§5.1 Kirby Calculus

Plan: 1) Handle cancellation/sticking in 2 and 3-dim.
2) How does this look in Kirby diagrams?
3) Different View on A-handles
4) Examples along the way

Concrete pictures in 2/3-dim. (abstract description later)

Cancellation:
2-dim:

3-dim:

Prop 4.2.9: A \((k-1)\)-handle \(h_{k-1}\) and a \(k\)-handle \(h_k\) \((1 \leq k \leq n)\)
can be cancelled, provided that the attaching sphere of \(h_k\) intersects the belt sphere of
\(h_{k-1}\) transversely in a single point. (regardless of framing)

Handle slides:

Careful with framings! Later

Def 4.2.10: Given two \(k\)-handles \(h_k\) and \(h_{k-1}\) \((0 \leq k \leq n)\) attached to \(\mathbb{S}X\), a handle slide of \(h_k\) over \(h_{k-1}\) is given as follows: Isotope the attaching sphere \(A\) of \(h_k\)
in \(\mathbb{S}(X \cup h_{k-1})\), pushing it through the belt sphere \(B\) of \(h_{k-1}\).
Intermediate stage: spheres will intersect in one point \(p\)

\[ \text{SLIDE} \]
Prop. 4.2.7: handles can be attached in order of increasing index -
dimension-counting and transversality argument: "Miss the last gate, miss it all."
\( \Rightarrow \) can push \( A \) off of \( B \) in two possible directions

- one direction: original picture
- other direction: result of the handle slide.

This is a complete set of moves:

**Thm. 4.2.12:** Given any two (relative) handle decompositions (ordered by increasing index) for a compact \( X \), it is possible to get from one to the other by a sequence of

1. handle slides
2. creating/annihilating canceling handle pairs
   "handle birth"
3. isotopies within levels

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Ex. of 1-handle slide in 2-dim:

**Handle slide changed:**
1. the attaching sphere of \( h_\alpha \) (untangle it from that of \( h_\beta \))
2. the framing (see element of \( \pi_0 \Gamma(1) \cong \mathbb{Z}^2 \))

1-handle slides in Kirby diagrams:

- take attaching ball of \( h_\alpha \)
- push it through the 1-handle \( h_\beta \)
- keep track of framings (e.g., by using double-strand notation)

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**Easier later when we change notation from**

- 0-handle
- 1-handle

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Ex. of 2-handle slide in 3-clim:

Core circles:

\[ \text{Attaching circles:} \]

2-handle slides in Kirby diagrams:

- 2-handles \( h^2_\alpha, h^2_\beta \) attached along
- Framed knots \( K_\alpha, K_\beta \)
- Parallel curve \( K'_\beta \) determining the framing on \( K_\beta \) bounding a disk.
- Parallel copy of the core of \( h^2_\beta \) lies in the upper boundary of the "rubber" handle \( h^2_\beta \).

Slide \( h^2_\alpha \) by isotoping \( K_\alpha \) over one such disk.

Form a band-sum of \( K_\alpha \) and \( K'_\beta \)

connected sum along some band

Ex. I:

Ex. II: (one possible alternative):

Change of links in \( H^2 \):

\[ h^2_\alpha \mapsto h^2_\alpha + h^2_\beta \quad \text{("handle addition")} \]

\[ h^2_\alpha \mapsto h^2_\alpha - h^2_\beta \quad \text{("handle subtraction")} \]
Aside: Homology from handles

Chain Complex: \[ C_k = \mathbb{Z} [k\text{-}handles \ h^k_\alpha] \]

boundary maps: \[ \partial_k : C_k \to C_{k-1} \text{ given by} \]

\[ \partial_k (h^k_\alpha) = \sum \left< h^k_\alpha \mid h^{k-1}_\beta \right> h^{k-1}_\beta = \text{incidence number of } h^k_\alpha \text{ with } h^{k-1}_\beta \]

\[ = \text{intersection number of the attaching sphere of } h^k_\alpha \text{ with the } (k-1)\text{ sphere of } h^{k-1}_\beta \]

Example: \[ \partial_1 h^k_\alpha = (+1) \cdot h^{k-1}_\beta \]

Algebraic effect of sliding: Sliding \( h^k_\alpha \) over \( h^k_\beta \) modifies \( \partial_k \) the same way

as would changing the basis of \( C_k \) by replacing \( h^k_\alpha \) by \( h^k_\alpha \pm h^k_\beta \)

For determining the framing on the slid handle:

If 2-handles are attached to \( h^2 \):

framing (new \( h_\alpha \)) = \((\alpha \pm \beta) \cdot (\alpha \pm \beta) = \alpha \cdot \alpha + \beta \cdot \beta \pm 2 \cdot \alpha \cdot \beta \)

= framing (\( h_\alpha \)) + framing (\( h_\beta \)) \pm 2 \cdot \text{linking (link \( h_\alpha \) and \( h_\beta \))}

\[ = \text{framing (new \( h_\alpha \)) + framing (old \( h_\beta \)) \pm 2 \cdot \text{linking (link \( h_\alpha \) and \( h_\beta \))} \]

Alternative notation:

Double strand notation:

i.e. isotope both strands \()\) over parallel disks in \( \partial h^2 \) by making two parallel band sums.
Fun example: $\mathbb{C}P^2 \# S^2 \times S^2 \cong \mathbb{C}P^2 \# S^2 \times S^2$

**Proof:**

$s^2 \times s^2$

\[\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}\]

\[\begin{array}{c}
\mathbb{C}P^2 \\
\circ \\
\circ \\
\end{array}\]

\[\begin{array}{c}
\text{slide} \\
\uparrow \\
\text{isotopy}
\end{array}\]

In general, Lemma: If $M^k$ has odd intersection form, then there is a diffeomorphism that is closed, simply connected, oriented, smooth.

$M \# S^2 \times S^2 \cong M \# S^2 \times S^2$

**Example:** Analogous to the 2-dim. case $S^1 \times S^1 \cong \mathbb{R}P^2 \# \mathbb{R}P^2$ we have

$S^2 \times S^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

Algebraically: Intersection matrix

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]

\[-5\]
**Example:** $S^1$-bundle over $S^2$

\[ X = \mathbb{D}^2 \text{bundle over } S^2 \text{ with Euler number } n \]

Adding 0-framed meridian gives
\[ DX = X \cup_{d \ast} X \]
(double)

$S^2$-bundle over $S^2$

\[ n - 2 \]

recovery Hopf-link with $n$ reduced by 2

**Interaction matrix:**

\[ \mathbb{P} \begin{pmatrix} n & 1 \\ 1 & \sigma \end{pmatrix} \rightarrow \begin{pmatrix} n - 1 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} n - 2 & 1 \\ 1 & \sigma \end{pmatrix} \]

**Corollary:** Differentiation type of $D \left( S^2 \text{-bundle with Euler } n \text{ over } S^1 \right)$ depends only on $n$ modulo 2

**Handle cancellation:**

Diagrams for a pair of cancelling 1- and 2-handles:

**Formally:** A $(k-1)$-handle and a k-handle can be cancelled if the attaching sphere of $h^{k-1}$ intersects the last sphere of $h^k$ transversely in a unique point (regardless of framing).

**Note:** Unique isotopy class of framed unknot.

Move complicated: If there are other 2-handles running over the 1-handle, slide the extra handles over the 2-handle that we wish to cancel. This removes them from the 1-handle.

(then untangle and erase the cancelling pair)

A handle we want to cancel

other 2-handles running over the 1-handle
Model for a cancelling 2-handle/3-handle pair:

Fact: Any cancelling 2-3 pair can be made to look like this.

Alternative way of attaching a 1-handle:

In 2-dim:

In 3-dim:

"Adding a 1-handle is the same as removing the 2-handle that cancels it."

In 4-dim: Sphere:

Alternative diagram for attaching an (orientation-preserving) 1-handle:

"Dotted circle notation", introduced by [Alkalai]

New diagram for a pair of cancelling 1- and 2-handles:
Attaching a 1-handle in 3-foo:

\[ S^2 = \partial D^3 \]

- Remove tubular neighborhood \( D^1 \times D^2 \) of core:
  - Circles running over the 1-handle → now contained in \( S^2 \)
  - Circles going under the 1-handle → now split in \( S^2 \setminus \text{attaching region} \) \( S^2 \times D^1 \)

- 2-handle running over 1-handle → comes fully into \( S^3 \)
- Part of surface running between spheres (under the 1-handle) → moves into \( \text{int}(D^3) \)