2019-10-31, Geometric Group Theory Study Group

Talk 2:

$\boxed{\text{Dehn functions}}$

Benjamin Matthias Ruppik, MPIM Bonn

Plan: Perimeter vs. enclosed area

- Using relations to measure area
- Word problems
- Isoperimetry
- Dehn function Landscape

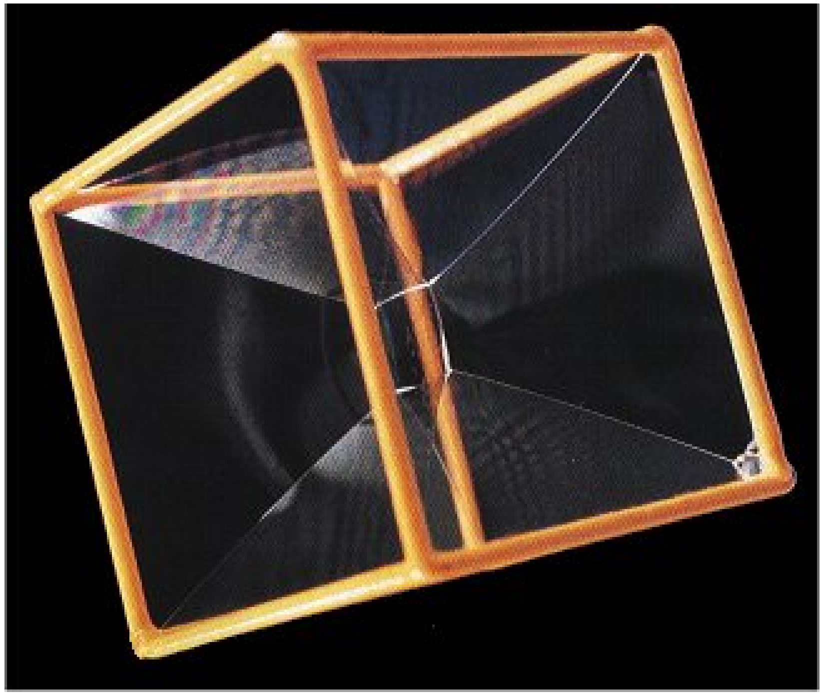
References:

- [Clay, Margalit: Office Hours with a Geometric Group Theorist]
  Office Hour 8 written by Timothy Riley
- [Clara Löh: Geometric Group Theory]

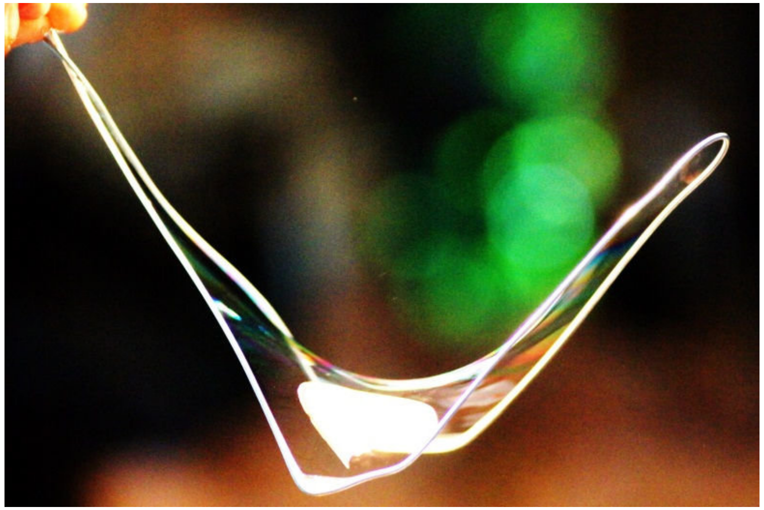
Soap films spanning wire frames / loops

Physical manifestation of isoperimetry:

A wire loop lifted out of a soap solution is spanned by an area-minimizing soap film.



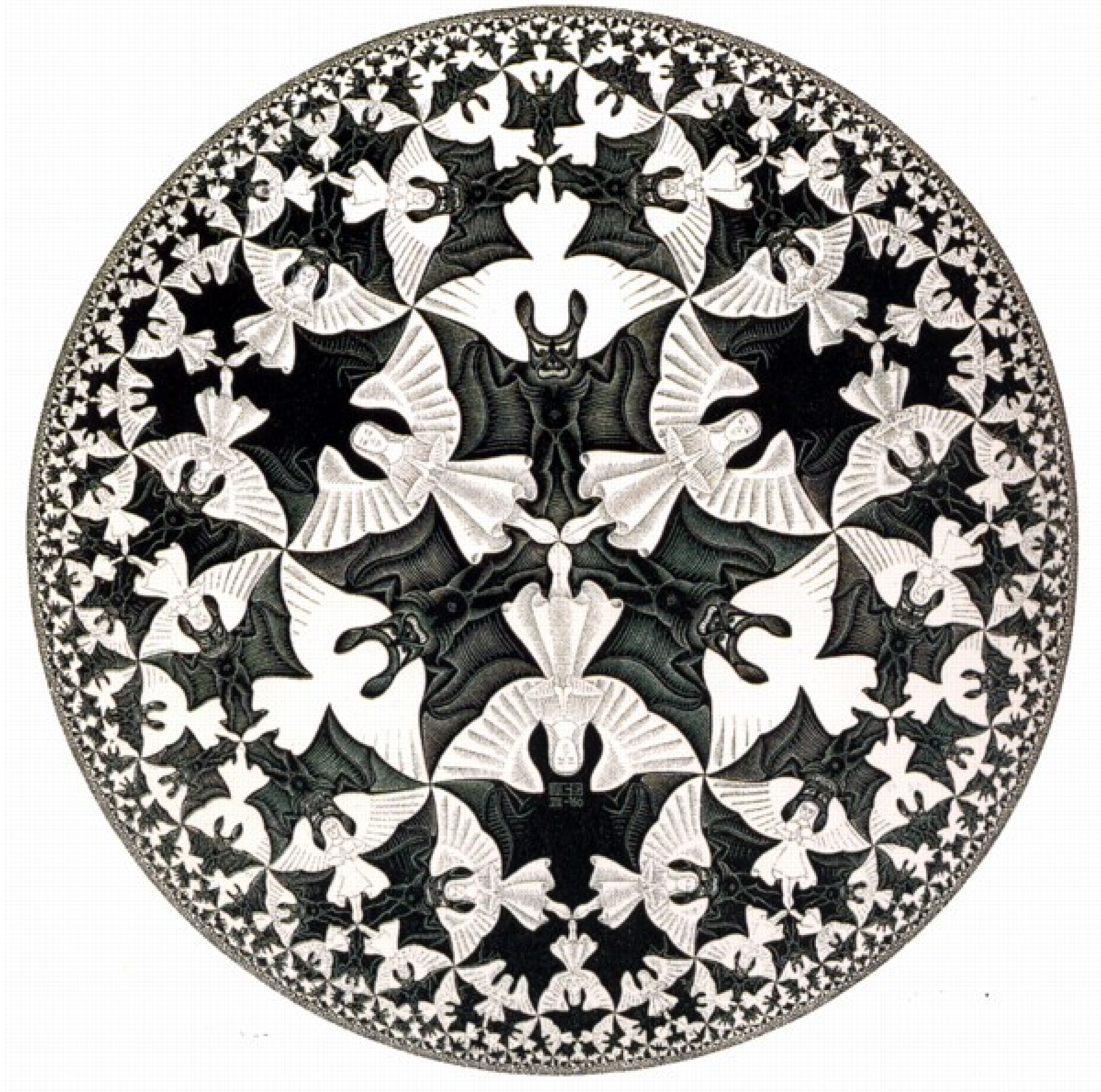
https://www.fam-bundgaard.dk/SOMA/NEWS/N020107.HTM



https://www.researchgate.net/figure/Minimal-surface-formed-by-a-soap-film-inside-an-irregular-tridimensional-wire-frame_fig1_315449646

Euclidean geometry:

A perimeter of length n encloses $\approx n^2$ area



[Escher, Circle Limit IV (Heaven and Hell), 1960]

Hyperbolic geometry: A perimeter of length n encloses $\approx n$ area

Relations have a 2-dimensional flair $\rightsquigarrow$ Area of a relation

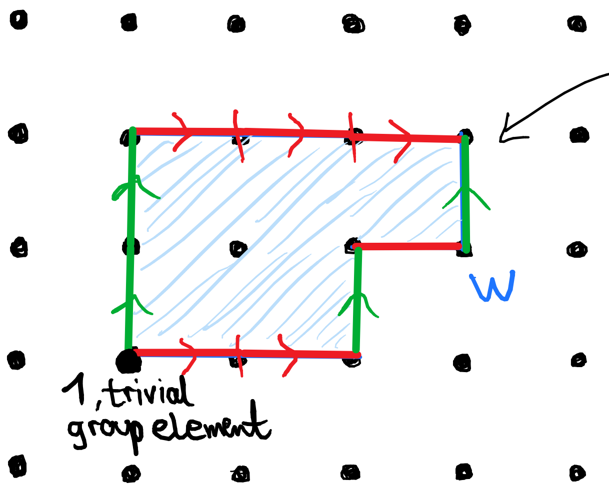
Def.: ·) $\langle S \mid R\rangle$ finite presentation of G

·) w reduced word in the generators that represents the trivial element in G

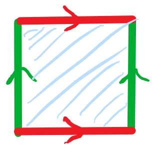
$$\rightsquigarrow \quad \mathrm{Area}_{\langle S\mid R\rangle}(w) := \min\left\{ n\in\mathbb{N} \;\middle|\; w = \begin{array}{l}\text{product of } n \text{ conjugates}\\ \text{of relators}\end{array} \right\}$$

$$= \min\left\{ n\in\mathbb{N} \;\middle|\; \begin{array}{l}\exists a_1,\dots,a_n \in Fr(S),\ \exists r_1,\dots,r_n\in R\cup R^{-1}\\ w = a_1 r_1 a_1^{-1}\cdot\ldots\cdot a_n r_n a_n^{-1} \text{ in } Fr(S)\end{array} \right\}$$

$\langle a, b \mid aba^{-1}b^{-1}\rangle$



Solve the puzzle with the minimal number of pieces



$\longleftrightarrow aba^{-1}b^{-1}$

A completed puzzle is called a van Kampen diagram for w.

Dehn function encodes the maximal area that can be enclosed by a given Length

Def.: **Dehn function** of a finite presentation $\langle S \mid R \rangle$

$= \langle s_1, \dots, s_k \mid r_1, \dots, r_\ell \rangle$

$$\mathrm{Dehn}_{\langle S \mid R \rangle} : \mathbb{N} \longrightarrow \mathbb{N}$$

$$n \longmapsto \max_{\substack{\text{words } w \text{ of} \\ \text{length} \leq n \\ \text{representing} \\ \text{the trivial group element}}} \mathrm{Area}(w)$$

Ex.

- $\langle a \mid a^m \rangle \rightsquigarrow \mathrm{Dehn}(n) = \lfloor \frac{n}{m} \rfloor$
- $\langle a, b \mid aba^{-1}b^{-1} \rangle \rightsquigarrow (n-3)^2 \leq 16 \cdot \mathrm{Dehn}(n) \leq n^2$

$\underbrace{\text{Solving the word problem}}$ for a finite presentation $\langle S \mid R \rangle$

INPUT: A word in the generators $S \cup S^{-1}$

TASK: Decide whether that word represents the identity

Thm.: For a finite presentation $\langle S \mid R \rangle$ of a group with Dehn function $f: \mathbb{N} \to \mathbb{N}$, TFAE:

1) There is an algorithm which solves the word problem

2) There is a $\underbrace{\text{recursive fct.}}$ $g: \mathbb{N} \to \mathbb{N}$ s.th. $\forall n \in \mathbb{N}: f(n) \leq g(n)$

There is a Turing machine which on input n computes $g(n)$

3) The Dehn function f itself is recursive.

But: Dehn function is not a good measure of the difficulty of the word problem

worst-case measure of how long a brute force attack on the word problem (by blindly applying defining relators, free reductions and expansions) will take

Often you can find shortcuts to solve the word problem:

- $\langle a,b \mid ab=ba\rangle \cong \mathbb{Z}\times\mathbb{Z}$, Dehn function grows like $n \mapsto n^2$

  To solve the word problem, just look at the exponent sum of $a^{\pm 1}, b^{\pm 1}$

  $\underbrace{a^3 b^{-2} a b a^{-4} b}_{\Rightarrow = e}$ $\qquad 3+1-4=0$, $\; -2+1+1=0$

- Heisenberg group $H_3 \cong \langle a,b,c \mid [a,c],[c,b],[a,b]=c\rangle$, Dehn fct. grows like $n \mapsto n^3$

  Can view this as a matrix group $\quad a \leftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $b \leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $c \leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

There are examples where the gap between the Dehn fct. & running time of an efficient solution of the word problem can be very Large, see

The mathematician who had little wisdom: a story and some mathematics

DANIEL E. COHEN

1. The Story

If any of you who read this volume do not like stories, then I am sorry for you. Stories are the thread from which the fabric of the world is woven, and to dislike stories is to dislike life. But, to any such people, I would also say that if you read this story you will also learn some mathematics.

Once there was a mathematician who had little wisdom. One spring he attended a group theory conference in Scotland, which may or may not have been wise of him. Since the conference was long, he decided to take a couple of days off, which was certainly wise. He had heard much about the beauty of Scotland's rivers, and the fine salmon that swam in them, so he decided to go salmon-fishing. He did not think of the need for a licence, nor that a large charge is made for the right to fish for salmon in most places; indeed, he had not even checked whether there were salmon in the rivers at that time of year. This may seem foolish of him, but turned out not to be so.

Isoperimetric Problems:

Determine the maximal area/volume/... that a shape can have when its boundary is constrained in some way.

Dehn fct. of a finitely presented group $\longleftrightarrow$ isoperimetric problem where we span loops with disks in a suitable space

G with the word metric is "too discrete", instead look at

① combinatorial version: Cayley 2-complex

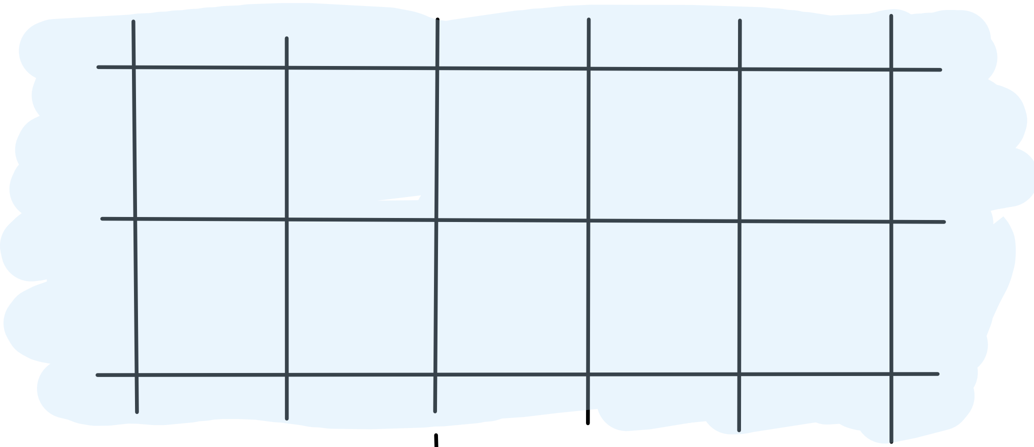
② continuous version: group actions on mflds.

Cayley 2-complex $\tilde{K}$: Universal cover of

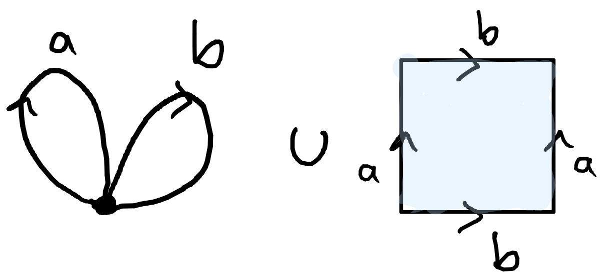
or rather a length(r)-sided polygon

$$K = \left(\text{one } 0\text{-cell}\right) \cup \left(\begin{array}{c}\text{directed 1-cell} \\ \text{for each gen.} \\ s \in S\end{array}\right) \cup \left(\begin{array}{c}\text{2-cell} \\ \text{for each rel} \\ r \in R\end{array}\right)$$

$$\tilde{K}_{\langle a,b \mid [a,b] \rangle} =$$



$$K_{\langle a,b \mid [a,b] \rangle} =$$



Metric: Edges have length 1, faces are regular Euclidean polygons whose sides all have length 1

(For one- and two-sides polygons take appropriate disks)

Finite presentation of a group G $\rightsquigarrow$ <u>Riemannian mfld. $\widetilde{M}$</u>

(should have $\pi_1 = \{1\}$ and coarsely resemble G)

<u>Example constr.:</u>

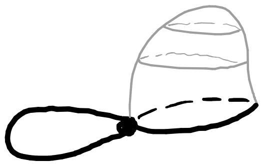
isometric action by deck-transformations

(even geometric)

$$G \circlearrowright \widetilde{M}_G$$

$$\Big\downarrow \text{universal cover}$$

4-mfld.

K_G =  $\hookrightarrow \mathbb{R}^5 \rightsquigarrow \partial(\mathrm{nbhd}_{\mathbb{R}^5}(K_G)) =: M_G$

$\widetilde{M}_G$ Riemannian means there is a notion of length and area of disks $\mathbb{D}^2 \to \widetilde{M}_G$

<u>Def.:</u> <u>Minimal isoperimetric function</u> $\mathrm{Area}_{\widetilde{M}} : [0,\infty) \longrightarrow [0,\infty)$

in fact minimum

$\mathrm{Area}_{\widetilde{M}}(l)$ = infimum s.th. any loop of length $\leq l$ can be spanned by a disk of at most this area

Functions growing at the same rate:

$f, g : [0, \infty) \longrightarrow [0, \infty)$

"f is Dehn dominated by g"

Write $f \preceq g$ when there is a constant $C > 0$ such that

$$\forall L \geq 0: \quad f(L) \leq C \cdot g(C \cdot L + C) + C \cdot L + C$$

important for later

Write $f \simeq g$ when $f \preceq g$ and $g \preceq f$

"Dehn equivalent"

<u>Examples:</u>

- All functions which grow at most linearly fast are equivalent
- $(n \mapsto n^\alpha) \simeq (n \mapsto n^\beta)$ where $\alpha, \beta \geq 1$ <u>iff.</u> $\alpha = \beta$
- (polynomially growing fcts.) $\not\simeq$ $\underbrace{(n \mapsto c^n)}_{\text{exponential}}$ where $c > 1$
- $(n \mapsto c^n) \simeq (n \mapsto d^n)$ for all $c, d > 1$

Filling theorem [Gromov]:

M compact Riemannian mfld., $\partial M = \emptyset$
with $P = \langle S \mid R \rangle$ a finite presentation of $\pi_1(M)$.

universal cover of M

Then: The minimal isoperimetric function $\mathrm{Area}_{\tilde{M}}(L)$ for $\tilde{M}$

is Dehn-equivalent to

the Dehn function $\mathrm{Dehn}_P(\pi_1(M))$

Pf. idea: ·) Map $\tilde{K}$ to $\tilde{M}$ beginning with the vertices (use the orbit map $G \longrightarrow \tilde{M}$)
extend to edges via minimal length paths, then to faces

·) This does not distort distances and areas by too much, van Kampen diagrams $\updownarrow$ filling disks

·) A loop in $\tilde{M}$ can be pushed into the 1-skeleton
while keeping control of distances and areas

$\square$

Large-scale geometric invariants:

Thm.: The equivalence class $\simeq$ of the Dehn function does not depend on the presentation.

Here we need the "$+Cn$" in the definition of $\simeq$:

$f \preceq g \Leftrightarrow \exists$ constant $C > 0$ s.th. $\forall L \geq 0$:
$f(L) \leq C \cdot g(C \cdot L + C) + C \cdot L + C$

Look at

$\langle a \mid \emptyset \rangle \cong \mathbb{Z} \cong \langle a, b \mid b \rangle$

Even better, the Dehn function is a quasi-isometry invariant!

Recall: Quasi-isometries are maps between metric spaces
which are coarsely surjective
and where we allow a bounded amount of stretching and tearing

Thm.: G, H finitely generated and quasi-isometric,
(w.r.t. some word metrics)
assume G finitely presented

$\Rightarrow$ H also finitely presentable, and their Dehn functions
(w.r.t. any finite presentation)
are equivalent in the sense of $\simeq$

Finite presentability is a geometric property of groups!

The Dehn function landscape

$$\boxed{\{\text{Hyperbolic groups}\} = \left\{\begin{array}{l}\text{groups whose Dehn function} \\ \text{is Dehn equivalent to } n\end{array}\right\}}$$

"Gromov's gap" – hyperbolic groups are qualitatively different in the universe of finitely presented groups

"Easy" direction: Hyperbolic groups have linear Dehn functions:

They have Dehn presentations which allow a very efficient solution of the word problem

(every application of a relation makes the word shorter – and for words representing the identity we can alway find a relation to apply)

$\rightsquigarrow$ algorithm for linear filling of loops in hyperbolic spaces

Two "hard" pieces:

- Any group with Linear Dehn function is hyperbolic
- Anything that is not hyperbolic has at least quadratic Dehn function

$\rightsquigarrow$ This is the gap! There is nothing between Linear and quadratic rates of filling, you can't find a group with $n \cdot \log n$ or $n^{\frac{3}{2}}$ Dehn fct.

For which $\alpha \in [1, \infty)$ are there finitely pres. groups with Dehn fct. equivalent to $n \mapsto n^\alpha$?

The set of such α is countable, as there are only countably many f.p. groups

Gap!

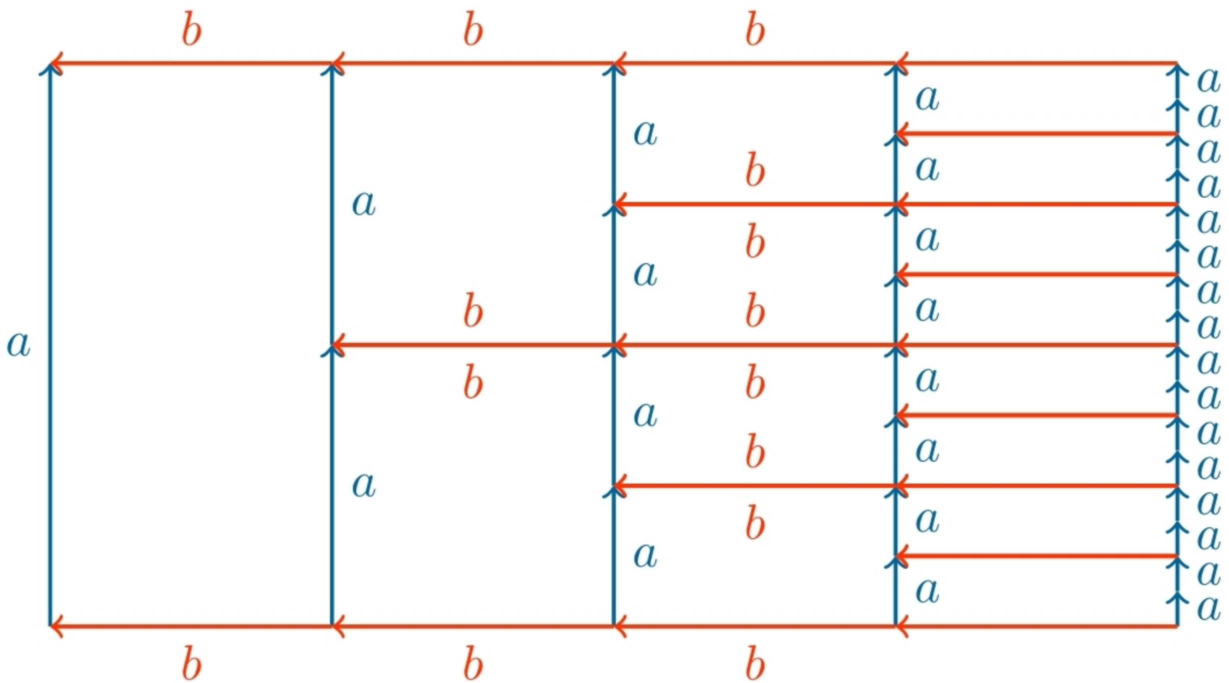
Groups with Dehn fct. $\lesssim n^2$ contain

CAT(0); finitely gen. abelian with subgroup $\mathbb{Z} \times \mathbb{Z}$;

$SL_n(\mathbb{Z})$ for $n \geq 5$; Thompson's group F;

examples of nilpotent groups of all nilpotency classes

Example: A group with exponential Dehn function



[Löh: Geometric Group Theory]

$$b^n a b^{-n} = a^{2^n}$$

Theorem: The Dehn function of the presentation $\langle a, b \mid b a b^{-1} = a^2 \rangle$ is Dehn-equivalent to 2^n.

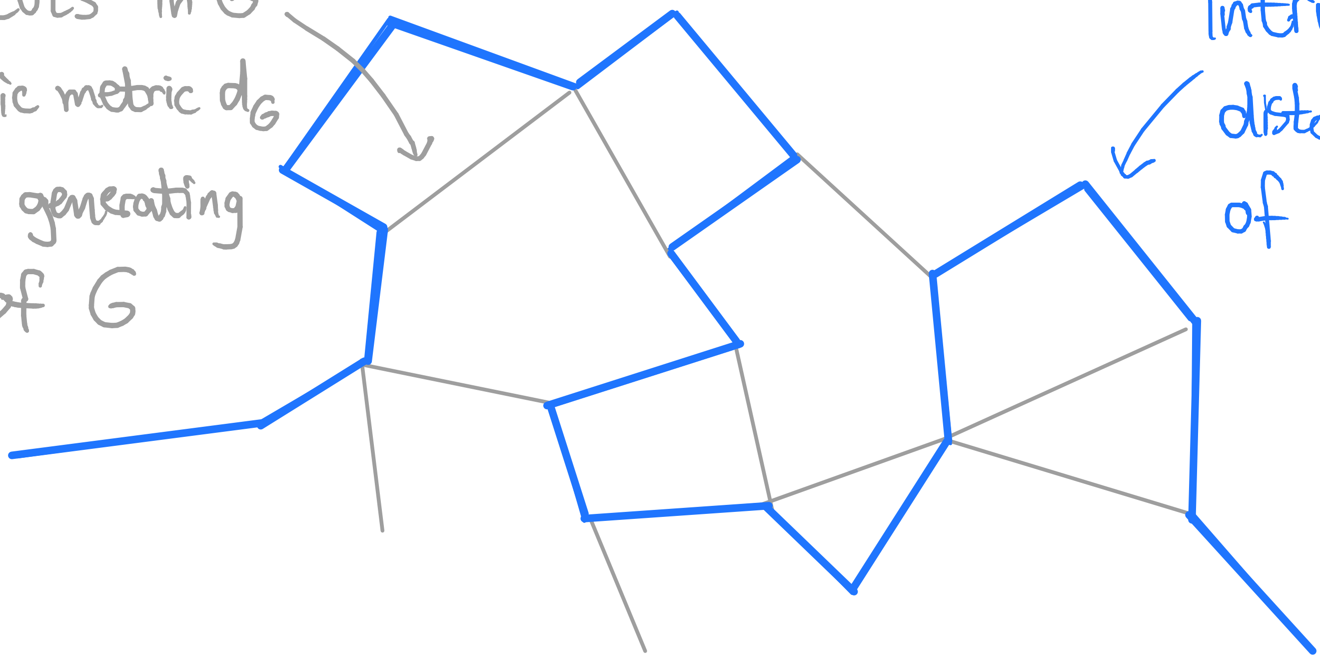
Baumslag – Solitar group $B(1,2)$

Subgroup distortion:

Measures how a finitely generated subgroup H sits inside an ambient finitely generated group G

"Shortcuts" in G

Extrinsic metric d_G from a generating set of G



Intrinsic metric d_H: Measures distances w.r.t. a generating set of H

Distortion function:

$$n \longmapsto \max_{\substack{h \in H \text{ with} \\ \mathrm{dist}_G(1,h) \leq n}} \mathrm{dist}_H(1,h)$$

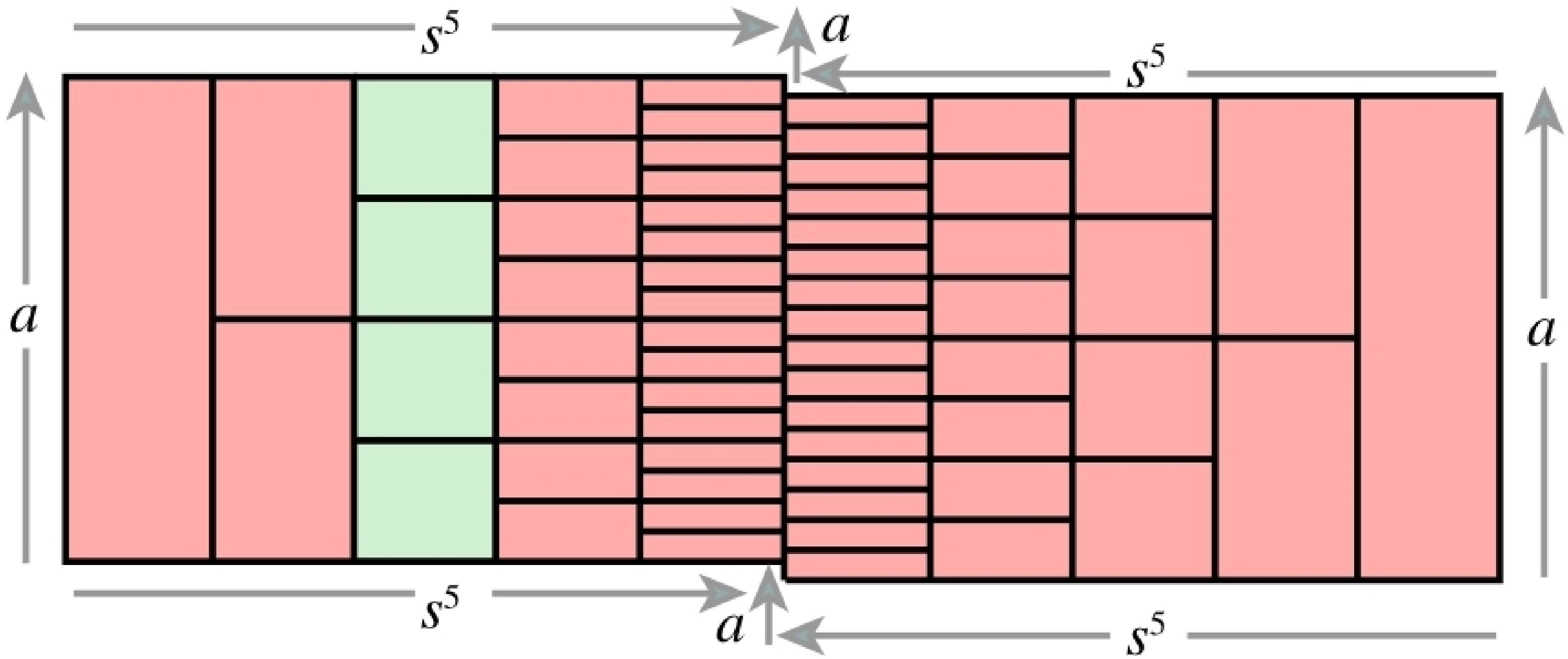


Figure 8.7 A van Kampen diagram for $as^{-5}as^5a^{-1}s^{-5}a^{-1}s^5$ with respect to $\langle a, s \mid s^{-1}as = a^2\rangle$. All vertical edges are labeled a and directed upwards. All horizontal edges are labeled s; those on the left half of the diagram are oriented to the right and those on the right half are oriented to the left. An example of a *corridor* is shown in green.

Exercise 13. Show that the Cayley 2–complex of $\langle a, s \mid s^{-1}as = a^2 \rangle$ is homeomorphic to the direct product of an infinite 3-valent (that is, three edges meet at each vertex) tree with a line, and so is contractible. For a picture of the Cayley 2–complex for $BS(1, 3)$ see Figure 12.1, which is similar except that the tree there is 4-valent instead of 3-valent. Show that the maps from the diagrams of Figure 8.7 to the Cayley 2–complex are embeddings.

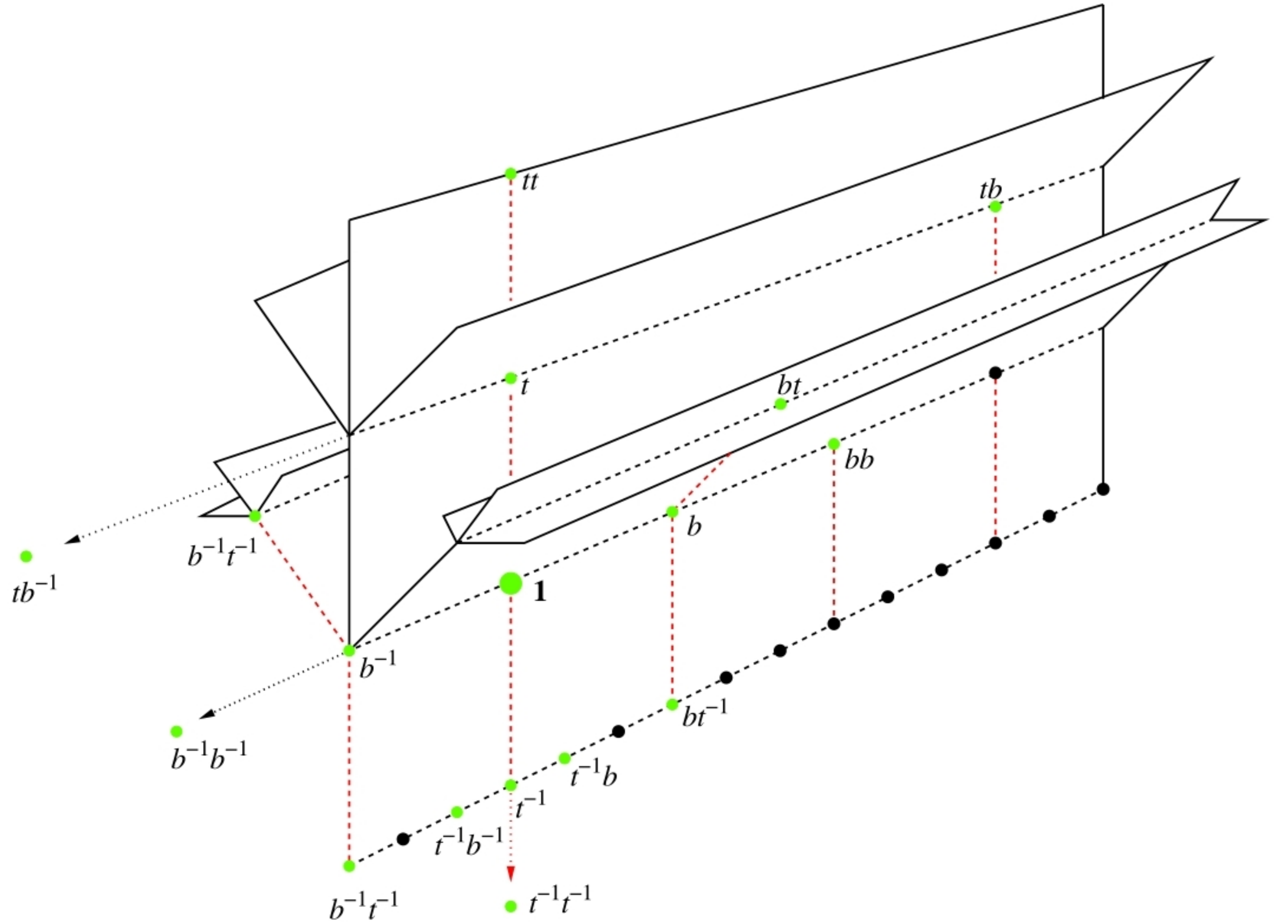


Figure 12.1 An interesting partial Cayley graph constructed from two generators. Dashed lines are graph edges, solid lines are cutaways. The green vertices comprise the ball of radius 2. Several of these vertices don't fit on the cutaway diagram and are indicated by arrows.

For example $\pi_1($  $) \cong \langle a_1, a_2, b_1, b_2, c_1, c_2 \mid [a_1,a_2][b_1,b_2][c_1,c_2] \rangle$

Project 7. Explore the Dehn functions of one-relator groups (groups that can be presented with a single defining relation). Magnus [196] solved the word problem for any one-relator group. So their Dehn functions are recursive. But how fast can they grow? A longstanding problem of Gersten [140, 142] is whether the example of Theorem 8.10 is the fastest possible.

THEOREM 8.10. *The Dehn function of*

$$\langle a, t \mid (t^{-1}at)^{-1}a(t^{-1}at) = a^2 \rangle$$

is equivalent to the function $n \mapsto \exp^{(\lfloor \log_2 n \rfloor)}(1)$.

The same strategy works for $\langle a, s \mid s^{-1}as = a^2\rangle$, whose Dehn function grows exponentially fast as we will see in Section 8.5. It can be represented by matrices via $a \mapsto \left(\begin{smallmatrix}1 & 0\\ 1 & 1\end{smallmatrix}\right)$ and $s \mapsto \left(\begin{smallmatrix}1/2 & 0\\ 0 & 1\end{smallmatrix}\right)$.

subgroups of $GL_n(\mathbb{C})$

Project 8. Another challenging open problem due to Gersten is to determine how fast Dehn functions of finitely presentable linear groups can grow. The group of Theorem 8.8 is linear and has an exponential Dehn function. Is there a faster growing example?

Theorem 8.8. *The Dehn function $f(n)$ of the presentation $\langle a, s \mid s^{-1}as = a^2\rangle$ satisfies $f(n) \simeq 2^n$.*

$= BS(1,2)$