Doubly slice knots

Benjamin Matthias Ruppik

MPIM Bonn

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Slice knots and Concordance

Definition

A knot $K \subset \mathbb{S}^3$ is called *slice* if it arises as the equatorial slice of a 2-sphere embedded[a] in $\mathbb{S}^4$.

Other definitions that you might have seen:
K is *slice* if ...

- ... it is concordant to the unknot.
- ... it is the boundary of a flat/smooth 2-disk in the 4-ball.

[a] There is a topological locally flat and smooth version of this.

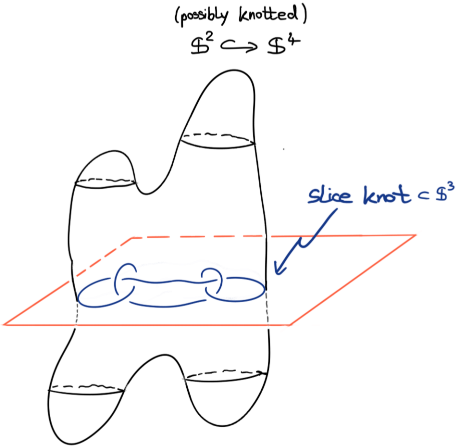


Figure 1: Slicing a 2-sphere

Doubly slice knots

Question [Fox, 1960s]

Which slice knots are cross sections of **unknotted** 2-spheres $\mathbb{S}^2 \hookrightarrow \mathbb{S}^4$?

Definition

We call such slices of a 2-unknot *doubly slice*.

- So named because a doubly slice knot slices in two ways.
- Example: 9_{46}

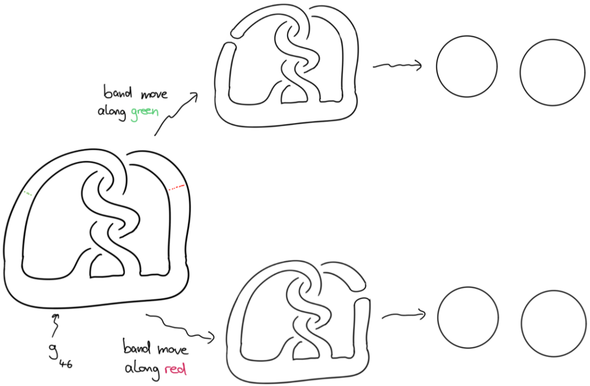


Figure 2: Claim: The union of the two slice disks in the picture is a trivial 2-sphere.

Another example

Proposition

$K\#(-K)$ is doubly slice.

- Careful: This is a cross-section of the sphere S obtained by *spinning* the knot K, but this $\mathbb{S}^2$ is **knotted** in $\mathbb{S}^4$! Its knot group $\pi_1(\mathbb{S}^4 \setminus S)$ agrees with $\pi_1(\mathbb{S}^3 \setminus K)$.
- Surprising observation [Zeeman, 1965]: The ± 1-spin of a knot K is unknotted.

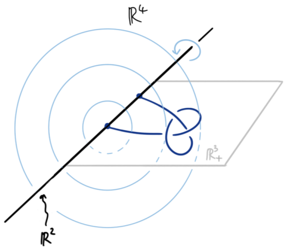


Figure 3: Spinning a knot to obtain a 2-sphere

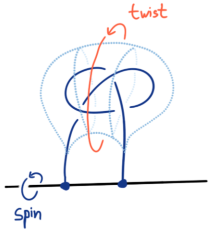
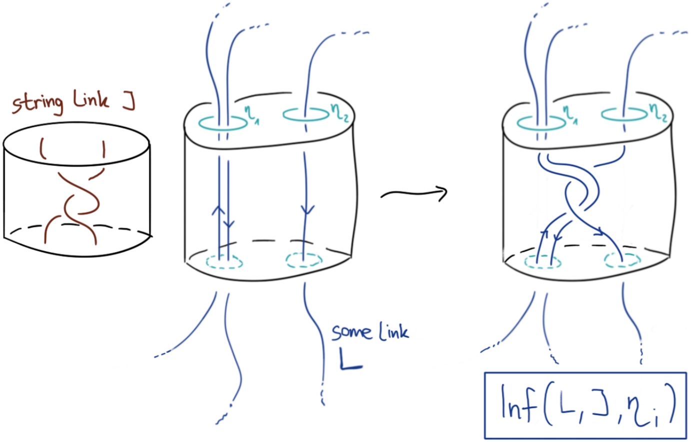


Figure 4: Twist the dotted 3-ball around an axis while performing the spinning in $\mathbb{R}^4$

Multi-infection by a string link

- Generalizes the notion of a satellite link



Connection

Proposition [Cochran, Friedl, Teichner: New constructions of slice links, 2009]

Any algebraically slice knot K is smoothly concordant to an infection of the form

$$\mathrm{Inf}(L, J, \eta_i)$$

with

- L ribbon
- J string link with linking numbers zero
- η_i in the intersection of the terms of the lower central series of $\pi_1(\mathbb{S}^3 \setminus L)$

Question

Is there a similar statement for algebraically doubly slice knots K?