Stable classification of 4-manifolds

Plan:

1. The Whitney trick - and why it does not work in dimension 4
2. Removing intersections by tubing into other things, \( \# S^2 \times S^2 \)
3. Spin thickenings & equivariant intersection forms

Everything in this talk will be smooth!

Sources:

[Scorpan: The wild world of 4-manifolds]
[Kasprzak, Powell, Teichner: Algebraic criteria for stable diffeomorphism of 4-manifolds]
[Arunima Ray, Peter Teichner: The topology of 4-manifolds]

Class taught at the University of Bonn in the winter of 2018]
Recall: The $h$-cobordism theorem (in dim $n \geq 5$)

\[ M_2 \xrightarrow{\sim} W^{n+1} \xrightarrow{\sim} M_0 \times [0,1] \]

"A homology cylinder in dim $\geq 5$ is already a cylinder"

Ingredients for the proof:
Take a handle decomposition of $W^{n+1}$ relative to $M_0$

```
\begin{array}{c}
M_4 \\
\vdots \\
\text{\small 2-handles} \\
\text{\small 1-handles} \\
\end{array}
\quad M_0 \times [0,1]
```

Deal with $0$, $1$-, $m$-, $(m+1)$-handles separately
handle trading, turning the decomposition upside down, ...

Want to use handle cancellation:

- belt sphere in red
- attaching sphere in green

For cancellation to work, the attaching sphere has to intersect the belt sphere geometrically once

2-dim. 3-dim.
The conditions on the boundary inclusions $M_i \hookrightarrow W$ yield that the $k$- and $(k+1)$-handles cancel algebraically.

\[ H_*(W, M_i) = 0 \]

Need to turn algebraic cancellation into geometric cancellation.
Whitney trick: turning algebraic into geometric cancellation

Past

Present

Future

two surfaces in a 4-manifold

two intersection points with different signs

Whitney disk

WHITNEY MOVE

The two algebraically cancelling intersection points between \( f_1 \) and \( f_2 \) are gone!

But ...
**Problems with this in 4-dimensions:**

1. $2 + 2 = 4 \implies$ Possibly can't find embedded Whitney disks. The disks might intersect other things, themselves or each other.

   $f_3 \leftarrow$ something might crash through the Whitney disk.

   $\uparrow$ Whitney trick removes the two cancelling intersections between $f_1$ and $f_2$, but introduces two new intersections between $f_2$ and $f_3$.

2. Even if we can find embedded disks, they might have the wrong framing (need this for the parallel copies of the Whitney disks).

   $\pi_1 \text{SO}(2) \cong \mathbb{Z}$

   Whitney framing: normal to $f_1$, tangent to $f_2$.

**Slogan:** 4-dimensional (smooth) topology is all about intersecting disks/spheres/surfaces!
Stable classification: Allow connected sum \( \neq S^2 \times S^2 \)

Schematic picture of \( S^2 \times S^2 \):

\[ S^2 \times \{ \text{pt} \} \]

transverse intersection in a single point

Kirby diagram of \( S^2 \times S^2 \):

\[ \sigma \quad (4\text{-handle}) \]

two zero-framed 2-handles

Intersection form:

\[ \lambda_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ \Lambda & 0 \end{pmatrix} \]

we sometimes call this a hyperbolic form

\[ \lambda_{S^2 \times S^2} : H_2(S^2 \times S^2; \mathbb{Z}) \times H_2(S^2 \times S^2; \mathbb{Z}) \to \mathbb{Z} \]

Removing intersections by tubing into things:

in our case, think of Whitney disks

Imagine two surfaces \( P, Q \) with an intersection point that we want to get rid of:

(P and Q transverse at this point)

\[ P \]

\[ Q \]

\[ \sigma \]

\[ \Lambda \]

Local model:

\[ \text{single intersection point} \]

\[ p \]

\[ q \]

Join \( P \) with an sphere by using tube

e such a thin

\[ P \text{ is now meeting the other sphere in exactly one point} \]

\[ \text{transverse/dual sphere for } P \]
Intersection point of P and Q has vanished

Pick a path in P from the intersection point with Q to the intersection point with the transverse sphere

\[\Rightarrow\] using a thin tube following this path, connect Q to a parallel copy of the sphere

"4D-picture":

Analogy: Removing an intersection of curves on a surface by connected summing with $S^2 \times S^2$:

Note: *) None of the maneuvers change the genus of either P or Q

\[\Rightarrow\] can use this to eliminate (self-) intersections of immersed Whitney disks

*) Can tube into the diagonal/anti-diagonal of an $S^2 \times S^2$ to change the framing of a disk by $\pm 2$
M, N closed, smooth simply connected 4-manifolds

\[ M \cong N \implies M \text{ h-cobordant to } N \]

\( \Rightarrow \) M diffeomorphic to N after connected summing with sufficiently many copies of \( S^2 \times S^2 \) stabilization.

Proof:

Of \( \Rightarrow \): The program for the proof of the h-cob. theorem now works:

Whenever we need to get rid of intersection points, add a copy of \( S^2 \times S^2 \).

Rem.: With luck, can use the same \( S^2 \times S^2 \) term to eliminate several intersections.

We don't know any example where more than one stabilization is necessary!
A complete classification of 4-manifolds is impossible!

Fact: Every finitely presented group appears as the fundamental group of a closed, smooth, orientable, spin, ...
(compact, no θ)

4-manifold.

Af. idea: \( \pi = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_m \rangle \)

1) Start with \( n \times S^3 \) one summand for each generator

\[ \pi \cong \text{free group on } n \text{ generators} \]

2) Realize the words of each relation by disjoint, embedded circles

3) Perform surgery on these curves:

Remove neighborhoods \( S^1 \times D^3 \) of each circle \( \Rightarrow \) boundary around each component is \( S^1 \times S^2 \)

this curve bounds a disk now

\[ \Rightarrow \text{introduces exactly the relation } r_n \text{ into the fundamental group} \]

Glue in \( m \) copies of \( D^2 \times S^2 \)

Exercise: Why doesn’t this construction work in dim. \( \leq 3 \)?
Matthias Kreck: Stable classification in the non-simply connected setting

\[
\begin{aligned}
&\text{smooth, closed, oriented, spin 4-manifolds} \\
&\text{with fundamental group } \pi_1 \text{ (finitely presented)} \\
&\text{stable diffeomorphism} \quad \left(= \#^k S^2 \times S^2 \right) \\
\end{aligned}
\]

\[\xrightarrow{1:1} \quad \frac{\Omega^\text{Spin}_4(B\pi_1)}{\text{Aut}}\]

\[\text{Aut} = H^1(B\pi_1; \mathbb{Z}_2) \times \text{Out}(\pi_1)\]

are the automorphisms of the so-called normal 1-type

\[\Omega^\text{Spin}_4(B\pi_1)\] is something we can sometimes calculate, for example using the Atiyah-Hirzebruch-Spectral Seq:

\[E_2 = H^p(B\pi_1; \Omega^\text{Spin}_{2+q}(\pi_1)) \Rightarrow \Omega^\text{Spin}_{p+q}(B\pi_1)\]

[Kosinski, Powell, Teichner]: Identified algebraic obstructions coming from the filtration of AHSS

- primary: \[c_\ast([M]) \in H^4(\pi_1; \mathbb{Z})\]
- secondary: related to intersection pairing on \(\pi_2(M)\)
- tertiary: higher order intersection data from Whitney disks

To prescribe the fundamental group we can give a (homotopy class of a) map

\[M \xrightarrow{c} B\pi_1 \quad \xrightarrow{\beta} \quad [M \xrightarrow{c} B\pi_1] \]

In my master thesis, I looked at

\[\left[O\right] \in \Omega^\text{Spin}_4(B\pi_1)\]

"a spin thickening of the group \(\pi_1\)" & its equivariant intersection form
Two constructions for a representative of $[O^3]$:

1. Take a presentation $2$-complex for $\pi$, embed into $\mathbb{R}^5 \hookrightarrow K_\pi$

   Look at the boundary of a tubular neighborhood of $K_\pi$: $\partial (\nu K_\pi)$ closed $4$-manifold with correct $\chi$.

2. Build a $4$-dim. handlebody (handles of index $\leq 2$) $M^{(k)}$ (even framings to get spin manifold)

   \[
   \text{Double} \left( M_\pi \right) = M_\pi \cup_{\partial M_\pi} M_\pi \quad \left( \cong \partial (M_\pi \times [0,1]) \right)
   \]

   It's not hard to draw a Kirby diagram for this!

   (doubling just adds $0$-framed meridians to all the $2$-handles)
Specifically, I looked at the equivariant intersection form of $[\sigma] = [\text{Double}(M_{g\text{c}})]$.

**Def:** $M$ closed, connected 4-manifold

$$\lambda_M : \pi_2(M, \ast) \otimes \pi_2(M, \ast) \to \mathbb{Z}[\pi_1 M]$$

**Diagram:**

- $(\text{immed})$ 2-spheres in $M$
- Sign $\pm 1$
- Fundamental group element
- Finite number of double points
- Basepoint

$$\lambda_M([\alpha : S^2 \to M], [\beta : S^2 \to M]) := \sum_{p \in \alpha \cap \beta} \varepsilon_p \cdot g_p$$

**Schematic:**

- Whiskers
- 2-spheres
- $g_p \in \pi_1 := \pi_1 M$

**Properties:**

- $\lambda_M$ sesquilinear
- $\lambda_M$ hermitian

**Involution on the group ring:**

$$: \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$$

$$g \mapsto \overline{g} = g^{-1}$$
**Question:** Parity?

**Def.:** $\lambda^\text{ad}_M$ is called even if $\lambda^\text{ad}_M = q + q^*$ for some $q \in \text{Hom}_{\mathbb{Z}/2}(\pi_2(M), \pi_2(M)^*)$.

**Ex.:** $\lambda^{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

**Question:** Is $\lambda_{\text{Diade}(M_{\Sigma})}$ even?

**Remark:** The answer does not depend on the presentation of $\pi_2$ or choice of $M_{\Sigma}$, because all involutions elements in $\Omega^{e, \text{Spin}}_4(B\Sigma)$ are stably diffeomorphic (by Kreck) and $\lambda^{S^2 \times S^2}$ is even.

**Proposition:** For $\Sigma = \mathbb{Z}/m \times \mathbb{Z}/n$, it is even!

**Rough idea:** It is actually enough to check the evenness on any closed, spin, 4-manifold with $\Sigma \cong \mathbb{Z}/m \times \mathbb{Z}/n$.

1. Use an action $\mathbb{Z}/m \times \mathbb{Z}/n$
   - $\to$ rotate
   - $S^3 \times S^1$

2. Then perform surgery on the quotient to get rid of the contribution of the $S^1$-factor to $\Sigma$

3. Compute $\pi_2$ by looking at $H_2$ of the universal cover, construct explicit representatives and count intersections.