2020-01-29

Some faces of the Poincaré Homology Sphere

Plan:
- ·) Story about singularities of complex algebraic curves
- ·) Poincaré's original conjecture (and his own counterexample)
- ·) Heegaard splittings & Dehn surgery
- ·) Many descriptions of the Poincaré sphere [↗ Handout]
  (and an indication why they describe the same 3-manifold)
- ·) Identifying opposite faces of a dodecahedron
  and Quaternion multiplication [↗ Slides]

Sources:

[Kirby, Scharlemann: Eight faces of the Poincaré Homology 3-Sphere]

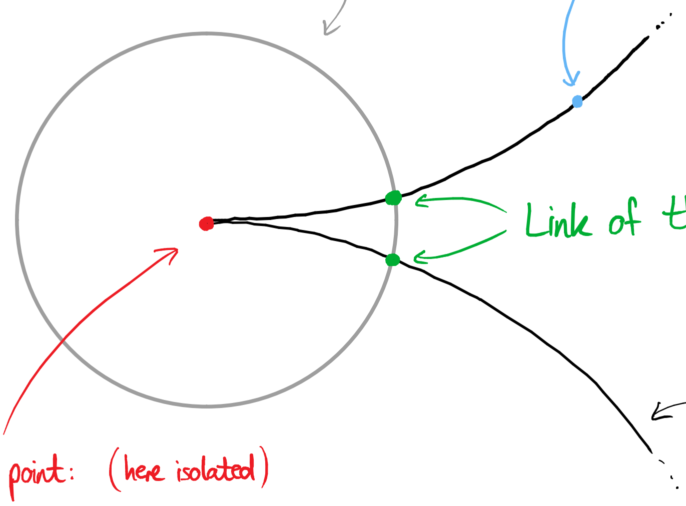
Pictures taken from
- ·) [Rolfsen: Knots and Links]
- ·) [Friedl: Algebraic topology]
- ·) **From Heegaard splittings to trisections; porting 3-dimensional ideas to dimension 4**
  DAVID T GAY
- ·) [Kauffman: On Knots]
- ·) [Poincaré (1904)]
- ·) [Seifert, Threlfall: Lehrbuch der Topologie (1934)]

Motivation: Singularities of complex algebraic curves

Ambient space $\mathbb{C}^n$

intersect the curve with a small sphere $\mathbb{S}^{2n-1}_\varepsilon$ centered around the singularity

non-singular/smooth point: Implicit function thm. tells us that the curve locally looks like $\mathbb{C}^{n-1} \times \sigma \subset \mathbb{C}^n$



Link of the singularity: $V(f) \cap \mathbb{S}^{2n-1}_\varepsilon =: \mathrm{Link}(f)$

Singular point: (here isolated)

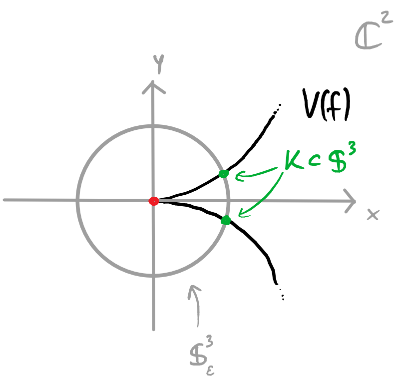
$p \in V(f)$ where the (complex) gradient $\nabla f := \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_n} \right)$ vanishes at p

Vanishing locus of a (non-const.) complex polynomial $f \in \mathbb{C}[z_1, \dots, z_n]$

$$V(f) := f^{-1}(\{0\}) \subset \mathbb{C}^n$$

Examples:

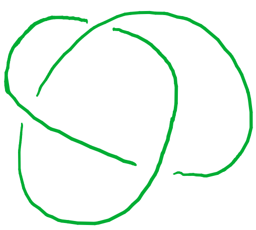
① $f(x,y) = x^2 + y^3$, $V(f) \subset \mathbb{C}^2$

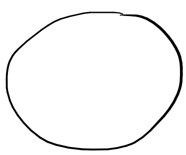


$V(f) \cap \mathbb{S}^3_\varepsilon =: K$ 1-manifold,

$K \underset{\text{diffeo}}{\cong} \mathbb{S}^1$ but with an interesting embedding in $\mathbb{S}^3$

$$(K^1 \subset \mathbb{S}^3) \underset{\text{isotopic}}{\not\cong} (\underbrace{\mathbb{S}^1 \subset \mathbb{S}^3}_{\text{unknot}})$$





Knots which arise as links of singularities of algebraic curves are called <u>algebraic knots</u>.

(2) $f(x,y,z) = x^2 + y^3 + z^5$, $V(f) \subset \mathbb{C}^3$ surface singularity at $(0,0,0)$

$P^3 = \mathrm{Link}(f) \subset \mathbb{S}^5$

3-manifold with the same integral homology groups as the 3-sphere, $H_*(P^3;\mathbb{Z}) \cong H_*(\mathbb{S}^3;\mathbb{Z})$

but

$P^3 \not\cong_{\text{homeo}} \mathbb{S}^3$ (for example, P^3 is <u>not</u> simply connected)

$\boxed{P^3 \text{ is the Poincaré homology sphere}}$

(3) $f = z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2$, $V(f) \subset \mathbb{C}^5$ singularity of a fourfold at 0

$K^7 = V(f) \cap \mathbb{S}^9_\varepsilon$

<u>Facts:</u>
- K^7 is an integral homology sphere
- K^7 is simply connected

$\xRightarrow[\text{theorem}]{\text{Smale's h-cobordism}}$ $K^7 \approx_{\text{homeo}} \mathbb{S}^7$

But K^7 is <u>not</u> diffeomorphic to $\mathbb{S}^7$.

$\boxed{K^7 \text{ is Milnor's original exotic 7-sphere}}$ [he constructed it using $\mathbb{S}^3$- bundles over $\mathbb{S}^4$]

$z_1^5 + z_2^3 + z_3^2 + z_4^2 + z_5^2 + z_6^2$ gives Kervaire's exotic 9-sphere , ...

Poincaré's original conjecture & his own counterexample & the fix

Observations: ·) A (closed) 1-manifold with the homology of $\mathbb{S}^1$ is already homeomorphic to $\mathbb{S}^1$.

·) Classification of surfaces:  ... $\mathbb{RP}^2$, $\mathbb{RP}^2 \# \mathbb{RP}^2$, ...

A homology 2-sphere is already $\mathbb{S}^2$

In 1900, Poincaré suspected that the same holds for $n=3$, but
... in 1903, he found a counterexample to his own claim!

Prop.: There is a closed 3-dim. mfld. M which is a homology 3-sphere, but such that $\pi_1(M)$ is a non-trivial group.

(Updated)

$\boxed{\text{Poincaré conjecture [1903]: If } M^3 \text{ is a topological homology 3-sphere that is simply connected, then } M \text{ is homeomorphic to } \mathbb{S}^3.}$

Solved in the positive almost 100 years later by Grigori Perelman!

(who refused a Fields medal in 2006 and 1 Mio. \$ for solving one of the seven Millenium Problems)

$\boxed{\text{Homotopy } n\text{-spheres are homeomorphic to spheres for all } n.}$

$n \leq 2$: Classification

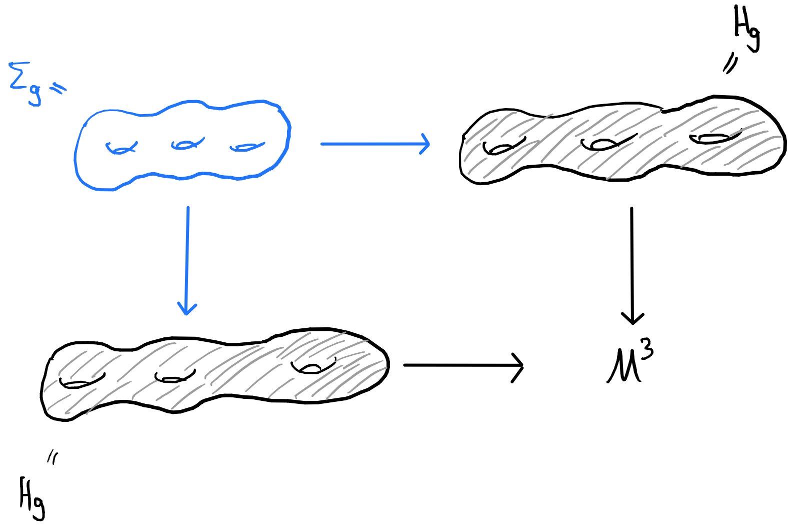
$n \geq 5$: Stephen Smale 1961 (Fields medal in 1966 for his proof of h-cobordism thm.)

$n = 4$: Michael Freedman 1981 (Fields medal in 1982)

Heegaard splittings of 3-manifolds

Any compact closed 3-mfld. can be cut into two simple pieces

3-dim. handlebodies $\cong \sqcup^g S^1 \times D^2 =: H_g$



$$M(\varphi) := H_g \cup_{\Sigma_g \xrightarrow[\varphi]{\cong} \Sigma_g} H_g$$

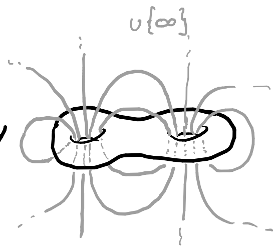
The "complexity" is hidden in this glueing map

Result of the glueing only depends on the isotopy class of the homeomorphism φ: → "symmetry"

⇝ Mapping class group!

Ex: ·) $S^3 \cong D^3 \cup_{S^2} D^3$

$\cong S^1 \times D^2 \cup_{\text{Longitude} \Leftrightarrow \text{meridian}} S^1 \times D^2$

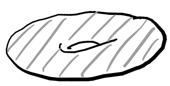
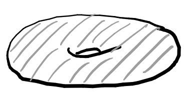
$\cong$  $\cup$ "outside" handlebody 

$\cong$ ... ($\exists$ Heegaard splitting of S^3 of every genus)

Waldhausen's theorem on uniqueness of Heegaard splittings of S^3:

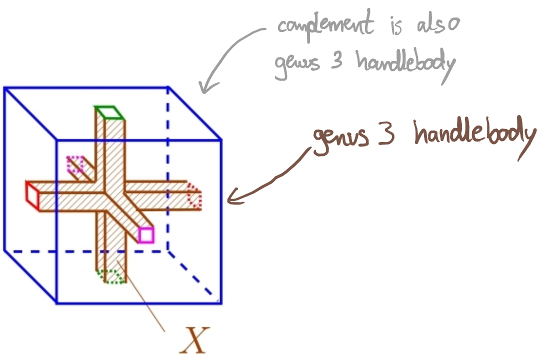
Every splitting of S^3 is obtained by stabilizing the genus 0 splitting.

·) Lens spaces

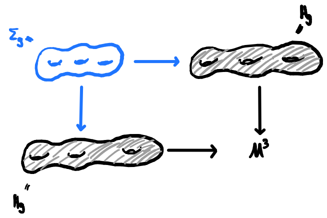
 $\cup$ self-diffeomorphism of the boundary torus 

e.g. any $A \in SL_2(\mathbb{Z})$ induces $\mathbb{R}^2/\mathbb{Z}^2 \xrightarrow{A\cdot} \mathbb{R}^2/\mathbb{Z}^2$

·) 3-Torus $S^1 \times S^1 \times S^1$



Aside: Algebraic formulations of the Poincaré conjecture



Seifert-van Kampen
$\rightsquigarrow$

$$\begin{tikzcd}
\langle a_1, a_2, a_3, a_4, a_5, a_6 \mid [a_1,a_2]\cdot[a_3,a_4]\cdot[a_5,a_6] \rangle \arrow[r] \arrow[d] & \langle x_1, x_2, x_3 \rangle \arrow[d] \\
\langle y_1, y_2, y_3 \rangle \arrow[r, "\text{pushout}"] & \pi_1(M)
\end{tikzcd}$$

$$\begin{tikzcd}
\text{surface group} \arrow[r] \arrow[d] & \text{free group} \arrow[d] \\
\text{free group} \arrow[r] & \text{3-mfld. group}
\end{tikzcd}$$

[Stallings: How not to prove the Poincaré Conjecture (1965)] suggested to study the "splitting homomorphisms"

$$\pi_g \longrightarrow Fr_g \times Fr_g$$

The following two statements are equivalent: surface group free group

(1) *Every 3-dimensional homotopy sphere is diffeomorphic to* S^3.
(2) *For any* $g \in \mathbb{N}$ *any epimorphism* $\alpha\colon \pi_g \to F_g \times F_g$ *factors through an* essential *monomorphism* $\beta\colon \pi_g \to A * B$ *from* π_g *to the free product of two groups* A *and* B.

[Stallings, Jaco]

also see Stefan Friedl's Algebraic Topology Notes

essential $\hat{=}$ not conjugate into one of the factors

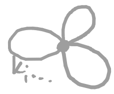
We say a homomorphism $\beta\colon G \to A * B$ *from a group* G *to the free product of two groups* A *and* B *is* essential, *if there is no* $h \in A * B$ *such that* $h\beta(G)h^{-1}$ *is contained in* A *or* B.

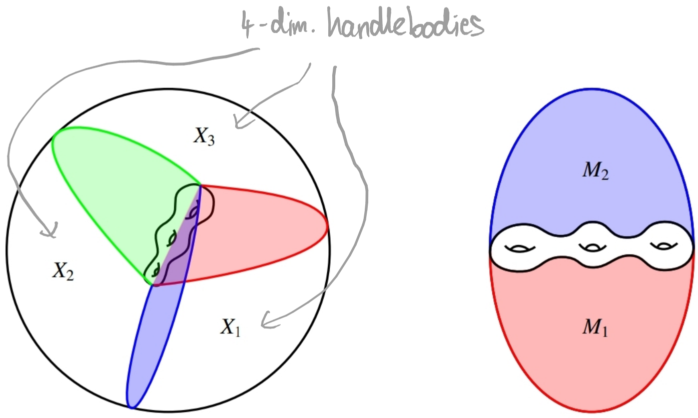
Perelman proved (1), and this is the only known proof of the (completely algebraic) statement (2).

Recently: [Gay, Kirby] Trisections of smooth 4-mflds.

They show that any smooth 4-manifold can be cut into **three** simple pieces $\natural^{k_i} S^1 \times D^3$ = 4-dimensional handlebody

= closed tubular neighborhood

of  $\subset \mathbb{R}^4$



(a) Trisection (b) Heegaard Splitting

Figure 1: Schematics of trisections and Heegaard splittings

[Abrams, Gay, Kirby] looked at the decomposition of $\pi_1(M)$

$$\begin{tikzcd}
& \pi_1(\natural^g S^1\times D^2) \arrow[rr] \arrow[dd] & & \pi_1(\natural^k S^1\times D^3) \arrow[dd] \\
\pi_1(\Sigma) \arrow[ur] \arrow[rr] \arrow[dd] & & \pi_1(\natural^g S^1\times D^2) \arrow[ur] \arrow[dd] & \\
& \pi_1(\natural^k S^1\times D^3) \arrow[rr] & & \pi_1(M) \\
\pi_1(\natural^g S^1\times D^2) \arrow[ur] \arrow[rr] & & \pi_1(\natural^k S^1\times D^3) \arrow[ur] &
\end{tikzcd}$$

Since any finitely presented group appears as π_1(compact, smooth 4-mfld.), from the existence of trisections we know that any f.p. group has a group trisection!

Group trisection of G:

$$\begin{tikzcd}
& \text{free group} \arrow[rr] \arrow[dd] & & \text{free group} \arrow[dd] \\
\text{surface group} \arrow[ur] \arrow[rr] \arrow[dd] & & \text{free group} \arrow[ur] \arrow[dd] & \\
& \text{free group} \arrow[rr] & & G \\
\text{free group} \arrow[ur] \arrow[rr] & & \text{free group} \arrow[ur] &
\end{tikzcd}$$

There is a stabilization move on trisections / group trisections ("uniqueness")

"Homotopy 4-spheres are diffeomorphic to S^4 ???"

Corollary 6 *The smooth 4–dimensional Poincaré conjecture is equivalent to the following statement: "Every $(3k, k)$–trisection of the trivial group is stably equivalent to the trivial trisection of the trivial group."*

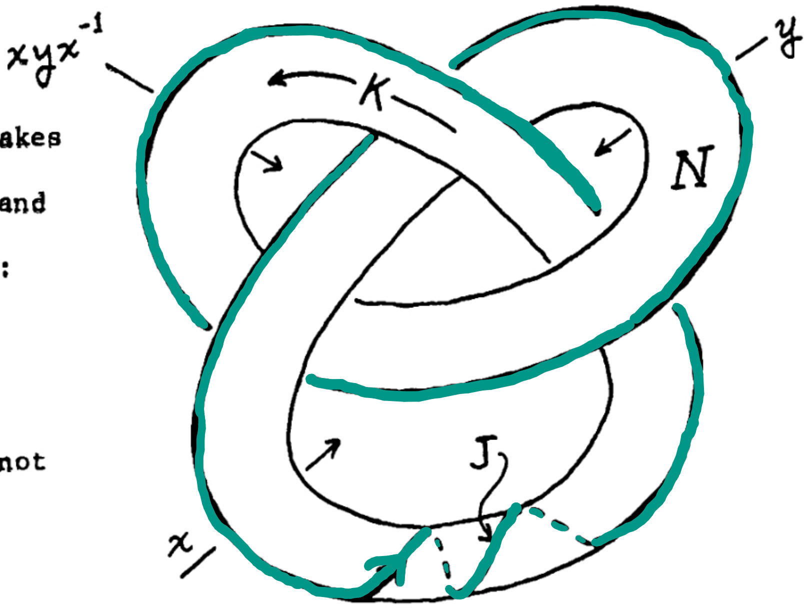
Dehn surgery

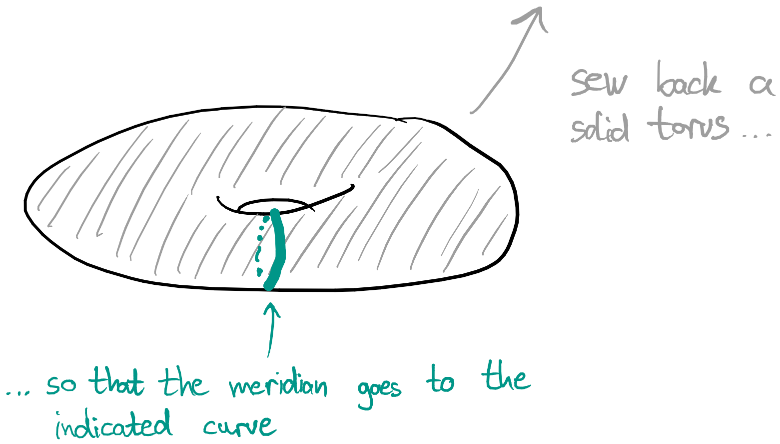
EXAMPLE : Dehn's construction of a homology sphere. Let N be a tubular neighbourhood of a right-handed trefoil K and let J be the curve on ∂N pictured here.

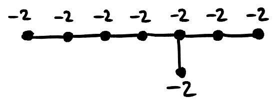
Now consider a homeomorphism $h: \partial(S^1 \times D^2) \to \partial N$ which takes a meridian $* \times S^1$ onto J and form the identification space:

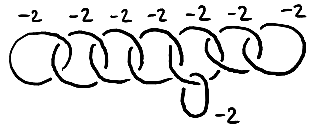
$$Q^3 = (S^3 - \mathring{N}) \bigcup_h (S^1 \times D^2)$$

sewing a solid torus to the knot exterior via h .









Freedman's fake 4-balls (*)

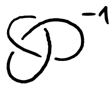
Glue to the boundary of the plumbing to obtain the E_8-manifold (non-smoothable 4-mfld.)

($\rightsquigarrow$ Rohlin's thm., $\frac{11}{8}$-conjecture, other open questions ...)

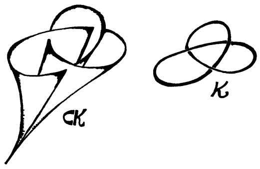
∂(Plumbing on the E_8-graph) $\longleftrightarrow$ surgery on the E_8-Link

Kirby calculus

(-1)-surgery on the left-handed trefoil



Link of the singularity of $f(x,y,z) = x^5 + y^3 + z^2$



An Immersion of CK in $\mathbb{R}^3$

$\boxed{\text{3-manifolds as boundaries of 4-manifolds}}$

$\Omega_3^{SO}(\{*\}) = 0$

Kirby's thm.

5-fold branched cover of the (2,3)-torus knot

or 2-fold branched cover of (3,5)-torus knot

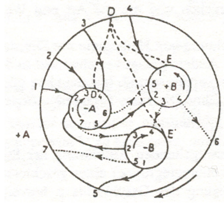
or 3-fold branched cover of (2,5)-torus knot

$\boxed{\text{Algebraic curves}}$

Poincaré homology sphere

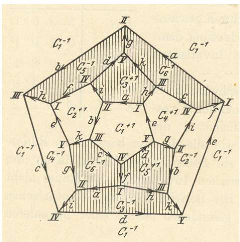
Heegaard-splitting

(Poincaré's original construction in 1904)



Seifert-fibered space

$\boxed{\text{Quotient space}}$



identify opposite faces of a dodecahedron via a $\frac{2\pi}{10} = 36°$ right-handed turn

dodecahedron as a fundamental domain

quotient of S^3 by the binary dodecahedral group: $S^3/_{Ico^*}$

opposite faces of a dodecahedron are misaligned by $\frac{1}{10}$ of a turn

$\rightsquigarrow$ gluing with $\frac{3}{10}$ rotation gives (hyperbolic) Seifert-Weber space

$\rightsquigarrow$ —"— $\frac{5}{10}$ rotation gives $\mathbb{RP}^3$

S^3 can be cut into 120 equal-sized dodecahedra, the Poincaré sphere has spherical (positively curved) geometry

120 elements

$$\begin{array}{ccccc} Ico^* & < & SU(2) & \approx & S^3 \\ \Big\downarrow{\scriptstyle 2:1} & & \Big\downarrow{\scriptstyle 2:1} & & \Big\downarrow{\scriptstyle 2:1} \\ Ico & < & SO(3) & \approx & \mathbb{RP}^3 \end{array}$$

60 elements

Cobordism groups of (rational) homology spheres:

$$\Theta^{sm.}_{\mathbb{Q}} := \frac{\{\text{diffeom. classes of rational homology 3-spheres}\}}{\text{smooth rational homology cobordism}}$$

$\Downarrow$

$$\Theta^{top.}_{\mathbb{Q}} := \frac{\{\text{homeom. classes of rational homology 3-spheres}\}}{\text{rational homology cobordism}} \twoheadrightarrow \mathbb{Z}/_2{}^{\infty} \oplus \mathbb{Z}/_4{}^{\infty} \oplus \mathbb{Z}/_8{}^{\infty}$$

$$\Theta^{sm.}_{\mathbb{Z}} := \frac{\{\text{diffeom. classes of integral homology 3-spheres}\}}{\text{smooth integral homology cobordism}} \geq \mathbb{Z}^{\infty}$$

$\Downarrow$

$$\Theta^{top.}_{\mathbb{Z}} = 0$$

[Freedman (*): Every integral homology 3-sphere bounds a 1-connected integral homology 4-ball]

this is the only of the four groups which has been fully calculated

The homology-sphere-cake: