Homology of the Little disks operad

24.04.2019
IMPRS Seminar
MPIM Bonn

Plan: ① The Poisson operad $\rightsquigarrow$ Poisson algebras

② Homology of configuration spaces via "planetary systems" $\rightsquigarrow H_*(\mathrm{Cube}^d) \cong \mathrm{Pois}^d$

Goal: $\Rightarrow$ Homology of a loop space $\Omega^d(X)$ is an algebra over Pois^d

Main source:

[Sinha: The (non-equivariant) homology of he Little disks Operad
arxiv: math/061023]

Warning:

We will switch between disks and cubes without further mentioning it!

<u>Recall:</u> An operad encodes multiplication.

Peter May 1970s: operation ← operad → monad

Sequence of objects $\{\mathcal{O}(n)\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ ← symmetric, monoidal, (unitary) category

+ multiplication maps
+ action of symmetric group Σ_n

s.th. ...

<u>Ex.:</u> •) Commutative operad Comm

has $\mathrm{Comm}(n) = \mathbb{1}_{\mathcal{C}}$

There is only one way to multiply n things commutatively (think of multiplying real numbers)

•) Associative operad Ass

Product of n things is determined by the order for an associative operation (think of multiplying matrices)

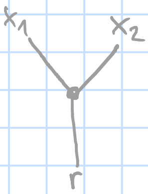
In (Top): $\mathrm{Ass}(n) = \Sigma_n$ ← the $n!$ orderings of n points

(CGWH)

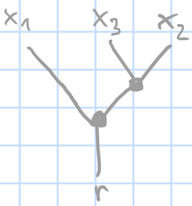
In $(\mathrm{Vect}_{\mathbb{K}})$: $\mathrm{Ass}(n) = \mathbb{K}[\Sigma_n]$ ← group ring of symmetric group

•) Lie operad Lie (in vector spaces)

To take commutators of matrices, we have to order and parenthesize them!



$\longleftrightarrow$ $[x_1, x_2]$



$\longleftrightarrow$ $[x_1, [x_3, x_2]]$

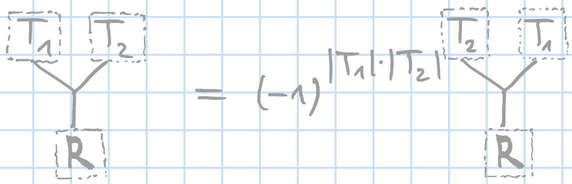
Labelled, rooted, uni-trivalent trees $\longleftrightarrow$ Lie bracketings "parathesizations"

$\mathrm{Lie}(n) := \mathbb{K}\langle n\text{-trees}\rangle \,/\,$ anti-symmetry & Jacobi identity

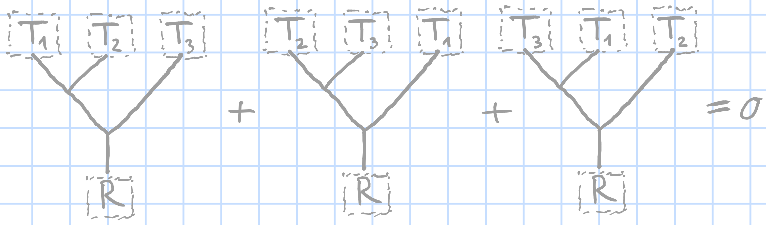
S-tree for a subset $S \subseteq \underline{n} = \{1,\dots,n\}$:
isotopy class of an acyclic uni-trivalent graph, rooted, embedded in upper half-plane with root at origin,
Leaves labelled by elements from the set S
(univalent vertices other than the root)

$[x,y] = -[y,x]$

(AS)



(Jac)



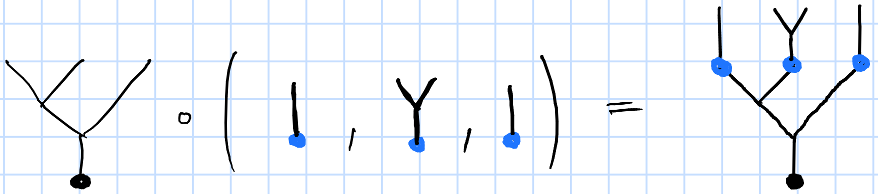
$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

here: T_1, T_2, T_3, R (possibly trivial) subtrees, not modified in these operations

$|T_i| :=$ number of internal vertices of T_i

Structure maps of Lie: "Grafting" trees, i.e. identifying the root of one with the leaf of another
(welldefined, because the (AS) & (Jac) relations are local)

Ex.: $\mathrm{Lie}(3) \otimes (\mathrm{Lie}(1) \otimes \mathrm{Lie}(2) \otimes \mathrm{Lie}(1)) \longrightarrow \mathrm{Lie}(4)$



•) degree-d graded Poisson operad Pois^d (in vector spaces)

In a Poisson algebra we can both

Lie multiply — represented by trees

and

multiply the results together — represented by placing the trees together in a forest

$[\_,\_]$ satisfies anticommutativity and Jacobi

$\_\cdot\_$ associative and graded commutative

& there is a Leibnitz rule: bracket is a derivation with respect to the multiplication

$$[X, Y\cdot Z] = (-1)^{|X|\cdot|Y|}\, Y\cdot[X,Z] + [X,Y]\cdot Z$$

$\mathrm{Pois}^d(n) = \mathbb{K}\langle n\text{-forests}\rangle \Big/$ anti-symmetry, Jacobi & commutativity:

n-forest: collection of S-trees, root vertices at points $(0,0), (1,0), (2,0), \dots$ in upper half-plane, each integer from 1 to n labels exactly one leaf.

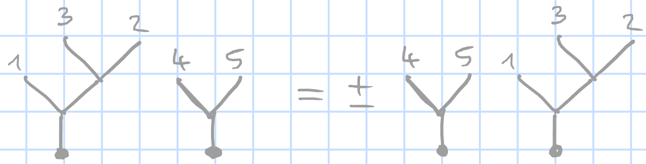
If $F_1 = (T_1, \dots, T_m)$

$F_2 = \dots$

consist of the same trees, then

$$F_1 = \sigma^{(d-1)} F_2$$

σ is the sign of the permutation which relates the ordering of the internal vertices of the trees in F_1 with those of F_2



Operad composition maps are defined so that the Leibnitz rule holds

(by the derivation property, we can always reduce to expressions that correspond to forests)

Example of a Poisson algebra (i.e an algebra over the Poisson operad):

$f, g \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R})$, think of elements of $\mathbb{R}^{2n}$ as a pair (x, p)

$x \in \mathbb{R}^n$ repr. position of a particle

$p \in \mathbb{R}^n$ repr. the momentum of a particle

$\rightsquigarrow$ Poisson bracket of f and g is the function on $\mathbb{R}^{2n}$ given by

$$[f, g](x,p) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right)$$

Together with pointwise multiplication $f \cdot g$ of functions, this gives a Poisson algebra.

This works more generally for $\mathcal{C}^\infty(M, \mathbb{R})$

M symplectic manifold

•) Little n-cubes operad Cube^n

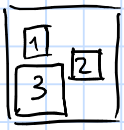
$$\mathrm{Cube}^{n}(j) = \Big\{ \text{Rectilinear embeddings } \underbrace{\mathbb{I}^n \amalg \mathbb{I}^n \amalg \dots \amalg \mathbb{I}^n}_{j \text{ many}} \longrightarrow \mathbb{I}^n \Big\}$$

(cubes are contained in int $\mathbb{I}^n$, have disjoint interiors)

n: dim. of the cubes

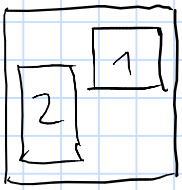
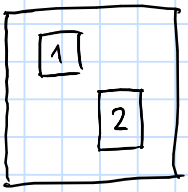
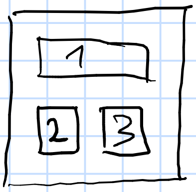
j: number of little cubes

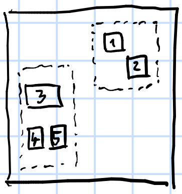
$\mathbb{I} = [0,1]$

Ex.: $\mathrm{Cube}^2(3) \ni$ 

Ex.: Compositions

$$\mathrm{Cube}^2(2) \otimes \left(\mathrm{Cube}^2(2) \otimes \mathrm{Cube}^2(3)\right) \longrightarrow \mathrm{Cube}^2(5)$$

 $\circ$ [ , ]

$=$ 

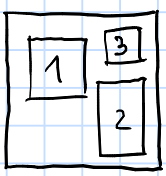
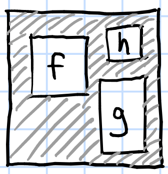
Observation: $\underbrace{d\text{-fold loopspaces}}$ are algebras over the little d-disks operad!

$$\Omega^d(X) = \{ \text{basepoint preserving maps } S^d \longrightarrow X \}$$

$$= \mathrm{Maps}\left((\mathbb{D}^d, S^{d-1}), (X, *) \right)$$

Action of Cube^d is given by

$$\mathrm{Cube}^d(n) \times \left(\Omega^d(X)^{\times n}\right) \longrightarrow \Omega^d X$$

 $\times f \times g \times h \longmapsto$ 

outside of the little cubes, the resulting map is constant at the basepoint of X

[Boardman-Vogt, May] Recognition principle:

The converse is essentially true:

If $\mathrm{Cube}^d \curvearrowright X$ and $\pi_0(X)$ (which is a monoid via $X \times X \to X$) is a group

$\Rightarrow$ $X \simeq_{w.e.}$ d-fold Loopspace

Coefficients always in a PID R

Slogan: Operad actions on spaces give operad actions on homology

Proposition: •) $\mathcal{O}$ operad of spaces $\overset{H_*(-)}{\rightsquigarrow}$ $H_*(\mathcal{O})$ operad of modules via

$$
\begin{array}{c}
H_*(\mathcal{O}(r)) \otimes H_*(\mathcal{O}(n_1)) \otimes \dots \otimes H_*(\mathcal{O}(n_r)) \\
\big\downarrow \text{ Künneth map/homology cross product} \\
H_*(\mathcal{O}(r) \times \mathcal{O}(n_1) \times \dots \times \mathcal{O}(n_r)) \\
\big\downarrow H_*(\text{operad comp.}) \\
H_*(\mathcal{O}(n_1 + \dots + n_r))
\end{array}
$$

•) X algebra over $\mathcal{O}$ $\overset{H_*(-)}{\rightsquigarrow}$ $H_*(X)$ algebra over $H_*(\mathcal{O})$

Goal: $H_*(\mathrm{Cube}^d) = \mathrm{Pois}^d$

$\rightsquigarrow$ Homology of a d-fold Loopspace $H_*(\Omega^d X)$ is an algebra over Pois^d

Upshot: •) Homology of Loop spaces has a rich additional structure

•) Example: Hurewicz map $\pi_n(X) \to H_n(X)$

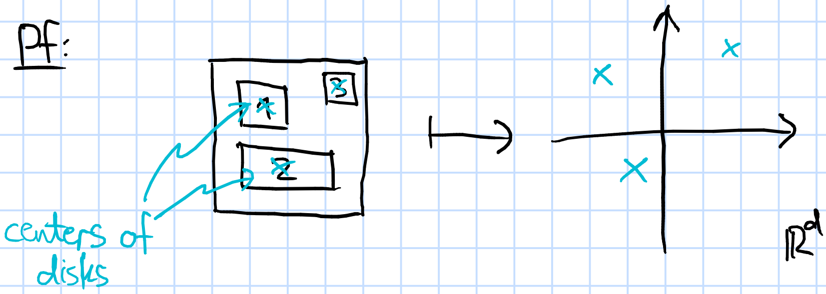
$\quad\quad \big\updownarrow$ k-fold looping

$\pi_{n-k}(\Omega^k X) \to H_{n-k}(\Omega^k X)$ might give additional information

Observation: $\mathrm{Cube}^d(n) \simeq \mathrm{Conf}_n(\mathbb{R}^d)$ ⟵ $\mathrm{Conf}_n(X) = \{n \text{ distinct points in a space } X\}$

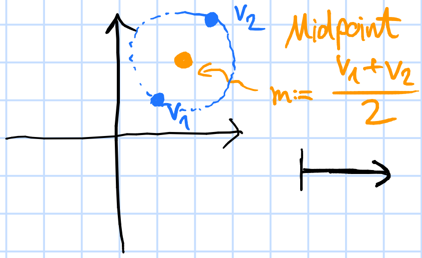
$= \{(x_1,\dots,x_n)\in X^{\times n} \mid x_i \neq x_j \text{ for } i \neq j\}$

Pf:



Fibre bundle where fibres $\hat{=}$ $\underbrace{\text{set of possible radii}}_{\text{convex} \Rightarrow \text{contractible}}$ $\square$

Important example: $\mathrm{Conf}_2(\mathbb{R}^d) \simeq S^{d-1}$



$$\longmapsto \left(\frac{v_1-m}{|v_1-m|}, \frac{v_2-m}{|v_2-m|}\right) \in \overbrace{\{(w_1,w_2) \mid w_1 = w_2,\ |w_i| = 1\}}^{=: P_{1 \mathrm{Y} 2}}$$

$$\cong S^{d-1}$$

$\rightsquigarrow H_*(\mathrm{Conf}_2(\mathbb{R}^d))$ free, rank $=1$ in dim 0, $d-1$
zero else

Moreover:

$$H_{d-1}(S^{d-1}) \longrightarrow H_*(\mathrm{Conf}_2(\mathbb{R}^d)) \text{ induced by } \begin{array}{ccc} S^{d-1} & & \mathrm{Conf}_2(\mathbb{R}^d) \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ v & \longmapsto & (v,-v) \end{array}$$

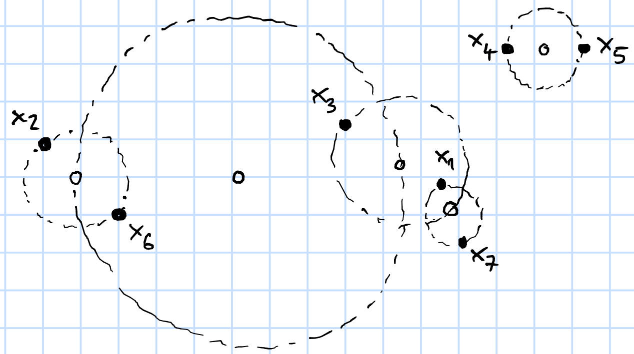
$[S^{d-1}] \longmapsto$ generating cycle

fundamental class

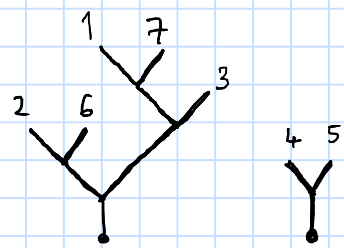
Easy example: $\mathrm{Conf}_1(\mathbb{R}^d) \cong \mathbb{R}^d \simeq pt.$

More generally can represent homology classes via

planetary systems:



Indexed by forests:



Bracket expression:

$$[[x_2, x_6], [[x_1, x_7], x_3]] \cdot [x_4, x_5]$$

For an S-Tree T: (recall that $|T|$ is the number of internal vertices of T)

$$P_T : (S^{d-1})^{\times |T|} \longrightarrow \mathrm{Conf}_n(\mathbb{R}^d)$$

$\rightsquigarrow$ homology class (pushforward of fundamental class $[(S^{d-1})^{\times |T|}]$)

$$[T] \in H_{|T|(d-1)}(\mathrm{Conf}_n(\mathbb{R}^d))$$

Similarly for a forest $F = (T_1, \dots, T_m)$:

The collection of the planetary systems P_{T_i} gives a submanifold of $\mathrm{Conf}_n(\mathbb{R}^d)$ (and thus a homology class)

Theorem: The planetary system map

$$P_{(-)} : \mathrm{Pois}^d(n) \longrightarrow H_*(\mathrm{Conf}_n(\mathbb{R}^d)) = H_*(\mathrm{Cube}^d(n))$$

is an isomorphism of operads.

Rough outline of the proof:

- show that the map is well-defined

  (i.e. on the left side we introduced anti-symmetry relations,
  Jacobi
  commutativity

  and need to show that those hold for the homology classes of configuration spaces)

- Use a perfect pairing between $Pois^d(n)$ and a dual complex $Siop^d(n)$ to conclude injectivity

  build from directed graphs, comes with a map $Siop^d(n) \longrightarrow H^*(Conf_n(\mathbb{R}^d))$ to cohomology and a combinatorial pairing between trees and graphs

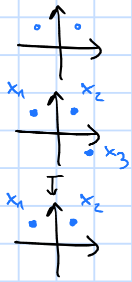
- estimates of the rank of $H_*(Conf_n(\mathbb{R}^d))$ to get surjectivity

$\rightsquigarrow$ $P_{(-)}$ is additive isomorphism

- Operad structure: need compatibility of geometric insertion str. of the Little disks & algebraic insertion str. of Poisson operad

Appendix: Homology of $\mathrm{Conf}_n(\mathbb{R}^d)$

Lemma: Have fibre bundles



$$\begin{tikzcd}
\mathbb{R}^d \setminus \{n-1 \text{ points}\} \simeq \bigvee_{n-1} S^{d-1} \arrow[d] & \\
\mathrm{Conf}_n(\mathbb{R}^d) \arrow[d, "p_i = \text{forget } i^{\text{th}} \text{ point}"] & (x_1, \dots, x_n) \arrow[d, mapsto] \\
\mathrm{Conf}_{n-1}(\mathbb{R}^d) & (x_1, \dots, \hat{x}_i, \dots, x_n)
\end{tikzcd}$$

⇝ Assemble these into Fadell–Neuwirth tower:

$$\begin{tikzcd}
\bigvee_{n-1} S^{d-1} \arrow[r] & \mathrm{Conf}_n(\mathbb{R}^d) \arrow[d, "p_n"] \\
\bigvee_{n-2} S^{d-1} \arrow[r] & \mathrm{Conf}_{n-1}(\mathbb{R}^d) \arrow[d, "p_{n-1}"] \arrow[u, bend right, "\text{split}"'] \\
& \vdots \arrow[d] \\
\bigvee_{2} S^{d-1} \arrow[r] & \mathrm{Conf}_3(\mathbb{R}^d) \arrow[d, "p_3"] \\
& \mathrm{Conf}_2(\mathbb{R}^d) \simeq S^{d-1}
\end{tikzcd}$$

split (a section adds a new n^{th} point "far away" or "doubles" a point)

L.e.s. of π_* for the fib. + Hurewicz $\longrightarrow$

Prop.: The first non-trivial homology group $H_{d-1}(\mathrm{Conf}_n(\mathbb{R}^d))$ is free of rank $\binom{n}{2}$.