Doubly slice knots

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Slice knots and Concordance

Definition
A knot $K \subset S^3$ is called slice if it arises as the equatorial slice of a 2-sphere embedded in $S^4$.

Other definitions that you might have seen:
- $K$ is slice if ...
  - ... it is concordant to the unknot.
  - ... it is the boundary of a flat/smooth 2-disk in the 4-ball.

\[\text{Figure 1: Slicing a 2-sphere}\]

\(^a\)There is a topological locally flat and smooth version of this.
Doubly slice knots

Question [Fox, 1960s]
Which slice knots are cross sections of unknotted 2-spheres $S^2 \hookrightarrow S^4$?

Definition
We call such slices of a 2-unknot doubly slice.

- So named because a doubly slice knot slices in two ways.
- Example: $9_{46}$

Figure 2: Claim: The union of the two slice disks in the picture is a trivial 2-sphere.
Another example

Proposition

$K \# (-K)$ is doubly slice.

- Careful: This is a cross-section of the sphere $S$ obtained by spinning the knot $K$, but this $S^2$ is knotted in $S^4$! Its knot group $\pi_1(S^4 \setminus S)$ agrees with $\pi_1(S^3 \setminus K)$.
- Surprising observation [Zeeman, 1965]: The $\pm 1$-spin of a knot $K$ is unknotted.

Figure 3: Spinning a knot to obtain a 2-sphere

Figure 4: Twist the dotted 3-ball around an axis while performing the spinning in $\mathbb{R}^4$
Multi-infection by a string link

- Generalizes the notion of a satellite link
Connection

**Proposition [Cochran, Friedl, Teichner: New constructions of slice links, 2009]**

Any algebraically slice knot $K$ is smoothly concordant to an infection of the form

$$\text{Inf}(L, J, \eta_i)$$

with

- $L$ ribbon
- $J$ string link with linking numbers zero
- $\eta_i$ in the intersection of the terms of the lower central series of $\pi_1(S^3 \setminus L)$

**Question**

Is there a similar statement for algebraically doubly slice knots $K$?