Ribbon concordances and doubly slice knots

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Ribbon concordances

Definition 2 (Ribbon concordance). A 2-sphere smoothly ribbon concordant to a knot, written \(J \sim_{ribbon} K\), is a smoothly embedded \(2\)-sphere \(S^2 \times \{0\} \subset \mathbb{R}^4\) such that \(J \times \{0\}\) is isotopic to \(K \times \{0\}\) in \(\mathbb{R}^4\).

Proposition. A handle decomposition of the concordance exterior into a handle decomposition of the concordance exterior \(X_C = (S^2 \times [0, 1]) \setminus \nu(K)\) relative to \(X_C(0) = S^2 \times \{0\}\) is complete whenever we pass through a level \(S^2 \times \{t\}\) corresponding to a critical point of index \(k\).

Homotopy ribbon

From the handle decomposition of the concordance exterior, we immediately see that \([\nu(S^2 \times \{0\})], \nu(S^2 \times \{1\})\) is injective. (2) is injective. This motivates the following homotopy analogue of the notion of a smooth ribbon concordance:

Definition 4. An oriented knot \(J \sim_{ribbon} K\) if \(J\) is a locally flat embedded concordance \(C \times [0, 1] \subset \mathbb{R}^4\), with \(C \times \{0\} \subset \nu(K)\) and \(C \times \{1\} \subset \nu(K)\) such that the induced maps on fundamental groups of exteriors satisfy \(\pi_1(C) \rightarrow \pi_1(K)\). We write \(J \sim_{ribbon} K\).

Doubly slice knots

Question (Fintushel, 1969). Which slice knots are cross sections of smooth \(S^2\)?

What is the most famous open problem in knot theory in its simplest form?

Relevant recently:

There has been an influx of papers dealing with the maps induced by ribbon concordances on various knot homology theories. For example, ribbon concordances yield injective maps on knot Floer homology \(\alpha_{ribbon}\) and on Khovanov homology \(\beta_{ribbon}\).

Proposition. A handle decomposition of the concordance annulus \(C\) translates into a recipe for a handle decomposition of the concordance exterior \(X_C = (S^2 \times [0, 1]) \setminus \nu(K)\) relative to \(X_C(0) = S^2 \times \{0\}\). Whenever we pass through a level \(S^2 \times \{t\}\) corresponding to a critical point of index \(k\), a handle is added to the concordance cylinder \(C\) and \((k+1)\)-handle is added to the exterior \(X_C\).

Doubly slice knots

We have to be careful when we want to define a doubly concordant group \(\mathbb{Z}\) whose natural choice would be to declare that \(K\) and \(J\) are doubly concordant iff \(K\sim_{ribbon} J\) is doubly slice, but it is not known whether this gives a transitive relation!

Open question (Stably doubly slice = doubly slice). Suppose \(K\) and \(J\) are doubly slice. Then, must \(J\) be doubly slice?

References

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* A concordance is a smoothly embedded annulus \(S^1 \times [0, 1] \subset \mathbb{R}^3\) with boundary \(\partial(S^1 \times [0, 1]) = \{K \times 0, J \times 1\}\).

* A sphere \(S^2\) is smooth if it admits a smooth embedding of a disk \(D^2 \to S^2\).

* In the general situation of multiple-infection one should start with a properly embedded multi-diske, thicken this and replace with a suitable cable of the string link.