Surgery on a framed link to describe a 3-dimensional manifold:

Analogy: Handle decomposition of a (compact, connected) 2-manifold.

Start with 0-handle $D^2$ and attach 1-handles $S^1 \times D^1$ along $S^1 \times \partial D^1$ framing $\phi$.

Result of attaching:

Handle slide

Only looking at attaching regions in the boundary of the 0-handle.

"Feet of the handle"

Think: The blackboard is $S^3$ and we are looking at the "feet" of the handles.

Example:

![Handle diagrams]

Think: [Lickorish] Any closed, connected, orientable 3-manifold $M^3$ bounds some $D^4 \cup_{q_1, \ldots, q_n} (k \text{ many } 2\text{-handles})$.

Different perspective: $M^3$ can be obtained from $S^3$ by a collection of 1-surgeries, that is, removing disjoint copies of $S^1 \times D^2$ and replacing by copies of $D^2 \times S^1$.

This surgery description of a 4-manifold is not unique!
1) Kirby type-I move:

\[ \begin{array}{c}
\text{Corresponds to} \\
W^4 \\ \Leftrightarrow \\ W^4 \neq \pm \Omega \mathbb{P}^2 \\
\text{closed}
\end{array} \]

2) Kirby type-II move:

Ex:

\begin{align*}
\text{(in general blue component can be located and linked)} \\
\text{sliding one 2-handle over another in } W^4
\end{align*}

\[ \begin{array}{c}
\text{corresponds to} \\
\text{regular isotopy yields a 3-manifold invariant.}
\end{array} \]

Fact: Type-I & Type-II do not change \( \exists W^4 = \mathbb{M}^3 \) and iterations of these two types of moves relate any two framed links representing the same 3-manifold

Ex:

- Linking matrix of a framed link \( L \) (entries are linking numbers between pairs of components of the link, diagonal entries are framing coefficients)
- A presentation matrix for \( H_4(M_L) \)
- Notation for result of surgery on \( L \)

Fact: Can "upgrade" Jones polynomial to a 3-mfld. invariant:

Take linear sums of the Jones polynomials of copies of the link where components are replaced by various parallels of the original components, and evaluate at a complex root of unity

\[ \text{\( \Leftrightarrow \) Witten's quantum } \text{SU}_q(2) \text{ invariants of 3-manifolds} \]