Eight faces of the Poincaré Homology Sphere

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Website: `http://ben300694.github.io`

Kirby & Scharlemann's paper from 1979

- **Based on:** Kirby, R.C. and Scharlemann, M.G., 1979. Eight faces of the Poincaré homology 3-sphere. In Geometric topology (pp. 113-146). Academic Press. [KS79]

EIGHT FACES OF THE POINCARÉ HOMOLOGY 3-SPHERE

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We give eight different descriptions of the Poincaré homology sphere, and show that they do define the same 3-manifold. The definitions are: (1) plumbing on the E_8 graph, (2) surgery on the E_8 link, (3) the link of the singularity $z_1^2 + z_2^3 + z_3^5 = 0$, (4) S^3/I^* where I^* is the binary icosahedral group, (5) the dodecahedral space, (6) the Seifert bundle, (7) surgery on the trefoil knot, (8) the p-fold cover of the (q,r)-torus knot, for $\{p,q,r\} = \{2,3,5\}$.

Poincaré's original construction from 1904

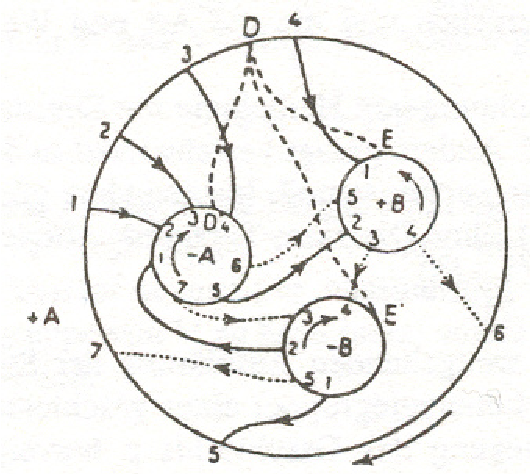
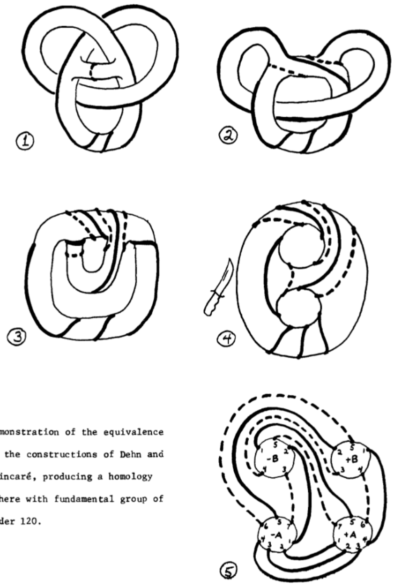


Figure 1: Genus 2 Heegaard diagram from [Poi04], more on the history at the Manifold Atlas

- Here Rolfsen [Rol03] demonstrates that (-1)-Dehn surgery on the trefoil describes the same 3-manifold as Poincaré's Heegaard diagram

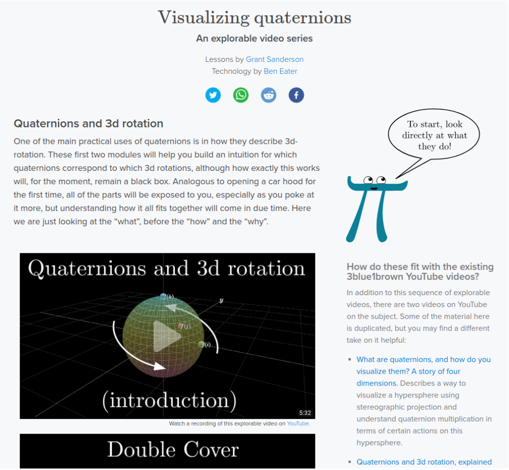


Demonstration of the equivalence of the constructions of Dehn and Poincaré, producing a homology sphere with fundamental group of order 120.

Quaternions

3blue1bown's interactive video series on visualizing quaternion multiplication

`https://eater.net/quaternions`



Filling a 3-sphere with 120 dodecahedrons

- Visualization of the 120-cell:
  `https://www.youtube.com/watch?v=MFXRRW9goTs`

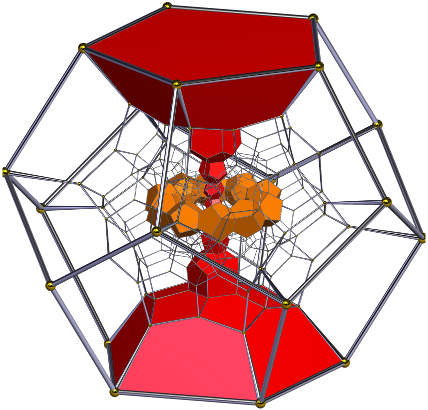


Figure 2: 120-cell, image by Robert Webb's Stella software

Figure 3: Leslie Valiant (Nevanlinna Prize), Michael Hartley Freedman, Gerd Faltings, Simon Donaldson (Fields Medalists), at the ICM 1986 in Berkeley. [Link]

Further reading

- Connections between Algebraic curve singularities and knot theory: [Poincaré homology sphere and exotic spheres from links of hypersurface singularities], [Knot Theory and Problems on Affine Space]
- Group trisections: [AGK18] with a purely group-theoretic formulation of the smooth 4-dimensional Poincaré conjecture
- Kirby calculus (graphical calculus for manipulating handle decompositions of 4-manifolds): [GS99]
- Rolfsen [Rol03]: The “Old Testament” of knot theory and low-dimensional topology

Bibliography

Aaron Abrams, David Gay, and Robion Kirby, *Group trisections and smooth 4–manifolds*, Geometry & Topology **22** (2018), no. 3, 1537–1545, https://arxiv.org/abs/1605.06731.

Robert E Gompf and András I Stipsicz, *4-manifolds and Kirby calculus*, no. 20, American Mathematical Soc., 1999.

Robion C Kirby and Martin G Scharlemann, *Eight faces of the Poincaré homology 3-sphere*, Geometric topology, Elsevier, 1979, Available from Martin Scharlemann's website: http://web.math.ucsb.edu/~mgscharl/Publications.html; in the first Google result https://www.maths.ed.ac.uk/~v1ranick/papers/kirbysch.pdf page 131 is missing!, pp. 113–146.

Jules Henri Poincaré, *Cinquième complément à l'analysis situs*, Rendiconti del Circolo Matematico di Palermo (1884-1940) **18** (1904), no. 1, 45–110, https://doi.org/10.1007/BF03014091.

Dale Rolfsen, *Knots and links*, vol. 346, American Mathematical Soc., 2003, https://www.maths.ed.ac.uk/~v1ranick/papers/rolfsen.pdf.