Ribbon concordances and doubly slice knots

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The knot concordance group

Definition 1. A knot $K \subset \mathbb{S}^3$ is called *(topologically/smoothly) slice* if it arises as the equatorial slice of a (locally flat/smooth) embedding of a 2-sphere in $\mathbb{S}^4$.
Proposition. Under the equivalence relation "$K \sim J$ iff $K\#(-J)$ is slice", the commutative monoid

$$\left(\mathcal{K} := \left\{\substack{\text{isotopy classes of}\\ \text{oriented knots } \mathbb{S}^1 \hookrightarrow \mathbb{S}^3}\right\}, \# := \substack{\text{connected sum}\\ \text{of knots}}\right)$$

becomes a group $\mathcal{C} := \mathcal{K}/\sim$, the *knot concordance group.*
The neutral element is given by the (equivalence class) of the unknot, and the inverse of $[K]$ is $[\overline{rK}]$, i.e. the mirror image of K with opposite orientation.
Remark. We should be careful and distinguish the smooth $\mathcal{C}^{\text{sm}}$ and topologically locally flat $\mathcal{C}^{\text{top}}$ version, because there is a huge difference between those: $\operatorname{Ker}(\mathcal{C}^{\text{sm}} \to \mathcal{C}^{\text{top}})$ is infinitely generated!

Some classical structure results:

- There are surjective homomorphisms $\mathcal{C}^{\text{sm}} \to \mathcal{C}^{\text{top}} \to \mathcal{AC}$, where $\mathcal{AC} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ is Levine's algebraic concordance group. **Open question:** Does this split? Nobody has observed 4-torsion in the concordance groups $\mathcal{C}^{\text{sm}}$ or $\mathcal{C}^{\text{top}}$; the currently best conjecture is that there is none!
- Every knot K with Alexander polynomial $\Delta_K(t) = 1$ is topologically slice by Freedman's work [Fre82]. But not all topologically slice knots are explained by this theorem, i.e. there are plenty of topologically slice knots not smoothly concordant to any Alexander polynomial $= 1$ knot [HLR12].

Still, many natural questions about the structure of $\mathcal{C}$ remain a mystery.
Application. We can use any topologically, but not smoothly slice knot to construct an exotic $\mathbb{R}^4$ (i.e. a 4-manifold which is homeomorphic, but not diffeomorphic to $\mathbb{R}^4_{\text{std}}$). In any other dimension $n \neq 4$, $\mathbb{R}^n$ has a unique smooth structure!

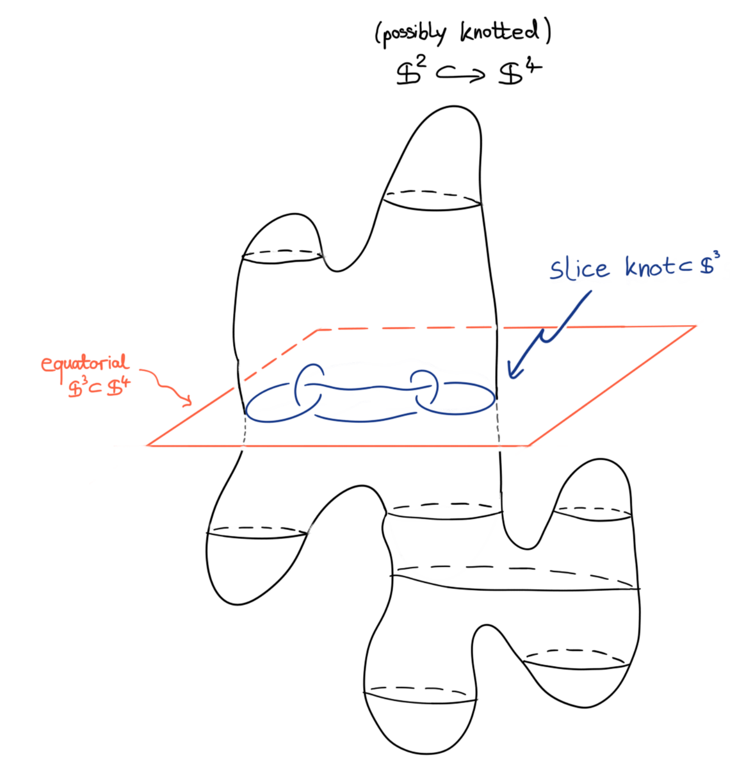

Figure 1: A *slice knot* appears when one uses a hyperplane to cut a 2-sphere in $\mathbb{S}^4$ in half. The lower (or upper) hemisphere of $\mathbb{S}^2$ forms a *slice disk* for the knot. In the picture we see a connected sum of a trefoil with its mirror image; there are also many slice prime knots.

Glossary

- **Abelian invariants:** Algebraic invariants extracted from the Alexander module $\mathcal{A}^{\mathbb{Z}}(K) = H_1(\mathbb{S}^3 \setminus K; \mathbb{Z}[t, t^{-1}])$ (i.e. from the homology of the universal abelian cover of the knot complement).
- **Blanchfield linking form:** Sesquilinear form on the Alexander module $\text{Bl}\colon \mathcal{A}^{\mathbb{Z}}(K) \times \mathcal{A}^{\mathbb{Z}}(K) \to \frac{\mathbb{Q}(\mathbb{Z})}{\mathbb{Z}[\mathbb{Z}]}$, the definition uses that $\mathcal{A}^{\mathbb{Z}}(K)$ is a $\mathbb{Z}[\mathbb{Z}]$-torsion module.
- **Levine-Tristram-signature:** For $\omega \in \mathbb{S}^1$, take the signature of the Hermitian matrix $(1-\omega)V + (1-\overline{\omega})V^T$, where V is a Seifert matrix for K. For any algebraically slice knot K and ω not a root of $\Delta_K(t)$, this is zero.
- **Derivative:** On a genus g Seifert surface a derivative is any nonseparating g-component link with linking and self-linking zero.
- **Metabelian invariants:** Their definition uses finer quotients of π_1 of the complement and signature defects of the associated covers, examples are the Casson-Gordon invariants.

Ribbon concordances

Definition 2 (Ribbon concordance)**.** J is smoothly *ribbon concordant* to K, written $J \geq_{\text{sm}} K$, if there is a smooth concordance[1] C from J to K such that the restriction of the projection map $\mathbb{S}^3 \times [0,1] \to [0,1]$ to C yields a Morse function without maxima. Note that this is not a symmetric relation!

	Index	slice disk	ribbon disk
Maximum	2	✓	✗
Saddle	1	✓	✓
Minimum	0	✓	✓

Table 1: Concordance vs. ribbon concordance

Conjecture (Gordon, [Gor81])**.** $\geq_{\text{sm}}$ is a partial ordering on the set of (isotopy classes of) knots in $\mathbb{S}^3$.
Remark. The difficult part is the anti-symmetry, i.e. showing that from the existence of two ribbon concordances $J \geq_{\text{sm}} K$ and $K \geq_{\text{sm}} J$ we get that the knots J and K are already isotopic.
Definition 3. $K \subset \mathbb{S}^3$ is called a *ribbon knot* if $K \geq_{\text{sm}}$ unknot.
Conjecture (Slice-ribbon, [Fox, 1960s])**.** Every smoothly slice knot is a ribbon knot.
Remark. This is probably the most famous open question in knot concordance. It is subtle, in part because there are examples of slice disks which are not even isotopic to ribbon disks (e.g. start with a nontrivial 2-knot and take a connected sum with the trivial disk on the unknot).

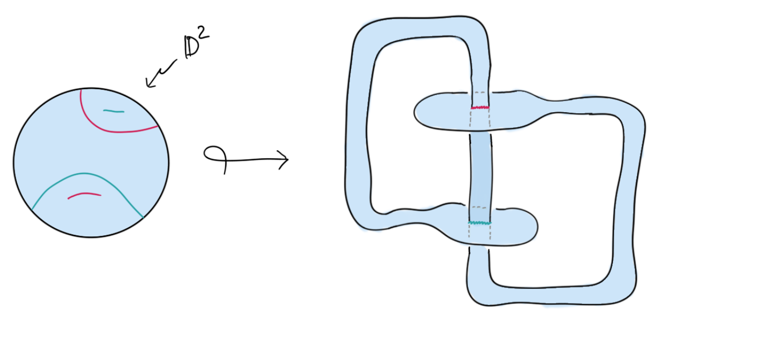

Figure 2: A ribbon disk for the stevedore knot 6_1, immersed into $\mathbb{S}^3$ with ribbon singularities.

Recently, there has been an influx of papers dealing with the maps induced by ribbon concordances on various knot homology theories: For example, ribbon concordances yield injective maps on knot Floer homology [Zem19] and on Khovanov homology [LZ19].
Proposition. A handle decomposition of the concordance annulus C translates into a recipe for a handle decomposition of the concordance exterior $X_C := (\mathbb{S}^3 \times [0,1]) \setminus \nu C$ relative to $X_J := \mathbb{S}^3 \setminus \nu J$: Whenever we pass through a level $\mathbb{S}^3 \times \{t\}$ corresponding to a critical point of index k, a k-handle is added to the concordance cylinder C and a $(k+1)$-handle is added to the exterior X_C.

$$X_C = X_J \times [0,1] \cup (\text{2-handles}) \cup (\text{3-handles})$$

and dually

$$X_C = X_K \times [0,1] \cup (\text{1-handles}) \cup (\text{2-handles})$$

Conjecture. Ribbon concordance exteriors are aspherical.
Remark. Since the exterior of a ribbon disk is homotopy equivalent to a 2-complex, the asphericity is equivalent to $\pi_2(X_C) = 0$. The Whitehead conjecture (**Conjecture:** Any subcomplex of an aspherical 2-complex is aspherical.) would imply this, because adding a meridian 2-disk of C to X_C produces a contractible space. For example, we know that the exterior is a $K(\pi,1)$ for ribbon disks with a single saddle, in this case the statement follows from the one-relator-theorem.

Homotopy ribbon

From the handle decomposition of the concordance exterior we immediately see that

$$\pi_1(\mathbb{S}^3 \setminus \nu J) \twoheadrightarrow \pi_1((\mathbb{S}^3 \times [0,1]) \setminus \nu C) \qquad (1)$$

is surjective. Gordon showed that on the other end, the induced map

$$\pi_1(\mathbb{S}^3 \setminus \nu K) \hookrightarrow \pi_1((\mathbb{S}^3 \times [0,1]) \setminus \nu C) \qquad (2)$$

is injective. This motivates the following homotopy analogue of the notion of a smooth ribbon concordance:
Definition 4. An oriented knot J in $\mathbb{S}^3$ is *homotopy ribbon concordant* to a knot K if there is a locally flatly embedded concordance $C \cong \mathbb{S}^1 \times [0,1]$ in $\mathbb{S}^3 \times [0,1]$, restricting to $J \subset \mathbb{S}^3 \times \{1\}$ and $K \subset \mathbb{S}^3 \times \{0\}$ such that the induced maps on fundamental groups of exteriors satisfy (1) and (2). We write $J \geq_{\text{top}} K$.

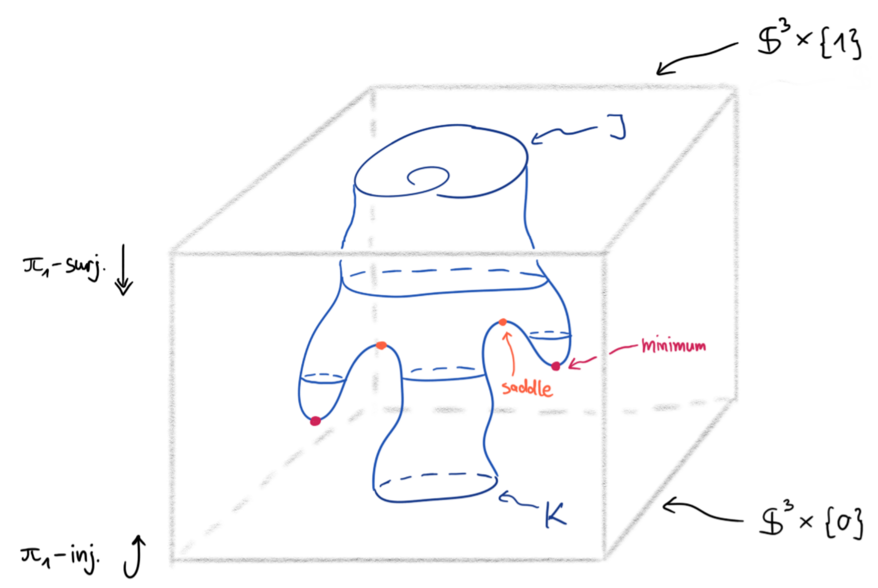

Figure 3: Going upwards in a ribbon concordance, we at first see the birth of some circles, which subsequently are connected via band-sums at the saddles.

Friedl and Powell observed [FP19] that a homotopy ribbon concordance between J and K implies the divisibility of Alexander polynomials:

$$J \geq_{\text{top}} K \Rightarrow \Delta_K \mid \Delta_J$$

Maybe this statement can be extended to an injection of Blanchfield forms (in the spirit of [Gil84]) in the **homotopy** ribbon setting.

Doubly slice knots

Question ([Fox, 1960s])**.** Which slice knots are cross sections of **unknotted**[2] 2-spheres $\mathbb{S}^2 \hookrightarrow \mathbb{S}^4$?
Definition 5. We call such slices of a 2-unknot *doubly slice.*
Warning. As in the case of slice knots, there is an important distinction between the smooth and the topologically locally flat category.

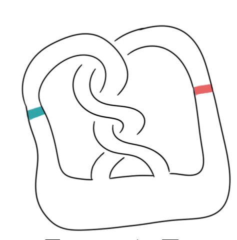

Figure 4: Two band-moves for 9_{46}.

Examples of doubly slice knots:
- Any knot of the form $K\#(-K)$ (this is a slice of the (± 1)-spin of K)
- 9_{46} (perform a saddle move on one of the two "arms", in any case this yields two unlinked unknotted circles which can be capped off by disks. **Claim:** These slice disks fuse to an unknotted $\mathbb{S}^2$ in $\mathbb{S}^4$)

Warning. We have to be careful when we want to define a double concordance group $\mathcal{C}_{\text{ds}}$. A natural choice would be to declare that K and J are doubly concordant iff $K\#(-J)$ is doubly slice, but it is not known whether this gives a transitive relation!
Open question (Stably doubly slice $\stackrel{?}{=}$ doubly slice)**.** Suppose K and $J\#K$ are doubly slice. Then, must J be doubly slice?

Multi-infection

We can take different parts of a knot R and tie them into the shape of a string link P, the resulting knot is called the *infection* of R with the pattern P. The following picture shows a very special example where the infection sites don't interact with each other[3]:

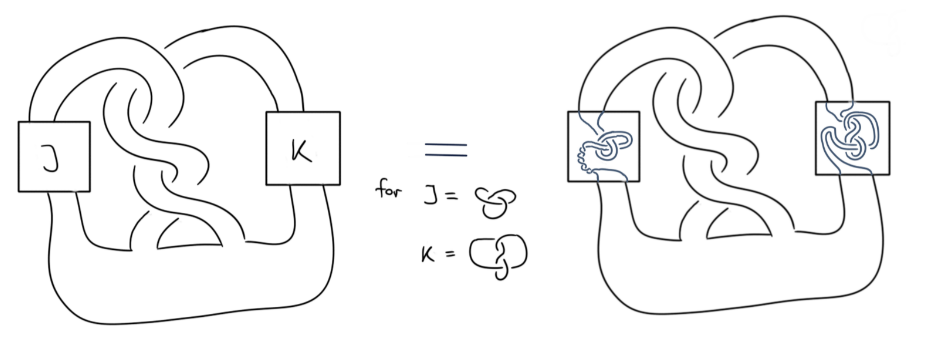

Figure 5: The boxes labelled with J and K indicate that the strands going through them are tied into the knots J and K (all parallel to each other). The extra full twists on the left arise from untwisting the writhe of the projection of the trefoil J.

If there is only a single box, this is a satellite construction. The operation is well-defined on concordance classes ("trace out the concordance annulus with the pattern"), and so any satellite operator P descends to a function (usually not a homomorphism!) $P\colon \mathcal{C} \to \mathcal{C}$, where $[K] \mapsto [P(K)]$.

- With this, one can construct (n)-solvable knots, i.e. knots lying in the n-th level $\mathcal{F}_n$ of the Cochran–Orr–Teichner solvable filtration [COT03] of the knot concordance group $\mathcal{C}$. The interesting but difficult part is to find examples which are not slice (or not even in $\mathcal{F}_{n.5}$)!
- Large classes of satellite operators are (weakly) injective on concordance classes. There are several ways of making $\mathcal{C}$ into an interesting metric space, and one can investigate how these metrics play together with the operations on the concordance group.

I would like to look at string link infection in the context of doubly slice knots: Many results should have an analog formulation for the double concordance group. For example, there is already work on the algebraic double concordance group $\mathcal{AC}_{\text{ds}}$, and Kim [Kim06] extended the COT filtration to topologically doubly slice knots.

References

[CFT09] Tim D. Cochran, Stefan Friedl, and Peter Teichner. New constructions of slice links. *Commentarii Mathematici Helvetici*, 84(3):617–638, 2009. https://arxiv.org/abs/0709.2148.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner. Knot concordance, Whitney towers and L2-signatures. *Annals of Mathematics*, pages 433–519, 2003. https://arxiv.org/abs/math/9908117.

[FP19] Stefan Friedl and Mark Powell. Homotopy ribbon concordance and Alexander polynomials. *arXiv preprint*, 2019. https://arxiv.org/abs/1907.09031.

[Fre82] Michael H. Freedman. A surgery sequence in dimension four; the relations with knot concordance. *Inventiones mathematicae*, 68(2):195–226, 1982.

[Gil84] Patrick M Gilmer. Ribbon concordance and a partial order on S-equivalence classes. *Topology and its Applications*, 18(2-3):313–324, 1984. https://core.ac.uk/download/pdf/82051356.pdf.

[Gor81] C. McA. Gordon. Ribbon concordance of knots in the 3-sphere. *Mathematische Annalen*, 257(2):157–170, 1981.

[HLR12] Matthew Hedden, Charles Livingston, and Daniel Ruberman. Topologically slice knots with nontrivial Alexander polynomial. *Advances in Mathematics*, 231(2):913–939, 2012. https://arxiv.org/abs/1001.1538.

[Kim06] Taehee Kim. New obstructions to doubly slicing knots. *Topology*, 45(3):543–566, 2006. https://arxiv.org/abs/math/0411126.

[LZ19] Adam Simon Levine and Ian Zemke. Khovanov homology and ribbon concordance. *arXiv preprint*, 2019. https://arxiv.org/abs/1903.01546.

[Zem19] Ian Zemke. Knot Floer homology obstructs ribbon concordance. *arXiv preprint*, 2019. https://arxiv.org/abs/1902.04050.

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[1] A *concordance* is a smoothly embedded annulus $\mathbb{S}^1 \times [0,1]$ in $\mathbb{S}^3 \times [0,1]$ with boundary $-\{0\} \times J \cup \{1\} \times K$.
[2] A sphere $\mathbb{S}^2 \hookrightarrow \mathbb{S}^4$ is by definition *unknotted* if it extends to an embedding of a disk $\mathbb{D}^3 \hookrightarrow \mathbb{S}^4$.
[3] In the general situation of multi-infection one should start with a proper embedded multi-disk, thicken this and replace with (a suitable cable of) the string link