhomologous cobordism between two μ-colored links L and L', then*

Murasugi-Tristram signature of a link or knot

nullity

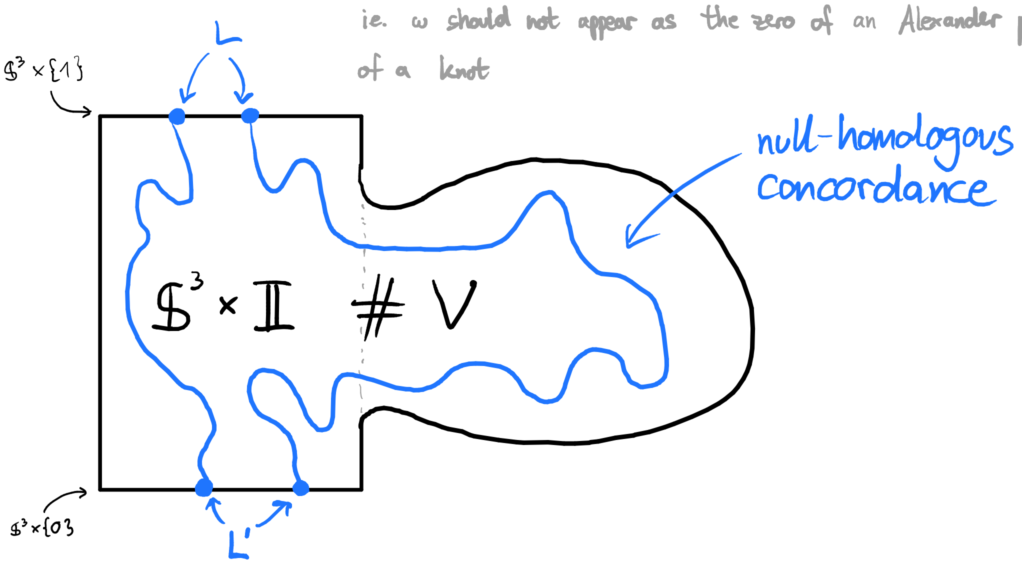
$$|\sigma_{L'}(\omega) - \sigma_L(\omega) + \operatorname{sign}(V)| + |\eta_{L'}(\omega) - \eta_L(\omega)| - \chi(V) + 2 \leq c - \sum_{i=1}^{\mu} \chi(\Sigma_i)$$

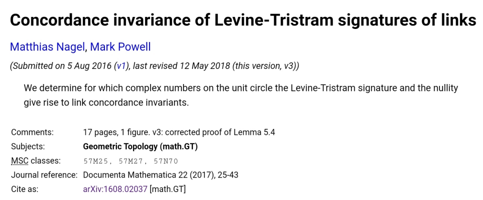
signature of the 4-mfld. V

ignore this (we don't care about cobordisms with singularities here)

for all $\omega \in \mathbb{T}^\mu_!$.

unit complex numbers which are not "Knotennullstellen", ie. ω should not appear as the zero of an Alexander polynomial of a knot




Concordance invariance of Levine-Tristram signatures of links

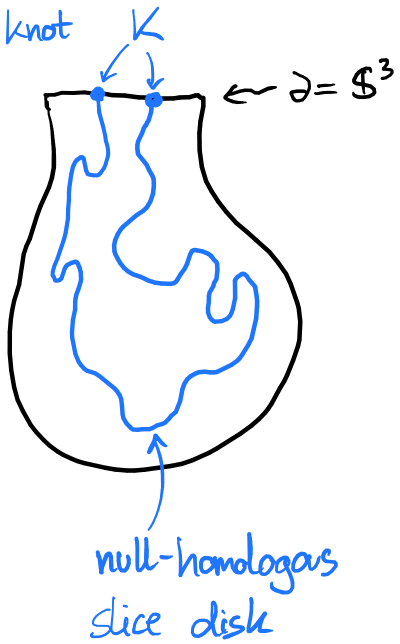
Matthias Nagel, Mark Powell

(Submitted on 5 Aug 2016 (v1), last revised 12 May 2018 (this version, v3))

We determine for which complex numbers on the unit circle the Levine-Tristram signature and the nullity give rise to link concordance invariants.

Comments:	17 pages, 1 figure. v3: corrected proof of Lemma 5.4
Subjects:	Geometric Topology (math.GT)
MSC classes:	57M25, 57M27, 57N70
Journal reference:	Documenta Mathematica 22 (2017), 25-43
Cite as:	arXiv:1608.02037 [math.GT]

In our special case:



punctured 4-mfld. $V^4 \setminus \operatorname{int} D^4$

(where $H_1(V) = 0$)

Then: $$\left| \sigma_K(\omega) + \operatorname{sign}(V) \right| - \chi(V) + 2 \leq 0$$

$\exists$ knots with arbitrary high signature, so for any 4-mfld. V^4 (with $H_1 V = 0$) we can find $K \subset S^3$ for which this inequality is violated, thus K cannot be slice in V $\square$