From knots to quantum $SU_q(2)$-invariants of 3-manifolds

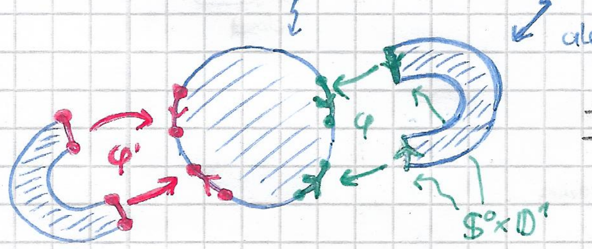
8.10.18

10 min talk

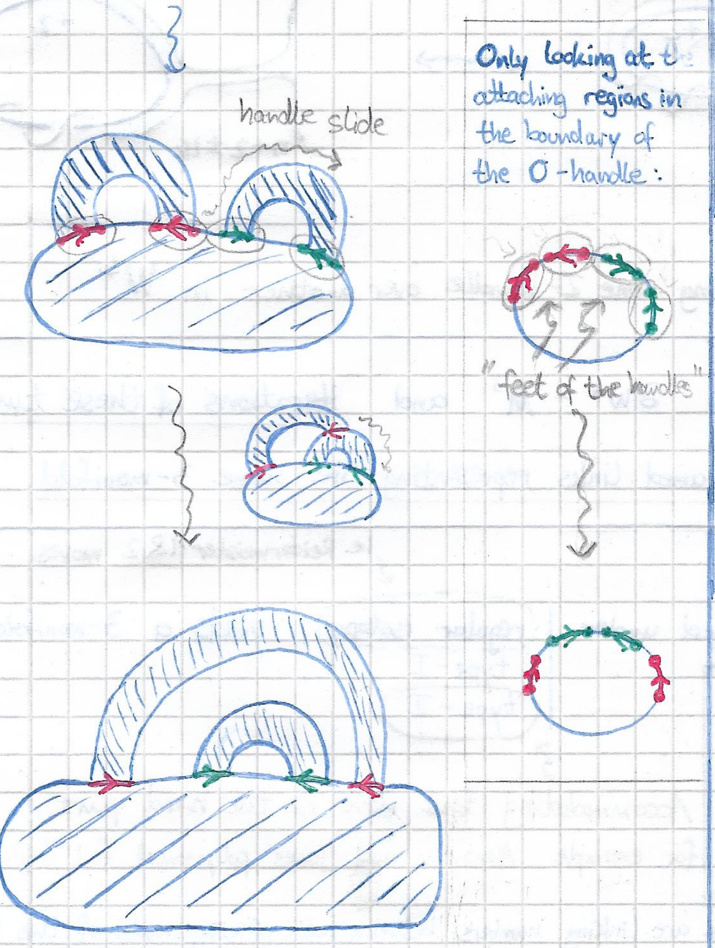
Surgery on a framed link to describe a 3-dimensional manifold:

Analogy: Handle decomposition of a (compact, connected) 2-mfld.:

Start with 0-handle $\mathbb{D}^2$ and attach 1-handles $\mathbb{D}^1 \times \mathbb{D}^1$ along $\mathbb{S}^0 \times \mathbb{D}^1 \xrightarrow{\varphi} \partial\mathbb{D}^2$

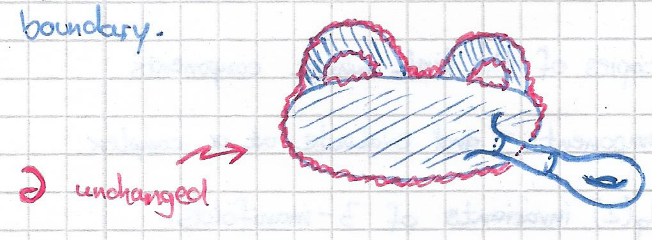


Result of attaching:



Observe:

- Handle sliding does not change our manifold and its boundary.
- Connected sum with a closed manifold does not change the boundary.



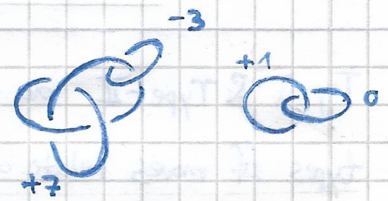
Our 3-manifolds will arise as the boundary of 4-manifolds of the form

$$\mathbb{D}^4 \cup (\text{4-dim. } \underbrace{\text{2-handles}}_{\mathbb{D}^2 \times \mathbb{D}^2 \text{ attached along } \mathbb{S}^1 \times \mathbb{D}^2 \xrightarrow{\varphi} \partial\mathbb{D}^4 = \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}})$$

framing corresponds to an integer on each link component

Think: The blackboard is $\mathbb{S}^3$ and we are looking at the "feet" of the handles.

Ex.:

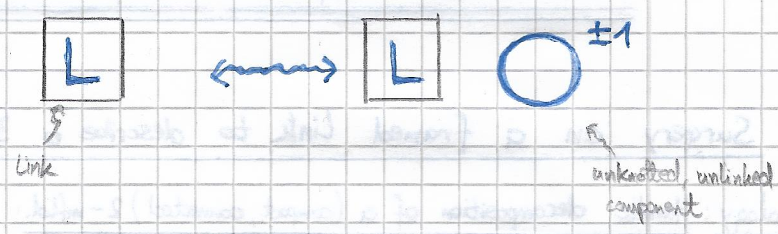


> **Thm.: [Lickorish]** Any closed, connected, orientable 3-manifold M^3 bounds some $\mathbb{D}^4 \cup_{\varphi_1, \dots, \varphi_k} (k \text{ many 2-handles})$

Different perspective: M^3 can be obtained from $\mathbb{S}^3$ by a collection of 1-surgeries, that is removing disjoint copies of $\mathbb{S}^1 \times \mathbb{D}^2$ and replacing by copies of $\mathbb{D}^2 \times \mathbb{S}^1$.

This surgery description of a 4-manifold is not unique!

•) Kirby type-I-move:

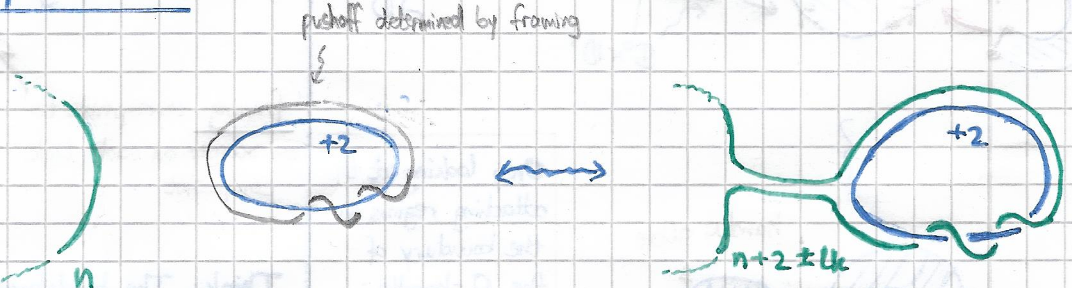


Corresponds to $W^4 \rightsquigarrow W^4 \# \underbrace{\pm \mathbb{CP}^2}_{\text{closed}}$

•) Kirby type-II-move:

Ex.:

(In general blue component can be knotted and linked)



corresponds to sliding one 2-handle over another in W^4

Fact: Type-I & Type-II do not change $\partial W^4 = M^3$ and iterations of these two types of moves relate any two framed links representing the same 3-manifold.

Upshot: Any link invariant unchanged under
- regular isotopy (ie Reidermeister 1&2 moves)
- type-I
- type-II

yields a 3-manifold invariant!

Accommodating this move is the hard part. for example Alexander and Jones polynomial fail here

$$\begin{tikzcd}
\{\text{framed links}\}/_{\text{reg. isotopy}} \arrow[rr, "\text{surgery}"] \arrow[dr] \arrow[dd, "\text{we know many link invariants}"'] & & \{\text{c.c.o 3-mflds.}\} \\
& \{\text{framed links}\}/_{\text{reg. isotopy, I, II}} \arrow[ur, "\cong"] \arrow[dl, dashed, "\text{want this}"] & \\
A & &
\end{tikzcd}$$

A ← something we understand, e.g. a polynomial ring

Ex: •) Linking matrix of a framed link L (entries are linking numbers between pairs of components of the link, diagonal entries are framing coefficients)

$\rightsquigarrow$ presentation matrix for $H_1(M_L)$

M_L: notation for result of surgery on L

•) Can "upgrade" Jones polynomial to a 3-mfld. invariant:

Take linear sums of the Jones polynomials of copies of the link where components are replaced by various parallels of the original components, and evaluate at a complex root of unity

↗ Witten's quantum $SU_q(2)$ invariants of 3-manifolds