Homology of the Little disks operad

Plan:
1. The Poisson operad $\Rightarrow$ Poisson algebras
2. Homology of configuration spaces
   via "planetary systems" $\Rightarrow H_*(\text{Cube}^d) \cong \text{Pois}^d$

Goal:
$\Rightarrow$ Homology of a loop space $\Omega^d(X)$
is an algebra over $\text{Pois}^d$

Main source:
[Sinha: The (non-equivariant) homology of the Little disks operad
arxiv: math/061023]

Warning:
We will switch between disks and cubes without further mentioning it!
Recall: An operad encodes multiplication.

Peter May (1970s): operation manad

Sequence of objects \( \{ G(n) \}_{n \in \mathbb{N}} \) in \( \mathcal{C} \).

- multiplication maps
- action of symmetric group \( \Sigma_n \)

Symmetric, monoidal, (unitary) category

Example 1) Commutative operad \( \text{Comm} \)

\[ \text{Comm}(n) = 1 \]

There is only one way to multiply \( n \) things commutatively (think of multiplying real numbers).

Example 2) Associative operad \( \text{Ass} \)

Product of \( n \) things is determined by the order for an associative operation (think of multiplying matrices).

\[ \text{(Top)}: \text{Ass}(n) = \Sigma_n \quad \text{the } n! \text{ orderings of } n \text{ points} \]

\[ \text{(Comm)}: \text{Ass}(n) = \mathbb{K} [ \Sigma_n ] \quad \text{group ring of symmetric group} \]

Example 3) Lie operad \( \text{Lie} \) (in vector spaces)

To take commutators of matrices, we have to order and parenthesize them!

\[ x_1 \leftrightarrow \{ x_1, x_2 \} \]

\[ x_1, x_3, x_2 \leftrightarrow \{ x_1, \{ x_3, x_2 \} \} \]

Labelled, un-trivalent trees \( \leftrightarrow \) Lie bracketings "parenthesizations"
\[ \text{Lie}(n) := \{ \text{trees} \} \]

anti-symmetry &
Jacobi identity

\( S \)-tree for a subset \( S \subseteq \{1, \ldots, n\} \):

isotopy class of an acyclic uni-trivalent graph,
rooted, embedded in upper half-plane with root at origin

Leaves labelled by elements from the set \( S \)
(undisplaced vertices other than the root)

\[ [x, y] = \mathbb{0} \]

\[ \text{Jac} \]

\[ (AS) \]

\[ \mathbb{0} \]

Structure maps of Lie: "Grafting" trees, i.e. identifying the root of one with the leaf of another

(well-defined, because the (AS) & (Jac) relations are local)

Ex:
\[
\text{Lie}(3) \otimes \left( \text{Lie}(1) \otimes \text{Lie}(2) \otimes \text{Lie}(1) \right) \rightarrow \text{Lie}(4)
\]

\[
\begin{array}{c}
\text{degree-0 graded Poisson operad} \\
\text{Pois}^0 \\
\text{(in vector spaces)}
\end{array}
\]

In a Poisson algebra we can both

Lie multiply and multiply the results together

represented by

represented by placing the

trees together in a forest

\([-,-]\) satisfies

antisymmetry and Jacobi

associative and graded commutative

& there is a Leibniz rule: bracket is a derivation with respect to the multiplication

\[
[X, Y \cdot Z] = (-1)^{\text{deg}[X, Y]} Y \cdot [X, Z] + [X, Y] \cdot Z
\]
\[\text{Pois } d(n) = \{ \text{n-forests} \}\]

- **n-forest**: collection of S-trees, root vertices at points \((0,0), (1,0), (2,0), \ldots\) in upper half-plane, each integer from 1 to \(n\) labels exactly one leaf.

- Anti-symmetry, Jacobi & Commutativity: If \(F_n = (T_1, \ldots, T_m)\)
  \[F_2 = \ldots\]
  consist of the same trees, then
  \[F_4 = \sigma^{-(d-1)} F_2\]
  \(\sigma\) is the sign of the permutation which relates the ordering of the internal vertices of the trees in \(F_4\) with those of \(F_2\).

- **Oprad composition maps** are defined so that the Leibnitz rule holds (by the derivation property, we can always reduce to expressions that correspond to forests).

**Example of a Poisson algebra** (i.e. an algebra over the Poisson operad):

\[f, g \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})\]

Think of elements of \(\mathbb{R}^{2n}\) as a pair \((x, p)\)

- \(x \in \mathbb{R}^n\) represents the position of a particle
- \(p \in \mathbb{R}^n\) represents the momentum of a particle

- Poisson bracket of \(f\) and \(g\) is the function on \(\mathbb{R}^{2n}\) given by

\[
\{f, g\} (x, p) = \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right)
\]

Together with pointwise multiplication \(fg\) of functions, this gives a Poisson algebra.

This works more generally for \(C^\infty(M, \mathbb{R})\), \(M\) a symplectic manifold.
Little $n$-cubes operad $\text{Cube}^n$

$\text{Cube}^n(j) = \{ \text{Reeb linear embeddings} \}
\prod_j^n \prod_j^n \prod_j^n \text{, $j$ many}
\text{cubes are contained in int } \mathbb{I}^n \text{, disjoint interiors}
\mathbb{I}^n = [0,1]

Ex: $\text{Cube}^2(3) = \begin{bmatrix} 1 & 2 \\ 3 & \end{bmatrix}$

Ex: Compositions
$\text{Cube}^2(2) \otimes \left( \text{Cube}^2(2) \otimes \text{Cube}^2(3) \right) \rightarrow \text{Cube}^2(5)$

Observation: $d$-fold loop spaces are algebras over the Little $d$-disks operad!

$\Omega^d(X) = \{ \text{basepoint preserving maps} \}
\mathbb{S}^d \rightarrow X
= \text{Maps} \left( (\mathbb{D}^d, \mathbb{S}^{d-1}), (X, \ast) \right)$

Action of $\text{Cube}^d$ is given by
$\text{Cube}^d(n) \times (\Omega^d(X)^{\times n}) \rightarrow \Omega^d X$

outside of the little cubes, the resulting map is constant at the basepoint of $X$
Recognition principle:

The converse is essentially true:

If $\text{Cube}^d \subseteq X$ and $\pi_0(X)$ (which is a monoid via $X \times X \to X$) is a group

$\Rightarrow X \cong \text{ve. } d$-fold Loop space

Coefficients always in a PID $R$

Slogan: Operad actions on spaces give operad actions on homology

**Proposition:** $\bigcirc$ operad of spaces $\xrightarrow{H_*(\mathcal{O})} H_*(\mathcal{O})$ operad of modules via

\[
H_*(\mathcal{O}(r)) \otimes H_*(\mathcal{O}(n_1)) \otimes \ldots \otimes H_*(\mathcal{O}(n_r))
\]

$\downarrow$ (operad comp.)

\[
H_*(\mathcal{O}(n_1 + \ldots + n_r))
\]

$\bigcirc$ $X$ algebra over $\mathcal{O} \xrightarrow{H_*(-)} H_*(X)$ algebra over $H_*(\mathcal{O})$

**Goal:** $H_*(\text{Cube}^d) = \text{Pois}^d$

$\Rightarrow$ homology of a $d$-fold Loop space $H_*\left(\Omega^d X\right)$ is an algebra over $\text{Pois}^d$

**Upshot:** homology of loop spaces has a rich additional structure

**Example:** Hurewicz map $\pi_n(X) \to H_n(X)$

$\xrightarrow{k\text{-fold looping}}$

$\pi_n(\Omega^k X) \to H_{n-k}(\Omega^k X)$ might give additional information
Observation: \( \text{Cube}^d(n) \cong \text{Conf}_n(\mathbb{R}^d) \)\

\[ \text{Conf}_n(X) = \{ n \text{ distinct points in a space } X \} = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j \} \]

Fibre bundle where fibres = set of possible radii

Convex \implies \text{contractible} \qed

Important example:

\[
\text{Conf}_2(\mathbb{R}^d) \cong S^{d-1} = \mathbb{P}^{d-2}
\]

\[ (v_1, m, v_2) \rightarrow \left( \frac{v_1 - m}{|v_1 - m|}, \frac{v_2 - m}{|v_2 - m|} \right) \in \left\{ (w_1, w_2) \mid w_1 = w_2, |w_1| = 1 \right\} \]

\[ \cong S^{d-1} \]

\[ \mapsto H_{\ast \ast}(\text{Conf}_2(\mathbb{R}^d)) \text{ free, rank } = 1 \text{ in dim } 0, d-1 \]

Moreover:

\[ H_{d-1}(S^{d-1}) \rightarrow H_{\ast \ast}(\text{Conf}_2(\mathbb{R}^d)) \text{ induced by } v \mapsto (v, -v) \]

[\( S^{d-1} \)] \mapsto \text{generating cycle}

Easy example: \( \text{Conf}_1(\mathbb{R}^d) \cong \mathbb{R}^d \cong \text{pt.} \)
More generally can represent homology classes via planetary systems:

\[ [x_2, x_6], [x_1, x_4, x_5], [x_4, x_5] \]

Indexed by forests:

\[ 1 \rightarrow 7 \rightarrow 3 \rightarrow 4 \rightarrow 5 \]

Bracket expression:

\[ [x_2, x_6], [x_1, x_4, x_5], [x_4, x_5] \]

For an S-Tree \( T \): (recall that \( |T| \) is the number of internal vertices of \( T \))

\[ P_T : \left( S^{d-1} \right)^{|T|} \rightarrow \text{Conf}_n(\mathbb{R}^d) \]

\( \rightarrow \) homology class

(pseudoforward of fundamental class \( [(S^{d-1})^{|TI|}] \))

\[ [T] \in H_{|TI|(d-1)} \left( \text{Conf}_n(\mathbb{R}^d) \right) \]

Similarly for a forest \( F = (T_1, \ldots, T_m) \):

The collection of the planetary systems \( P_{T_i} \) gives a submanifold of \( \text{Conf}_n(\mathbb{R}^d) \)

(and thus a homology class)

**Theorem:** The planetary system map

\[ P_{(-)} : \text{Pos}^d(n) \rightarrow H_* \left( \text{Conf}_n(\mathbb{R}^d) \right) = H_* \left( \text{Cube}^d(n) \right) \]

is an isomorphism of operads.
Rough outline of the proof:

1. Show that the map is well-defined
   (i.e. on the left side we introduced anti-symmetry relations, Jacoby, commutativity
   and need to show that those hold for the homology
classes of configuration spaces)

2. Use a perfect pairing between Pois\(d(n)\) and a dual complex \(\text{Siop}^d(n)\)
to conclude injectivity

3. Estimates of the rank of \(H_\ast(\text{Conf}_n(\mathbb{R}^d))\)
to get surjectivity

\(\sim \) \(\mathcal{P}_\ast\) is additive isomorphism

4. Operad structure: need compatibility of geometric insertion str. of the little disks
   & algebraic insertion str. of Poisson operad
Appendix: Homology of $\text{Conf}_n(\mathbb{R}^d)$

**Lemma:** Have fibre bundles

\[
\begin{align*}
\mathbb{R}^d \setminus \{n-1 \text{ points}\} & \xrightarrow{\sim} \bigvee_{n-1} S^{d-1} \\
\text{Conf}_n(\mathbb{R}^d) & \xrightarrow{p_i = \text{forget } i^{th} \text{ point}} \bigvee_{n-1} \text{Conf}_{n-1}(\mathbb{R}^d) \xrightarrow{(x_1, \ldots, x_n)} (x_1, \ldots, x_i, \ldots, x_n)
\end{align*}
\]

Assemble these into Fadell-Neuwirth tower:

\[
\begin{align*}
\bigvee_{n-1} S^{d-1} & \longrightarrow \text{Conf}_n(\mathbb{R}^d) \\
\bigvee_{n-2} S^{d-1} & \longrightarrow \text{Conf}_{n-1}(\mathbb{R}^d) \\
\bigvee_{n-3} S^{d-1} & \longrightarrow \text{Conf}_{n-2}(\mathbb{R}^d) \\
\bigvee_{n-4} S^{d-1} & \longrightarrow \text{Conf}_{n-3}(\mathbb{R}^d)
\end{align*}
\]

Split (a section adds a new $n^{th}$ point "for away" or "deletes" a point)

Less of $x_n$ for the fib. + Hurewicz

**Prop:** The first non-trivial homology group

\[H_{d-1}(\text{Conf}_n(\mathbb{R}^d))\] is free of rank \(n\).