Lyndon-Hochschild-Serre Spectral Sequence - Application

Benjamin Ruppik, s6berupp@uni-bonn.de

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Exercise. *[Wei95, 6.8.5] Let $n \geq 3$ be odd. Calculate the group homology with trivial $\mathbb{Z}$-coefficients of the dihedral group*

$$\begin{aligned} D_n &= \langle s, d \,|\, s^2, d^n, sd = d^{-1}s \rangle \\ &\cong C_n \rtimes C_2 \end{aligned}$$

How-to guide:

We use the (split) short exact sequence[1]

$$\{1\} \to C_n \to D_n \to C_2 \to \{1\}$$

The homological Lyndon-Hochschild-Serre Spectral Sequence takes the form

$$E^2_{p,q} \cong H_p(C_2; H_q(C_n; \mathbb{Z})) \Rightarrow H_*(D_n; \mathbb{Z})$$

The extension is not central (why?), so we really have to be careful to think about the action of C_2 on the homology groups $H_q(C_n; \mathbb{Z})$ later.

Recall: Homology of cyclic groups

The homology of cyclic groups with **trivial** $\mathbb{Z}$-coefficients is

$$H_q(C_m; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}/m & \text{for } q = 1, 3, 5, \ldots \\ 0 & \text{for } q = 2, 4, 6, \ldots \end{cases}$$

Below we will also need the more general statement[2] for coefficients in a C_m-module A:

$$H_q(C_m; A) \cong \begin{cases} A/(\sigma - 1)A & \text{for } q = 0 \\ A^{C_m}/NA & \text{for } q = 1, 3, 5, \ldots \\ \{a \in A \,|\, Na = 0\}/(\sigma - 1)A & \text{for } q = 2, 4, 6, \ldots \end{cases}$$

[1] In the following we will denote by C_m the (multiplicatively) written cyclic group $\langle \sigma \,|\, \sigma^m \rangle$ with m elements. We also use $\mathbb{Z}/m$ if we think of it as a $\mathbb{Z}$-module, of course switching to additive notation in this case.

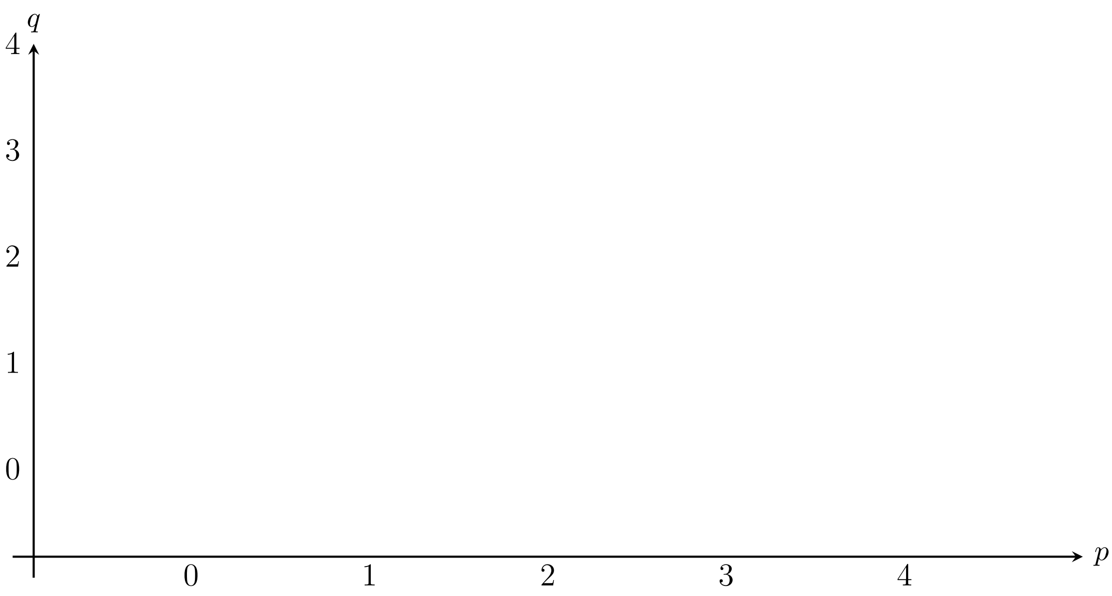
[2] Notation: $N = 1 + \sigma + \sigma^2 + \ldots + \sigma^{m-1}$ is the norm element in $\mathbb{Z}[C_m]$

Calculating the E^2-page

The lower left corner of the first quadrant[3] looks like this:

$$
\begin{tikzcd}[column sep=small]
q & & & & & \\
4 & H_0(C_2;H_4(C_n;\mathbb{Z})) & H_1(C_2;H_4(C_n;\mathbb{Z})) & H_2(C_2;H_4(C_n;\mathbb{Z})) & H_3(C_2;H_4(C_n;\mathbb{Z})) & \\
3 & H_0(C_2;H_3(C_n;\mathbb{Z})) & H_1(C_2;H_3(C_n;\mathbb{Z})) & H_2(C_2;H_3(C_n;\mathbb{Z})) & H_3(C_2;H_3(C_n;\mathbb{Z})) & \\
2 & H_0(C_2;H_2(C_n;\mathbb{Z})) & H_1(C_2;H_2(C_n;\mathbb{Z})) & H_2(C_2;H_2(C_n;\mathbb{Z})) & H_3(C_2;H_2(C_n;\mathbb{Z})) & \\
1 & H_0(C_2;H_1(C_n;\mathbb{Z})) & H_1(C_2;H_1(C_n;\mathbb{Z})) & H_2(C_2;H_1(C_n;\mathbb{Z})) & H_3(C_2;H_1(C_n;\mathbb{Z})) \arrow[ull, "d^2"'] \arrow[uulll, "d^3"] & \\
0 & H_0(C_2;H_0(C_n;\mathbb{Z})) & H_1(C_2;H_0(C_n;\mathbb{Z})) & H_2(C_2;H_0(C_n;\mathbb{Z})) & H_3(C_2;H_0(C_n;\mathbb{Z})) & \\
 & 0 & 1 & 2 & 3 & p
\end{tikzcd}
$$

- Try to write down as many of the E^2 terms as you can figure out!
- Then take a look at the information about the C_2 action on the homology of C_n given after the figure to fill in the rest.[4]



[3]Example d^2 and d^3 differentials are drawn to remind you of the direction of the maps in a homological spectral sequence (but you will see in a minute that for our case all the differentials are the zero map!). Also remember that d_3 is only well-defined on the E^3-page, so a priori not on the terms written in the diagram here but only after taking homology once.

[4]Remember that n is odd - where do we use this?

Local coefficients – C_2 acts on $H_q(C_n;\mathbb{Z})$

Using explicit calculations in a resolution one can show:

The C_2-action on $H_{2i-1}(C_m;\mathbb{Z})$ is multiplication by $(-1)^i$.

Take this on faith for now![5]

- For example, you will realize (why?) that on the vertical q-axis we have the co-invariants
$$E^2_{0,q} \cong H_q(C_n;\mathbb{Z})_{C_2}$$
Check that these are given by
$$H_q(C_n;\mathbb{Z})_{C_2} \cong \begin{cases} \mathbb{Z} & \text{for } q = 0 \\ \mathbb{Z}/n & \text{for } q \equiv 3 \pmod 4 \\ 0 & \text{else} \end{cases}$$
- On the p-axis we have $\mathbb{Z}$, $\mathbb{Z}/2$ and 0 (topological interpretation?)
- All terms away from the axes are zero.

Almost done

- Now argue that there are no nontrivial differentials: The spectral sequence collapses and $E^2 \cong E^\infty$.
- Can you solve the extension problems?

In the end we get the following

Proposition (Homology of the odd dihedral groups)**.** *The homology of the dihedral group D_n for $n \geq 3$ odd and trivial $\mathbb{Z}$-coefficients is given by*
$$H_*(D_n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } * = 0 \\ \mathbb{Z}/2 & \text{for } * \equiv 1 \pmod 4 \\ \mathbb{Z}/2n & \text{for } * \equiv 3 \pmod 4 \\ 0 & \text{else} \end{cases}$$

References

[Wei95] Charles A Weibel. *An introduction to homological algebra.* Number 38. Cambridge University Press, 1995.

[5] See [Wei95, Example 6.7.10] for the details