Deeply Slice Knots
[with Michael Klug]

"Surprising" fun fact:
If a local knot $K$ in a 3-mfld. $Y^3$ bounds a disk in $Y \times [0,1]$ then $K$ is already slice in $D^4 \setminus \partial$
(in other words, also bounds a disk)
(in $S^3 \times [0,1]$)

1) "Surprising", because you might expect that the "extra room" in $Y$ would allow you to construct more disks in $Y \times [0,1]$ than in $S^3 \times [0,1]$, but apparently this is not the case.

2) Proof uses the non-trivial fact that the universal cover $\tilde{Y}$ of every (punctured) compact 3-mfld. embeds in $S^3$.

This was probably known to Perelman and is a corollary of his Geometrization theorem, but appears in print later in

Concordance group of virtual knots
Hans U. Boden, Matthias Nagel
(Submitted on 27 Jun 2019)
We study concordance of virtual knots. Our main result is that a classical knot $K$ is virtually slice if and only if it is classically slice. From this we deduce that the concordance group of classical knots embeds into the concordance group of long virtual knots.

Subjects: Geometric Topology (math.GT)
MSC classes: 57K10 , 57N75
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Smooth and topological almost concordance
Matthias Nagel, Patrick Orson, Junghwan Park, Mark Powell
(Submitted on 4 Jul 2017 (v1), last revised 4 Jan 2018 (v2))
We investigate the disparity between smooth and topological almost concordance of knots in general 3-manifolds $Y$. Almost concordance is defined by considering knots in $Y$ modulo concordance in $Y \setminus \{0\}$ and the action of the concordance group of knots in the 3-sphere that lies in local knots. We prove that the trivial free homotopy class in every 3-manifold other than the 3-sphere contains an infinite family of knots, all topologically concordant, but not smoothly almost concordant to one another. Then, in every lens space and for every free homotopy class, we find a pair of topologically concordant but not smoothly almost concordant knots. Finally, as a topological counterpart to these results, we show that in every lens space every free homotopy class contains infinitely many topological almost concordance classes.

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\[
\begin{align*}
\text{Pf. (of fun fact):} & \quad \text{universal over} \\
\tilde{\mathcal{Y}}^3 \times I & \xrightarrow{\alpha} S^3 \times I \\
\downarrow & \\
D^2 & \xrightarrow{\Delta} \tilde{\mathcal{Y}}^3 \times I \\
\text{with } \alpha = \text{local knot } K = \mathcal{Y} \times [0,1]
\end{align*}
\]

We can remove \((3\text{-ball}) \times I\) missing the disk \(\Delta\) from \(\mathcal{Y} \times I\)
so that we can cite the embedding result
for the punctured 3-mfd. \(\mathcal{Y} \times I\).
Def: $K \subset \partial W^4$ is deeply slice in $W^4$ if it bounds a properly embedded disk $D^2 \to W$, $\partial D^2 = K$ but $K$ is not null-concordant in a collar $\partial W \times I$ of the boundary of $W$.

All the slice disks of $K$ have to "go deep" into $W$. Knots which are null-concordant in a neighborhood of the boundary could be called shallowly slice.

Stupid observation:
Slice knots in $W$ which are not even nullhomotopic in $\partial W$ will always be deeply slice, since they won't even bound an immersed disk in a collar. e.g. the $\{pt\} \times S^1 \subset S^2 \times D^2$ is deeply slice.

$\Rightarrow$ From now on, only consider nullhomotopic knots $K \subset \partial W$ in the boundary.

Prop: 4-dim solid 1-handlebodies $\#^k S^1 \times D^3$ don't contain deeply slice knots.

Pf: By general position, can make any (2-dim.) disk disjoint from the "core"

$$\infty = \bigvee S^1$$

then push disk (radially) towards boundary collar
**Algebraic linking numbers of knots in 3-manifolds**

Rob Schneiderman
(Submitted on 4 Feb 2022 v1, last revised 4 Oct 2022 (this version, v4))

Reflect self-linking and linking "numbers" for pairs of knots in oriented 3-manifolds are defined in terms of intersection invariants of immersed surfaces in 4-manifolds. The resulting concordance invariants generalize the usual homological notion of linking by taking into account the fundamental group of the ambient manifold and often map onto finitely presented groups. The knot invariants generalize the cyclic (type 1) invariants of Kirk and Livingston and when taken with respect to certain preferred knots, called spherical knots, relative self-linking numbers are characterized geometrically as the complete obstruction to the existence of a singular concordance which has all singularities paired by Whitney discs. This geometric equivalence relation, called $\mathcal{W}$-equivalence, is also related to finite type-1 equivalence (in the sense of Habiro and Goussarov) via the work of Constantin and Teichner and represents a 'first order' improvement to an arbitrary singular concordance. For null-homotopic knots, a slightly weaker geometric equivalence relation is shown to admit a group structure.

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**A Note on Knot Concordance**

Eylem Zelilha Yildiz
(Submitted on 6 Jul 2017 v1, last revised 28 May 2018 (this version, v3))

We use classical techniques to answer some questions raised by Daniele Celoria about almost-concordance of knots in arbitrary closed 3-manifolds. We first prove that, given $Y^3 \neq S^3$, for any non-trivial element $g \in \pi_1(Y)$ there are infinitely many distinct smooth almost-concordance classes in the free homotopy class of the unknot. In particular we consider these distinct smooth almost-concordance classes on the boundary of a Mazur manifold and we show none of these distinct classes bounds a PL-disk in the Mazur manifold, but all the representatives we construct are topologically slice. We also prove that all knots in the free homotopy class of $S^1 \times pt$ in $S^1 \times S^2$ are smoothly concordant.

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Thm: Any 2-handlebody contains deeply slice knots in its boundary.

4-dim. $\pi$-handle v some number of 2-handles

**Pf.** Splits up in two cases

1. $\mathcal{E}V^4 \cong S^3$ and thus not simply-connected

   $\Rightarrow$ Wall's self-intersection

   concordance invariant

   $\mu(K) \in \mathbb{Z}[\mathcal{E}V]$

2. $\mathcal{E}V \cong S^3$, here we consider whether some homology classes can be represented by embedded surfaces and apply one of Rohlin's thms.

$V^4 = 2$-handlebody = $H^0 \sqcup \bigcup_{\infty} H^2$

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A non-trivial topology of $\mathcal{E}V$

K in $\mathcal{E}V$ oriented, null-homotopic knot

$\mathcal{E}V \times \sigma$

Track of nullhomotopy: immersed $\mathbb{D}^2 \varphi \rightarrow \mathcal{E}V \times I$ with $\varphi = K$

and finite number of self intersections
$\mu(K) := \sum_{p \in \text{double points in the track of null-homotopy}} \text{sign}(p) \cdot g_p \in \frac{\mathbb{Z}[\pi_1(\mathcal{N})]}{<g, g^{-1}> \oplus \mathbb{Z}[e]}$ (quotient as abelian groups)

- Changing order of sheets at a double point transforms a group element from $g_p$ to $g_p^{-1}$
- Cuts introduce interactions with trivial group elements

Independent of choice of:
- Null homotopy
- Orderings of sheets
- Whiskers on the disk (which is simply-connected)

\( \mu \) is concordance invariant:

\[ \to \]
Remember we are trying to find a deeply slice knot \( \gamma \) in \( \partial V = \partial ( h^0 \cup h_\alpha^2 ) \) under the assumption that \( \pi_1(\partial V) \neq 1 \) described by framed link (attaching circles of the 2-handles).

\[ \pi_1(\partial V) \neq 1, \text{ so at least one of the meridians is nontrivial in } \pi_1. \]

Let's suppose it's this one.

Then \( \gamma = \mu_1(\gamma) \) is deeply slice!

**Null-homotopy of \( \gamma \) in \( \partial V \):**

\[ \Rightarrow \mu(\gamma) = m + 5 \in \mathbb{Z}[x \cap \partial V] / \langle g - g^{-1} \rangle \otimes \mathbb{Z}[x] \]

and thus \( \gamma \) not null-concordant in \( \partial V \times \mathbb{I} \).
Suppose $\partial V = \partial (h^0 \cup h_x^2)$ simply connected

$\Rightarrow \partial V = S^2$

[$\nu$ is of no use since it takes values in the trivial module]

But now we can close $V$ off with a 4-handle: $\hat{V} = V \cup_{S^2} D^4$

Theorem 3.5 (Rohlin, [Roh71]). Let $X$ be an oriented closed smooth 4-manifold with $H_1(X; \mathbb{Z}) = 0$. Let $\psi \in H_2(X; \mathbb{Z})$ be an element that is divisible by 2, and let $A$ be a closed oriented surface of genus $g$ smoothly embedded in $X$ that represents $\psi$. Then

$$4g \geq |\psi \cdot \psi - 2\sigma(X)| - 2b_2(X)$$

Lemma 3.6. Let $X$ be a closed smooth 4-manifold with $H_1(X; \mathbb{Z}) = 0$, and $H_2(X; \mathbb{Z}) \neq 0$. Then there exists a homology class $\psi \in H_2(X; \mathbb{Z})$ that cannot be represented by a smoothly embedded sphere.

We’ll use this: Let $\psi \in H_2(\hat{V}; \mathbb{Z})$ be a homology class that cannot be represented by an embedded sphere.

$\hat{V}$ is a 4-manifold with $\partial \hat{V} = S^2$.

We have $\partial V = S^2$.

$F \cap V = \text{disk } D$

$K = \partial D$ is deeply slice in $V$, otherwise we could replace the part of $F$ in the 4-handle with an embedded disk and get an (impossible) sphere representative of $\psi$.

\[ \Box \]
"Universal slicing" - manifolds

Norman's trick: Any knot \( K \subset S^3 \) bounds a properly embedded disk in \( S^2 \times S^2 \setminus \text{int } D^4 \)

**Pf.:** Track of nullhomotopy gives immersed disk, tube into \( S^2 \times \text{pt.}, \text{pt} \times S^2 \) to remove double points.

Accurate picture in 2 dim:

Tubing into the coordinate spheres of \( S^2 \times S^2 \) changes the homology class, i.e. this process creates embedded disks which are usually not null-homologous

Most people require that a (properly embedded) slice disk \( \Delta: \partial D^2 \to V' \) satisfies \( [\Delta, \partial \Delta] = 0 \in H_2(V', \mathbb{Z}) \)

\( \Rightarrow \) \( \Delta \) intersects all oriented closed surfaces algebraically zero times
Prop: K slice (via a null-homologous disk) in a
\[
(\#^n \mathbb{S}^2 \times \mathbb{S}^2) \setminus \text{int } D^4
\]
iff \( \text{Arf}(K) = 0 \).

pf: Use the coordinate spheres of \( \mathbb{S}^2 \times \mathbb{S}^2 \)
to perform band passes

or equivalently

Since we are always sliding pairs of oppositely oriented strands, the track of this movie is a null-homologous cobordism.
**Theorem:** Any knot \( K \subset S^3 \) is slice (via a null-homologous disk) in some \( \left( \#^k \mathbb{CP}^2, \#^l \overline{\mathbb{CP}^2} \right) \setminus \text{int } D^+ \).

**Abstract.** Results of S. K. Donaldson, and others, concerning the intersection forms of smooth 4-manifolds are used to give new information on the unknotting numbers of certain classical knots. This information is particularly sensitive to the signs of the knot crossings changed in an unknotting process.

**Proof:**

1. Sequence of positive and negative crossing changes leads from \( K \) to unknot.
2. Use the “correct” \( \mathbb{CP}^2 \) or \( \overline{\mathbb{CP}^2} \) to remove the double points.

**Just like before:** Since we are always sliding pairs of oppositely oriented strands, the track of this move is a null-homologous cobordism.

More generally, by sliding over the \((+1)\)-2-sphere \( \mathbb{CP}^2 \cong \overline{\mathbb{CP}^2} \), \((-1)\)-2-sphere \( \mathbb{CP}^2 \cong \overline{\mathbb{CP}^2} \),

- Add a positive full twist to a bundle of parallel strands.
- If the track of this cobordism/move should give a null-homologous disk, we better only do this to bundles where the orientations algebraically sum up to zero.

There are examples of knots which are slice in \( \#^m \mathbb{CP}^2 \), but not in \( \#^{m-1} \mathbb{CP}^2 \).

It is probably very interesting to think about a "stabilizing number" for slicing in connected sums of projective planes.
Q: Is there a (closed, sm.) $4$--mfld. $V_{\text{univ}}$ s.t. every knot $K = S^3$ is slice in $V_{\text{univ}} \setminus \text{int } D^4$? (via a nullhomologous disk)

No!

Prop: Any compact (sm.) $4$--mfld. $W$ with $\partial W = S^3$ contains a knot in its boundary which is not even topologically slice in $W$.

Let's construct such a non-slice knot under the additional assumption that $H_1(W) = 0$.

We'll use the following generalization of the Murasugi--Tristram inequality:

The next theorem provides an obstruction for two links to cobound a nullhomologous cobordism.

Theorem 3.8. Let $V$ be a closed topological $4$--manifold with $H_1(V; \mathbb{Z}) = 0$. If $\Sigma \subset (S^3 \times I) \# V$ is a nullhomologous cobordism between two $\mu$--colored links $L$ and $L'$, then

$$|\sigma_{L'}(\omega) - \sigma_L(\omega) + \text{sign}(V)| + |\eta_{L'}(\omega) - \eta_L(\omega)| - \chi(V) + 2 \leq c - \sum_{i=1}^{\mu} \chi(\Sigma_i)$$

for all $\omega \in \mathbb{T}^4$.

In the next theorem, we use the fact that the null-homologous concordance of $S^3 \times \mathbb{I}$ is an obstruction to concordance invariance of Levine--Tristram signatures of links.

Concordance invariance of Levine--Tristram signatures of links

Matthias Nagel, Mark Powell

(Submitted on 3 Aug 2016; last revised 12 May 2018)

We determine for which complex numbers $\omega$ the concordance on the Levine--Tristram signature and the nullity give the link concordance invariants.

Concordance invariance of Levine--Tristram signatures of links

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We determine for which complex numbers $\omega$ the concordance on the Levine--Tristram signature and the nullity give the link concordance invariants.
In our special case:

\[
\text{null-homologous slice disk}
\]

Then:

\[
\left| \sigma_K(\omega) + \text{sign}(V) \right| - \chi(V) + 2 \leq 0
\]

For any 4-mfld. \( V^4 \) (with \( H_4(V) = 0 \)) we can find \( K = S^3 \) for which this inequality is violated, thus \( K \) cannot be slice in \( V \) \( \square \)