

# Eight faces of the Poincaré Homology Sphere

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# Kirby & Scharlemann's paper from 1979

EIGHT FACES OF THE POINCARÉ HOMOLOGY 3-SPHERE

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- ▶ **Based on:** Kirby, R.C. and Scharlemann, M.G., 1979. Eight faces of the Poincaré homology 3-sphere. In Geometric topology (pp. 113-146). Academic Press. [KS79]

*We give eight different descriptions of the Poincaré homology sphere, and show that they do define the same 3-manifold. The definitions are: (1) plumbing on the  $E_8$  graph, (2) surgery on the  $E_8$  link, (3) the link of the singularity  $z_1^2 + z_2^3 + z_3^5 = 0$ , (4)  $S^3/I^*$  where  $I^*$  is the binary icosahedral group, (5) the dodecahedral space, (6) the Seifert bundle, (7) surgery on the trefoil knot, (8) the  $p$ -fold cover of the  $(g,r)$ -torus knot, for  $\{p,q,r\} = \{2,3,5\}$ .*

# Poincaré's original construction from 1904

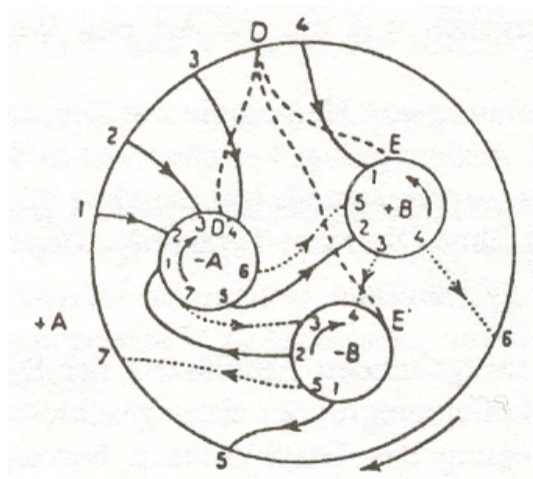
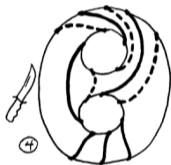
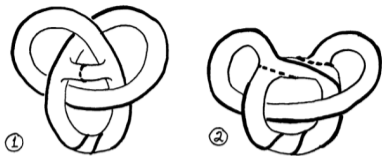
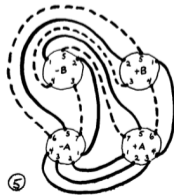


Figure 1: Genus 2 Heegaard diagram from [Poi04], more on the history at the Manifold Atlas

- ▶ Here Rolfsen [Rol03] demonstrates that  $(-1)$ -Dehn surgery on the trefoil describes the same 3-manifold as Poincaré's Heegaard diagram



Demonstration of the equivalence of the constructions of Dehn and Poincaré, producing a homology sphere with fundamental group of order 120.



# Quaternions

3blue1brown's interactive video series  
on visualizing quaternion  
multiplication

<https://eater.net/quaternions>

## Visualizing quaternions

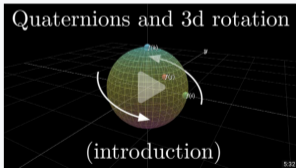
An explorable video series

Lessons by Grant Sanderson  
Technology by Ben Eater



### Quaternions and 3d rotation

One of the main practical uses of quaternions is in how they describe 3d rotation. These first two modules will help you build an intuition for which quaternions correspond to which 3d rotations, although how exactly this works will, for the moment, remain a black box. Analogous to opening a car hood for the first time, all of the parts will be exposed to you, especially as you poke at it more, but understanding how it all fits together will come in due time. Here we are just looking at the "what", before the "how" and the "why".



Watch a recording of this explorable video on YouTube

Double Cover

How do these fit with the existing 3blue1brown YouTube videos?

In addition to this sequence of explorable videos, there are two videos on YouTube on the subject. Some of the material here is duplicated, but you may find a different take on it helpful:

- [What are quaternions, and how do you visualize them? A story of four dimensions.](#) Describes a way to visualize a hypersphere using stereographic projection and understand quaternion multiplication in terms of certain actions on this hypersphere.
- [Quaternions and 3d rotation, explained](#)

## Filling a 3-sphere with 120 dodecahedrons

- ▶ Visualization of the 120-cell:  
<https://www.youtube.com/watch?v=MFXRW9goTs>

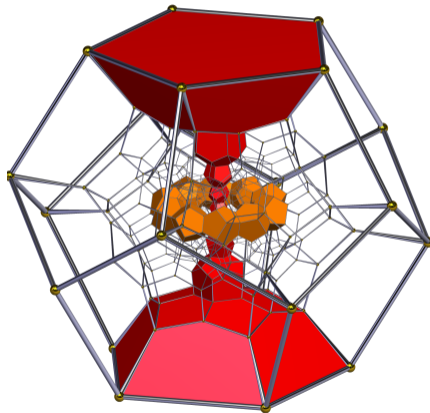


Figure 2: 120-cell, image by Robert Webb's Stella software








Figure 3: Leslie Valiant (Nevanlinna Prize), Michael Hartley Freedman, Gerd Faltings, Simon Donaldson (Fields Medalists), at the ICM 1986 in Berkeley. [Link]

## Further reading

- ▶ Connections between Algebraic curve singularities and knot theory: [Poincaré homology sphere and exotic spheres from links of hypersurface singularities], [Knot Theory and Problems on Affine Space]
- ▶ Group trisections: [AGK18] with a purely group-theoretic formulation of the smooth 4-dimensional Poincaré conjecture
- ▶ Kirby calculus (graphical calculus for manipulating handle decompositions of 4-manifolds): [GS99]
- ▶ Rolfsen [Rol03]: The “Old Testament” of knot theory and low-dimensional topology



# Bibliography

-  Aaron Abrams, David Gay, and Robion Kirby, *Group trisections and smooth 4-manifolds*, *Geometry & Topology* **22** (2018), no. 3, 1537–1545, <https://arxiv.org/abs/1605.06731>.
-  Robert E Gompf and András I Stipsicz, *4-manifolds and Kirby calculus*, no. 20, American Mathematical Soc., 1999.
-  Robion C Kirby and Martin G Scharlemann, *Eight faces of the Poincaré homology 3-sphere*, *Geometric topology*, Elsevier, 1979, Available from Martin Scharlemann's website: <http://web.math.ucsb.edu/~mgscharl/Publications.html>; in the first Google result <https://www.maths.ed.ac.uk/~v1ranick/papers/kirbysch.pdf> page 131 is missing!, pp. 113–146.
-  Jules Henri Poincaré, *Cinquième complément à l'analysis situs*, *Rendiconti del Circolo Matematico di Palermo (1884-1940)* **18** (1904), no. 1, 45–110, <https://doi.org/10.1007/BF03014091>.
-  Dale Rolfsen, *Knots and links*, vol. 346, American Mathematical Soc., 2003, <https://www.maths.ed.ac.uk/~v1ranick/papers/rolfsen.pdf>.