Stable classification of 4-manifolds

2019-06-14
Regensburg
LKS-Seminar

Plan:

1. The Whitney trick - and why it does not work in dimension 4
2. Removing intersections by tubing into other things, $\_ \# S^2 \times S^2$
3. Spin thickenings & equivariant intersection forms

Everything in this talk will be smooth!

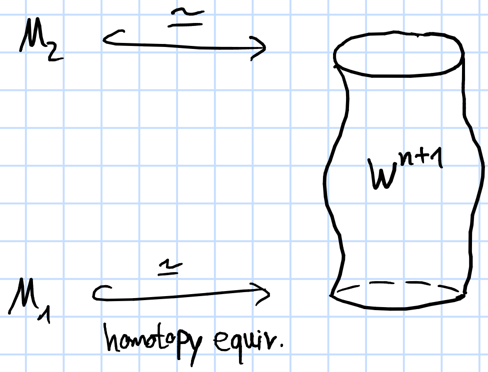
Sources:

[Scorpan: The wild world of 4-manifolds]

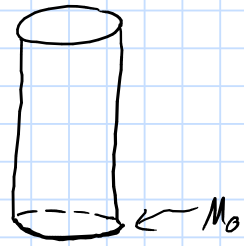
[Kasprowski, Powell, Teichner: Algebraic criteria for stable diffeomorphism of 4-manifolds]

[Arunima Ray, Peter Teichner: The topology of 4-manifolds
Class taught at the university of Bonn in the winter of 2018]

Recall: The h-cobordism theorem (in dim $n \geq 5$)



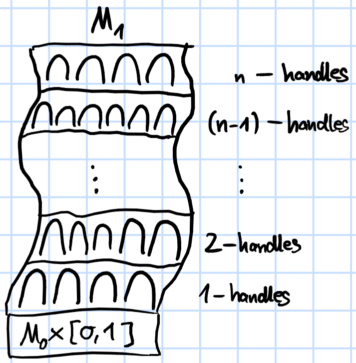
$\Rightarrow \quad W \cong M_0 \times [0,1]$



"A homology cylinder in dim ≥ 5 is already a cylinder"

Ingredients for the proof:

Take a handle decomposition of W^{n+1} relative to M_0



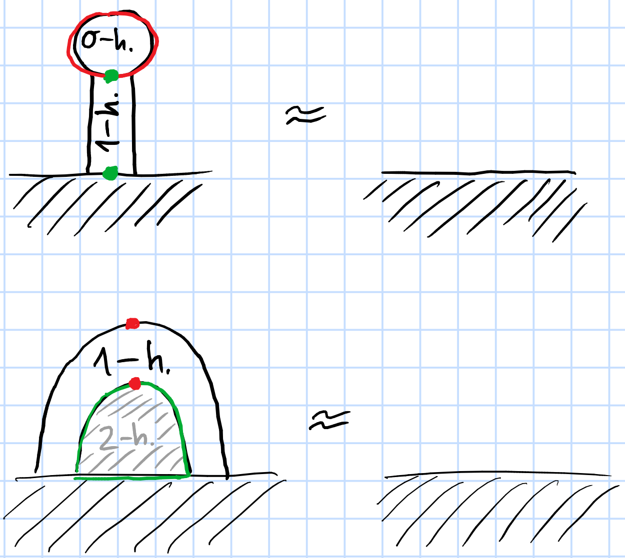
[Deal with 0-, 1-, n-, (n+1)-handles separately
handle trading, turning the decomposition upside down, ...]

Want to use handle cancellation:

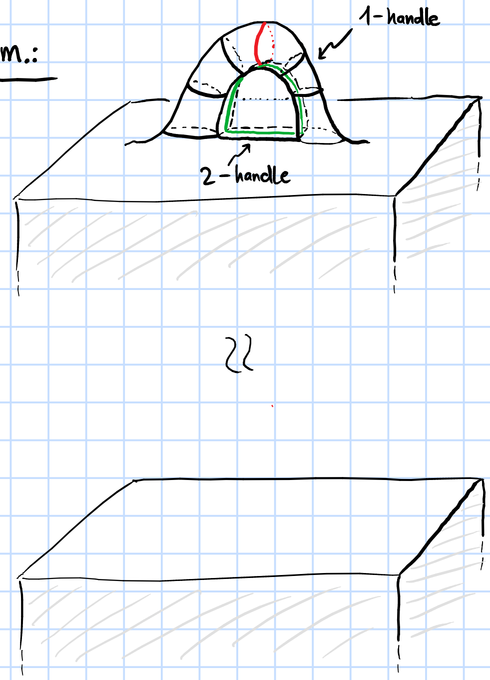
belt sphere in red
attaching sphere in green

← for cancellation to work, the attaching sphere has to intersect the belt sphere geometrically once

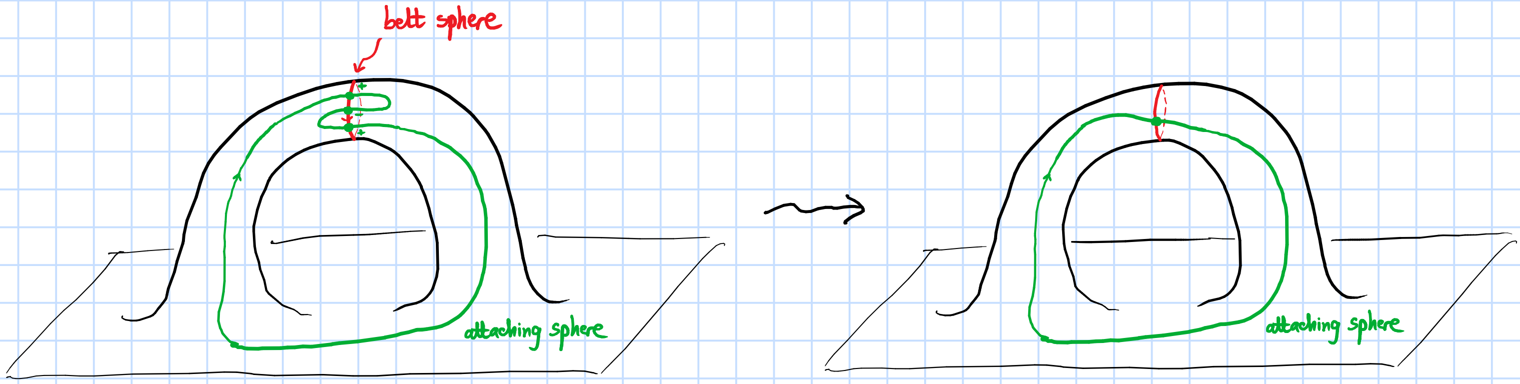
2-dim.:



3-dim.:

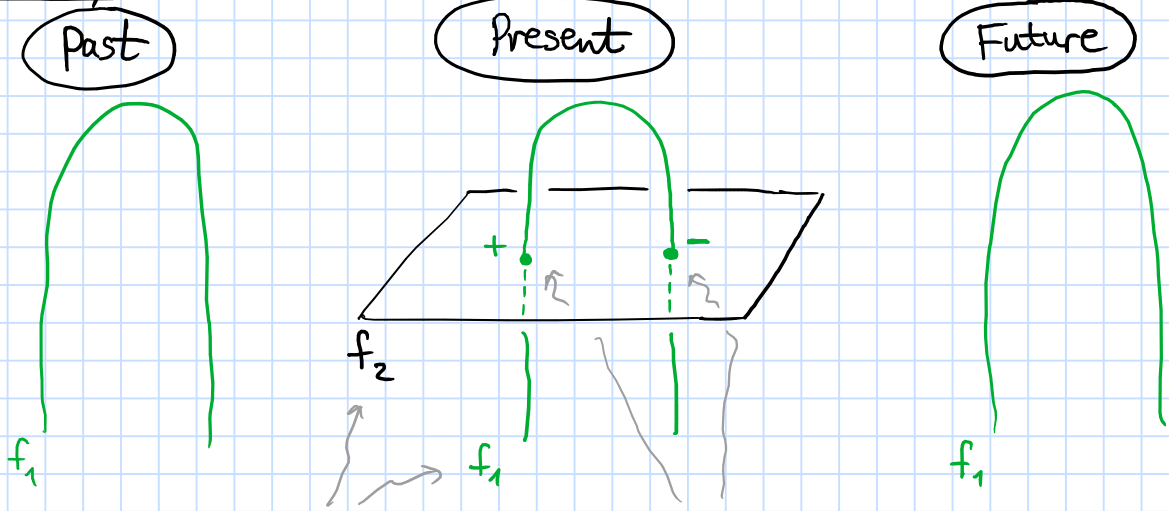


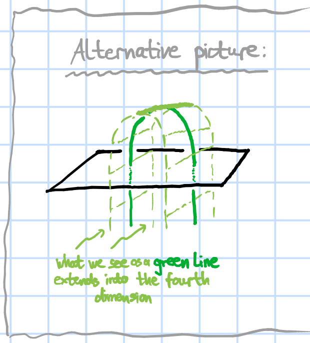
The conditions on the boundary inclusions $M_i \hookrightarrow W$ yield that the k- and $(k+1)$-handles cancel algebraically $\Big\}\ H_*(W, M_i) = 0$



Need to turn algebraic cancellation into geometric cancellation

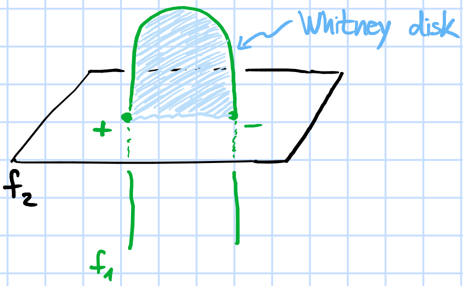
Whitney trick: turning algebraic into geometric cancellation



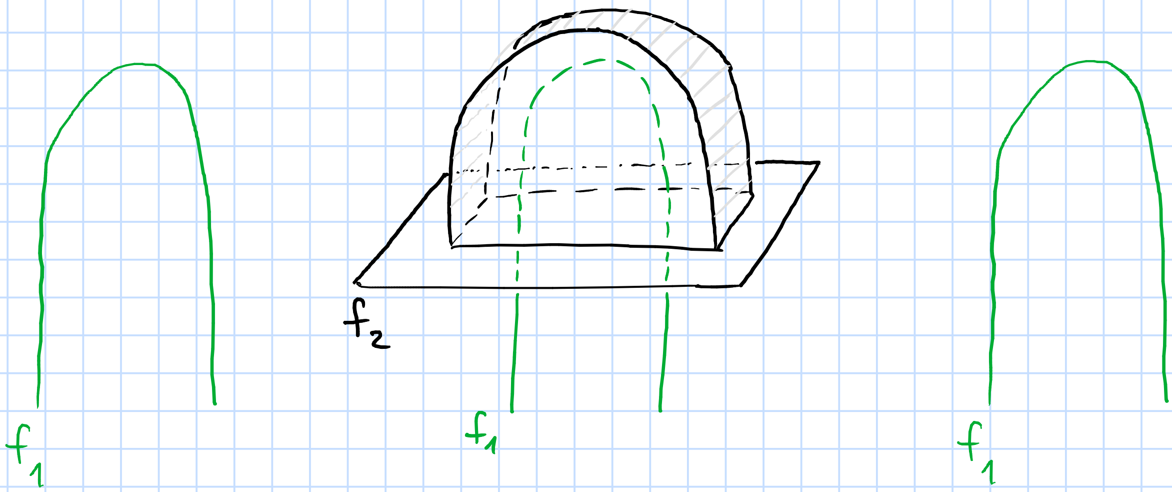


two surfaces in a 4-manifold

two intersection points with different signs



WHITNEY MOVE



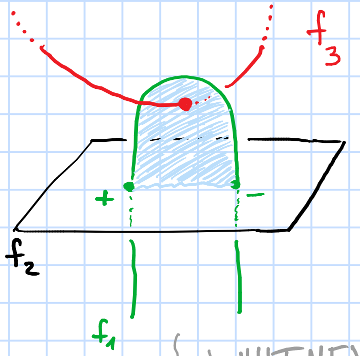
The two algebraically cancelling intersection points between f_1 and f_2 are gone!

But ...

Problems with this in 4-dimensions:

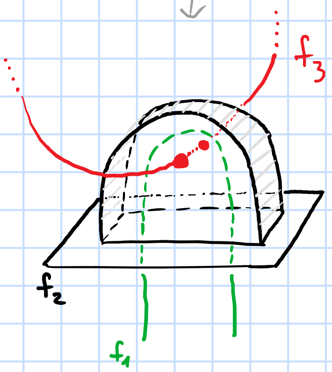
·) $2+2=4 \rightsquigarrow$ Possibly can't find embedded Whitney disks,

The disks might intersect other things, themselves or each other



$\leftarrow$ something might crash through the Whitney disk

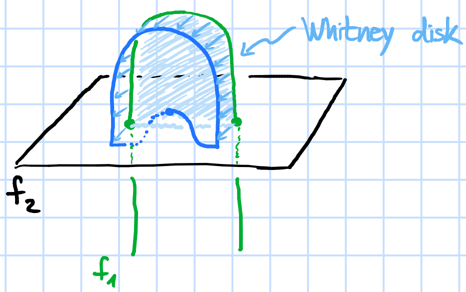
WHITNEY MOVE



$\rightsquigarrow$ Whitney trick removes the two cancelling intersections between f_1 and f_2, but introduces two new intersections between f_2 and f_3

·) Even if we can find embedded disks, they might have the wrong framing

(need this for the parallel copies of the Whitney disks) $\qquad \pi_1 SO(2) \cong \mathbb{Z}$

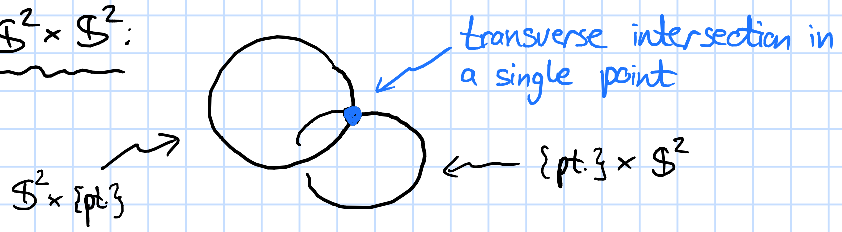


Whitney framing: Normal to f_1, tangent to f_2

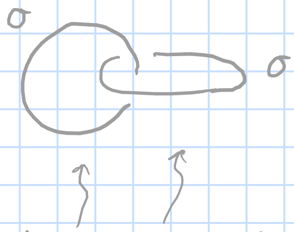
Slogan: 4-dimensional (smooth) topology is all about intersecting disks / spheres / surfaces !

Stable classification: Allow connected sum $\_\ \# S^2 \times S^2$

Schematic picture of $S^2 \times S^2$:



Kirby diagram of $S^2 \times S^2$:



$\cup$ (4-handle)

two zero-framed 2-handles

Intersection form:

$$\lambda_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

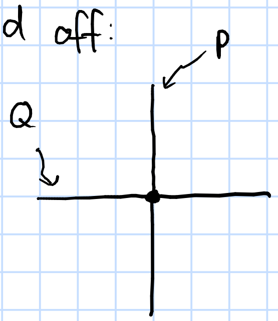
(we sometimes call this a hyperbolic form)

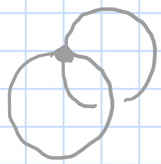
$$\lambda_{S^2 \times S^2} : H_2(S^2 \times S^2; \mathbb{Z}) \times H_2(S^2 \times S^2; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

Removing intersections by tubing into things:

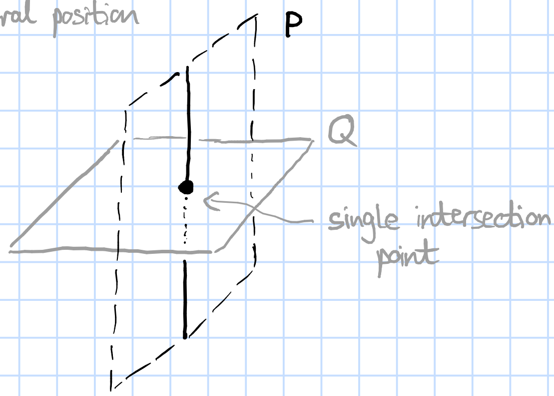
Imagine two surfaces P, Q (in our case, think of Whitney disks) with an intersection point that we want to get rid off:
($P \pitchfork Q$ transverse at this point)

Remember: A schematic like this shows two 2-dimensional objects intersecting in general position



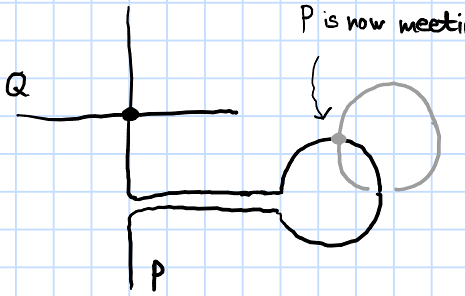


Local model:



e such
a thin

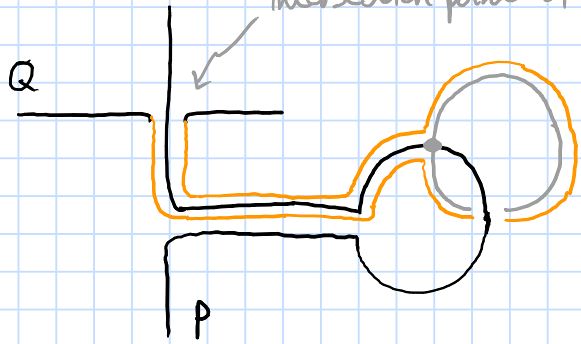
join P with an sphere by using tube



P is now meeting the other sphere in exactly one point

transverse / dual sphere for P

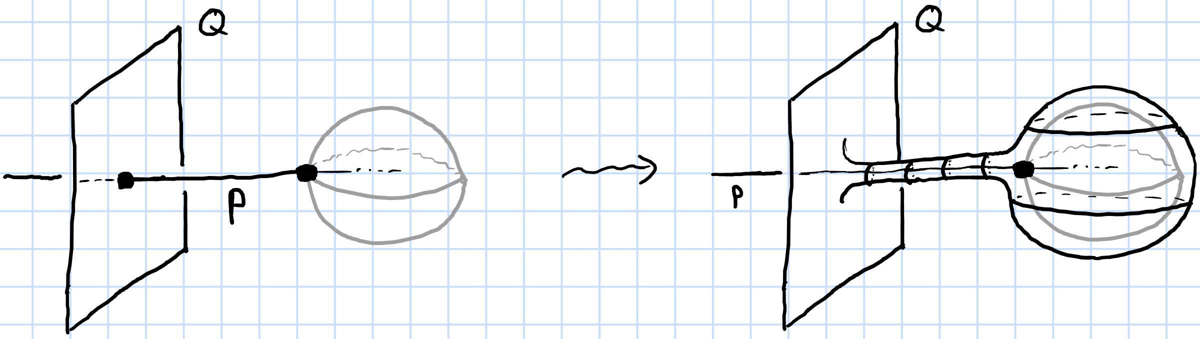
intersection point of P and Q has vanished



Pick a path in P from the intersection point with Q to the intersection point with the transverse sphere

⇝ using a thin tube following this path, connect Q to a parallel copy of the sphere

"4D-picture":



Analogy: Removing an intersection of curves on a surface by connected summing with $S^1 \times S^1$:

Note: •) none of the maneuvers change the genus of either P or Q

⇝ can use this to eliminate (self-) intersections of immersed Whitney disks

•) Can tube into the diagonal / anti-diagonal of an $S^2 \times S^2$ to change the framing of a disk by ± 2

Wall [1960s]: M, N closed, smooth simply connected 4-manifolds

$M \simeq N \implies M$ h-cobordant to N

homotopy equivalent

$\implies M$ diffeomorphic to N after $\underbrace{\text{connected summing with sufficiently many copies of } S^2 \times S^2}_{\text{stabilization}}$

Proof:

Of $\implies$: The program for the proof of the h-cob. theorem now works: (Assuming that we believe that the only problem are self-intersections of Whitney disks)

Whenever we need to get rid of intersection points, add a copy of $S^2 \times S^2$. $\square$

Rem.: With luck, can use the same $S^2 \times S^2$-term to eliminate several intersections.

We don't know any example where more than one stabilization is necessary!

A complete classification of 4-manifolds is impossible!

<u>Fact:</u> Every finitely presented group appears as the fundamental group of a closed (compact, no ∂), smooth, orientable, spin, ... 4-manifold.

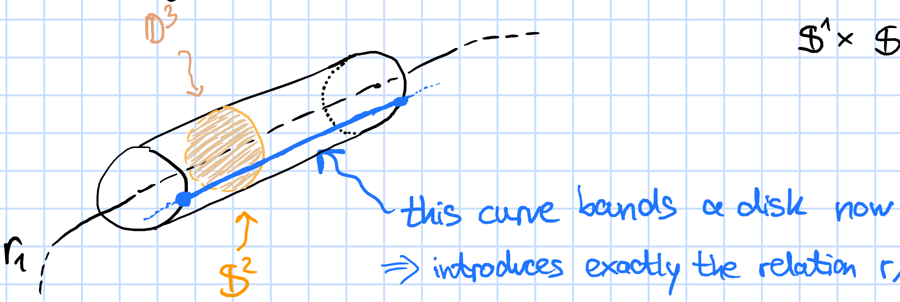
<u>Pf. idea:</u> $\pi = \langle g_1, \dots, g_n \mid r_1, \dots, r_m \rangle$

- Start with $\#^n S^1 \times S^3$ ⟵ $\pi_1 \cong$ free group on n generators

  one summand for each generator

- Realize the words of each relation by <u>disjoint</u>, <u>embedded</u> circles
- Perform surgery on these curves:

  Remove neighborhoods $S^1 \times \mathring{D}^3$ of each circle ⇝ boundary around each component is $S^1 \times S^2$



⟹ introduces exactly the relation r_1 into the fundamental group

Glue in m copies of $D^2 \times S^2$

<u>Exercise:</u> Why doesn't this construction work in dim. ≤ 3?

Matthias Kreck: Stable classification in the non-simply connected setting

$$\frac{\left\{\begin{matrix}\text{smooth, closed, oriented, spin 4-manifolds} \\ \text{with fundamental group } \pi \ (\text{finitely presented})\end{matrix}\right\}}{\text{stable diffeomorphism} \quad (-\#^k S^2\times S^2)} \xleftrightarrow{1:1} \frac{\Omega_4^{Spin}(B\pi)}{Aut}$$

$Aut := H^1(B\pi; \mathbb{Z}/2) \rtimes Out(\pi)$ are the automorphisms of the so-called normal 1-type

$\Omega_4^{Spin}(B\pi)$ is something we can sometimes calculate, for example using the Atiyah-Hirzebruch-Spectral Seq.

$$E^2_{p,q} = H_p(B\pi; \Omega_q^{Spin}(\{*\})) \Rightarrow \Omega_{p+q}^{Spin}(B\pi)$$

[Kasprowski, Powell, Teichner]: Identified algebraic obstructions coming from the filtration of AHSS

- primary: $c_*([M]) \in H_4(\pi; \mathbb{Z})$
- secondary: related to intersection pairing on $\pi_2(M)$
- tertiary: higher order intersection data from Whitney disks

To prescribe the fundamental group we can give a (homotopy class of a) map

$$M \xrightarrow{c} B\pi \longmapsto [M \xrightarrow{c} B\pi]$$

bordism class

In my master thesis, I looked at

$$[\mathcal{O}] \in \Omega_4^{Spin}(B\pi)$$

"a spin thickening of the group π"

& its equivariant intersection form

Two constructions for a representative of $[\sigma]$:

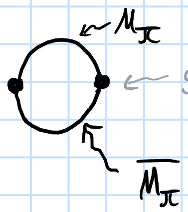
① Take a presentation 2-complex for π, embed into $\mathbb{R}^5 \hookleftarrow K_\pi$
2-dim. CW-complex K_π

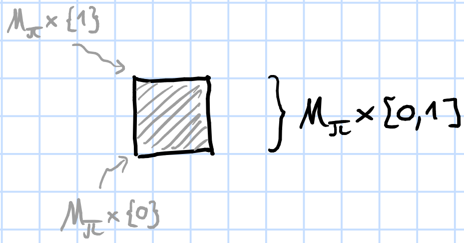
Look at the boundary of a tubular neighborhood of K_π: $\partial(\nu K_\pi)$ closed 4-mfld with correct π_1

② Build a 4-dim. handlebody (handles of index ≤ 2) $M_\pi^{(4)}$ (even framings to get spin mfld)

$$\text{Double}(M_\pi) = M_\pi \cup_{\partial M_\pi} \overline{M_\pi} \quad \left(\cong \partial(M_\pi \times [0,1])\right)$$

opposite orientation





It's not hard to draw a Kirby diagram for this! (doubling just adds 0-framed meridians to all the 2-handles)

Specifically, I looked at the equivariant intersection form of $[\mathcal{O}] = [\mathrm{Double}(M_\pi)]$

Def.: M closed, connected 4-manifold

$$\lambda_M : \pi_2(M,*) \otimes \pi_2(M,*) \longrightarrow \mathbb{Z}[\pi_1 M]$$

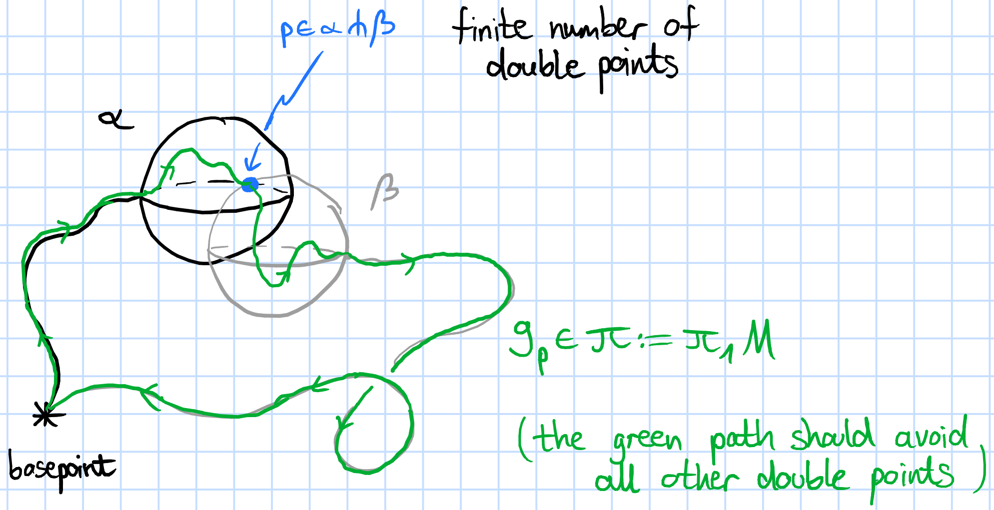
(immersed) 2-spheres in M

$$\lambda_M\left([\alpha : S^2 \looparrowright M], [\beta : S^2 \looparrowright M]\right) := \sum_{p \in \alpha \pitchfork \beta} \varepsilon_p \cdot g_p$$

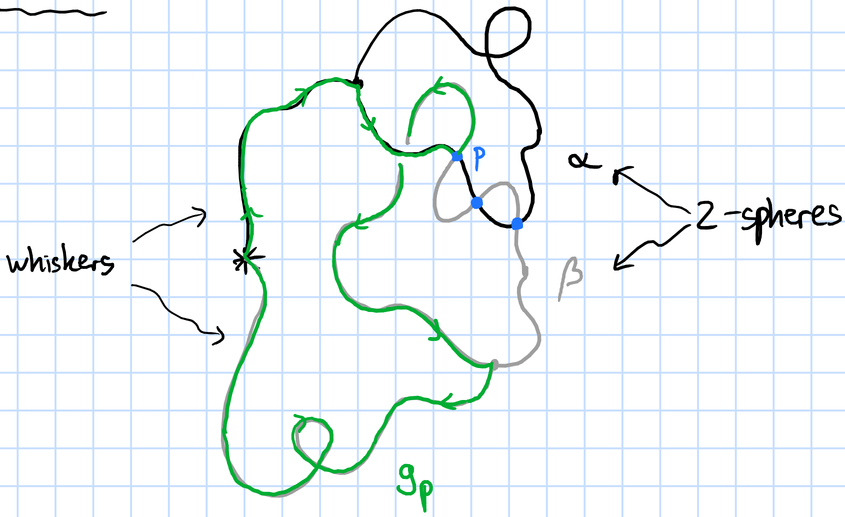
sign $\in \{\pm 1\}$

fundamental group element

finite number of double points



Schematic:



Properties: •) λ_M sesquilinear

•) λ_M hermitian

Involution on the group ring:

$$\overline{} : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi]$$

$$g \longmapsto \bar{g} = g^{-1}$$

Question: Parity ?

Def.: λ_M is called **even** if $\lambda_M^{ad} = \underbrace{q + q^*}$ "conjugate transpose"

for some $q \in \mathrm{Hom}_{\mathbb{Z}\pi}(\pi_2(M), \pi_2(M)^*)$

Ex.:

$$\lambda_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{q} + \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{q^*}$$

Question: π finitely presented group

Is $\lambda_{\mathrm{Double}(M_\pi)}$ even ?

Rem.: The answer does not depend on the presentation of π / choice of M_π, because all nullbordant elements in $\Omega_4^{Spin}(B\pi)$ are stably diffeomorphic (by Kreck) and $\lambda_{S^2 \times S^2}$ is even

Proposition: For $\pi = \mathbb{Z}/m \times \mathbb{Z}/n$, it is even!

Rough idea: •) It is actually enough to check the evenness on any closed, spin, ... 4-mfld. with $\pi_1 \cong \mathbb{Z}/m \times \mathbb{Z}/n$

•) Use an action

$$\begin{tikzcd} \mathbb{Z}/m \arrow[d, "free"'] & \times & \mathbb{Z}/n \arrow[d, "rotate"] \arrow[dll] \\ S^3 & \times & S^1 \end{tikzcd}$$

•) Then perform surgery on the quotient to get rid of the contribution of the S^1-factor to π_1

•) Compute π_2 by looking at H_2 of the universal cover, construct explicit representatives and count intersections $\square$