First Lie Theorem

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1 Motivation – From local to global

Intuitively the tangent space $T_{\mathbb{1}}(G)$ of a Lie group G at the identity consits of the infinitesimally small elements of the group, those elements which are just a slight nudge away from the neutral element. Formally one can see this space as the velocity vectors of curves through the identity, where two velocity vectors are identified if they give the same derivations on smooth real valued functions on the manifold. The tangent space is not only a finite dimensional vector space with addition and scalar multiplication, it also carries the structure of a Lie algebra with a Lie bracket $[\_,\_]$ containing information about the noncommutativity of G.

More formally we have a functor

$$\text{Lie}\colon (\text{LieGrp}) \to (\text{LieAlg}) \tag{1}$$

$$\text{on objects: } G \mapsto T_{\mathbb{1}}G = \text{Lie}(G) \tag{2}$$

$$\text{on morphisms: } (G \xrightarrow{F} H) \mapsto (T_{\mathbb{1}}G \xrightarrow{T_{\mathbb{1}}F} T_{\mathbb{1}}H) \tag{3}$$

where Lie(G) is equipped with the Lie bracket

$$[X,Y]_{\mathbb{1}} \colon \mathcal{O}^{\infty}_{\mathbb{1}} \to \mathbb{R} \tag{4}$$
$$\psi \mapsto X_{\mathbb{1}}(Y(\psi)) - Y_{\mathbb{1}}(X(\psi)) \tag{5}$$
$$\tag{6}$$

The derivative of the smooth group homomorphism F is a Lie algebra homomorphism, i.e. it is linear and it preserves the Lie bracket $T_{\mathbb{1}}F[X,Y]_G = [T_{\mathbb{1}}F(X), T_{\mathbb{1}}F(Y)]_H$.

What about the other direction? Given two Lie groups G, H and a Lie algebra homomorphism between their tangent spaces at the identity, $T_{\mathbb{1}}G \xrightarrow{\phi} T_{\mathbb{1}}H$ (i.e. ϕ is linear and $\phi([X,Y]) = [\phi(X), \phi(Y)]$), does this lift to a smooth group homomorphism

$$G \xrightarrow{\Phi} H \text{ such that } T_{\mathbb{1}}\Phi = \phi?$$

From the examples $SO(3)$ and $\mathbb{S}^S$ with the same local, but very different global structure, we know that this cannot be true without further assumptions on the global topology of the groups. The key property we need is that paths in the source are unique up to homotopy: The answer to the lifting question above is **yes** if we know that G is **simply connected**! (We even know that the lift is unique in this case). This is precisely the statement of the First Lie Theorem:

Theorem 1 (First Lie Theorem, [Stillwell, 2008, 9.6])**.** *Let G, H be Lie groups with Lie algebras $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{h} = \operatorname{Lie} H$ such that G is simply connected. If $\phi \colon \mathfrak{g} \to \mathfrak{h}$ is a morphism of Lie algebras, then there is a unique morphism $\Phi \colon G \to H$ of Lie groups lifting ϕ, i.e. such that $T_{\mathbb{1}}\Phi = \phi$.*

To prove this we will proceed in two steps:

1. Find a lift locally around $\mathbb{1} \in G$ (here we don't need that G is simply connected)
2. Use 1-connectedness of G to extend the local map to a global map

A small outlook on the other Lie theorems: The question remains whether for any Lie algebra there is a Lie group with exactly this algebra as its tangent space. The answer is positive:

Theorem 2 (Cartan-Lie-Theorem)**.** *Every finite dimensional Lie algebra comes from a (simply connected) Lie group.*

This shows that the functor

$$(\text{LieGrp}_{\text{simply conn.}}) \xrightarrow{\text{Lie}} (\text{LieAlg}_{\text{finite dim.}}) \tag{7}$$

is fully faithful (First Lie theorem) and essentially surjective (Cartan-Lie), so Lie gives an equivalence of categories.

2 Producing a local map

Theorem 3 (First Lie Theorem, local version [Tits, 1983, III.4.2, Satz 1])**.** *Let G, H be analytical Lie groups with Lie algebras $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{h} = \operatorname{Lie} H$ (no simple connectedness assumption!), $\phi\colon \mathfrak{g} \to \mathfrak{h}$ a morphism between their Lie algebras.*

- *Then there is a local (analytical) homomorphism Φ from G to H such that $T_{\mathbb{1}}\Phi = \phi$. More precisely: There is an open neighborhood U of $\mathbb{1}$ in G and an analytical map $\Phi\colon U \to H$ with the following properties:*
  - *For all $x, y, z \in U$ with $xy = z$ we have $\Phi(xy) = \Phi(x)\Phi(y)$.*
  - *$T_{\mathbb{1}}\Phi = \phi\colon \mathfrak{g} \to \mathfrak{h}$*
- *The local homomorphism is locally unique, i.e. if we have U_1, Φ_1 and U_2, Φ_2 as above then there is a neighborhood W of $\mathbb{1}$ in G where Φ_1 and Φ_2 agree.*
- *If ϕ is an isomorphism, then Φ is a local isomorphism from G to H, i.e. there is a $W \subset G$ such that the restriction of Φ to W gives an analytical isomorphism onto the image $\Phi(W)$.*

Remark 1. *From the second assertion we get: A local isomorphism $\Phi\colon G \to G$ with $T_{\mathbb{1}}\Phi = \mathrm{id}_{\mathfrak{g}}$ is locally equal to the identity.*

2.1 The Campbell-Hausdorff-Baker-Dynkin formula

We need to take a small excursion into algebra to derive an important formula connected to the exponential mapping in a Lie group.

Definition 1. *Let $\mathbb{K} \in \mathbb{R}, \mathbb{C}$ be the field of real or complex numbers (some of this works more generally, but we will not need this). Let $\mathbb{K}\langle\langle X, Y\rangle\rangle$ be the ring of formal power series in the* ***non****commuting variables X, Y. The elements are thus infinite sums of the form*

$$\sum_{i=0,j=0}^{\infty} a_{ij} X^i Y^j + b_{ij} Y^i X^j$$

with the usual degreewise addition and multiplication defined via the Cauchy-product of series (like one would expect). In $\mathbb{K}\langle\langle Z\rangle\rangle$ we have the special power series

$$\exp(Z) \coloneqq 1 + Z + \frac{Z^2}{2!} + \frac{Z^3}{3!} + \ldots + \frac{Z^n}{n!} + \ldots \tag{8}$$

$$\log(1+Z) \coloneqq Z - \frac{Z^2}{2} + \frac{Z^3}{3} \mp \ldots + (-1)^{n-1}\frac{Z^n}{n} + \ldots \tag{9}$$

One can define a notion of convergence in these power series rings and then $\exp, \log$ *are convergent (but we will not go there).*

Definition 2. *Let A be an algebra. The map*

$$[\_,\_]\colon A \times A \to A \tag{10}$$

$$[a,b] \coloneqq ab - ba \tag{11}$$

is called the commutator in A.

Theorem 4 (Algebraic Campbell-Hausdorff [Tits, 1983, III.3.4.2, Satz 3])**.** *The algebraic product in logarithmic coordinates is completely determined by the commutator in the algebra $\mathbb{K}\langle\langle X,Y\rangle\rangle$, i.e. there is a formula of the form*

$$\log(\exp(X)\cdot\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[[X,Y],Y] + \frac{1}{12}[[Y,X],X] + \dots \tag{12}$$

where all the following terms are iterated commutators of X and Y. We will write

$$\log(\exp(X)\cdot\exp(Y)) = \sum_{n=1}^{\infty} h_n(X,Y) \tag{13}$$

for this and call it the Campbell-Hausdorff-Series. $h_n(X,Y)$ is the sum of the homogenous terms of degree n. Explicitly the sum is given by the formula

$$\log(\exp(X)\cdot\exp(Y)) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum \frac{1}{\sum_{i=1}^{m}} \frac{\sigma(X^{p_1}Y^{q_1}X^{p_2}Y^{q_2}\dots X^{p_m}Y^{q_m})}{p_1!q_1!p_2!q_2!\dots p_m!q_m!} \tag{14}$$

The second summation is over $p_i, q_i \in \mathbb{N}, i = 1,2,\dots,m$ where $p_i + q_i > 0$, and σ is defined as

$$\sigma(X^{p_1}Y^{q_1}X^{p_2}Y^{q_2}\dots X^{p_m}Y^{q_m}) \tag{15}$$

$$\coloneqq [\underbrace{\dots[X,X],\dots],X}_{p_1\text{-times}}],\underbrace{Y],\dots],Y}_{q_1\text{-times}}],X],\dots],Y],\underbrace{X],\dots],X}_{p_m\text{-times}}],\underbrace{Y],\dots],Y}_{q_m\text{-times}}]. \tag{16}$$

2.2 The group product in logarithmic coordinates

$\mathbb{K}\langle\langle X,Y\rangle\rangle$ satisfies the universal property that for a finite dimensional Lie algebra A with commutator $[\_,\_]_A$ (bilinear, alternating, satisfying Jacobi identity) we can plug in elements of A into the variables X and Y. For $a,b \in A$ let $h_n(a,b)$ be the element in A if we replace X by a, Y by b and the commutator in $\mathbb{K}\langle\langle X,Y\rangle\rangle$ by the commutator $[\_,\_]_A$ in A. Via this mechanical process we get a series $\sum h_n(a,b)$ and we now want to look at its convergence properties. A is a finite dimensional vector space with norm $\|.\| \to \mathbb{R}$, since all norms on A are equivalent it does not matter which one we choose. The bilinear commutator $[\_,\_]_A$ is continuous with respect to the norm topologies on $A \times A$ and A, so there is a constant $C > 0$ such that $\|[a,b]\| \leq C\|a\|\|b\|$ for all $a,b \in A$. Let us define a modified norm $|.|$ by setting $|a| \coloneqq \frac{1}{C}\|a\|$, with this new norm $|[a,b]| \leq |a||b|$ (at least if $C \geq 1$).

Lemma 1 ([Tits, 1983, III.3.5, Lemma 1])**.** *Let $U = \{a \in A : |a| < \frac{1}{2}\ln(2)\}$, then for all $a,b \in U$ the series $\sum_{n=1}^{\infty} h_n(a,b)$ is absolutely convergent. Its sum $a \circ b \coloneqq \sum_{n=1}^{\infty} h_n(a,b)$ defines an analytical mapping $\circ\colon U \times U \to A$.*

From the algebraic Campbell-Hausdorff-formula $\sum_{n=1}^{\infty} h_n(X,Y) = \log(\exp(X)\cdot\exp(Y))$ we can conclude that on the level of formal power series

$$\exp(\sum_{n=1}^{\infty} h_n(X,Y)) = \exp(X)\cdot\exp(Y). \tag{17}$$

We now take the Lie algebra $\mathrm{Lie}(G)$ with the Lie bracket coming from the group structure on G as our finite dimensional algebra A. For $u, v \in \mathrm{Lie}(G)$ where $u \circ v := \sum_{n=1}^{\infty} h_n(u,v)$ is defined, the formal equation 17 transforms into

$$\exp(u \circ v) = \exp(u)\cdot\exp(v) \tag{18}$$

where we interpret exp as the exponential mapping $\exp\colon \mathrm{Lie}(G) \to G$ of the Lie group and the product on the right side as the product in G. The geometric version of the Campbell-Hausdorff-formula asserts that this equation is true indeed for all u, v in a small enough neighborhood of $0 \in \mathrm{Lie}(G)$.

Theorem 5 (Geometric Campbell-Hausdorff [Tits, 1983, III.3.5, Satz])**.** *Let G be an analytical Lie group, $\mathrm{Lie}(G)$ its Lie algebra and $\exp\colon \mathrm{Lie}(G) \to G$ its exponential mapping. There is a neighborhood U of 0 in $\mathrm{Lie}(G)$ with the following property: For all $u, v \in U$ the Campbell-Hausdorff-series $u \circ v = \sum_{n=1}^{\infty} h_n(u,v)$ converges absolutely and for its sum the following equation holds:*

$$\exp(u \circ v) = \exp(u)\cdot\exp(v). \tag{19}$$

In particular the group product in U is completely determined by the commutator in the Lie algebra of G.

2.3 Proof of the local First Lie theorem

Take a open neighborhoods $\mathbb{1} \in U \subset G$, $0 \in U' \subset \mathfrak{g}$ such that exp gives a diffeomorphism $\exp\colon U' \xrightarrow{\cong} U$. By Campbell-Baker-Hausdorff (the geometric version) we know that the product of $e^X, e^Y \in U$ is given by

$$e^X \cdot e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\text{further Lie bracket terms}}. \tag{20}$$

Define

$$\Phi\colon U \to H \tag{21}$$

$$\Phi(e^X) := e^{\phi(X)} \tag{22}$$

Now observe that Φ is a homomorphism:

$$\Phi(e^X \cdot e^Y) \overset{\text{CBH}}{=} \Phi(e^{X+Y+\frac{1}{2}[X,Y]+\text{further Lie bracket terms in } X,Y}) \tag{23}$$

$$\overset{\text{def}}{=} e^{\phi(X+Y+\frac{1}{2}[X,Y]+\text{further Lie bracket terms in } X,Y)} \tag{24}$$

$$\overset{\phi \text{ Lie alg. hom.}}{=} e^{\phi(X)+\phi(Y)+\frac{1}{2}[\phi(X),\phi(Y)]+\text{further Lie bracket terms in } \phi(X),\phi(Y)} \tag{25}$$

$$\overset{\text{CBH}}{=} e^{\phi(X)} \cdot e^{\phi(Y)} \tag{26}$$

$$\overset{\text{def}}{=} \Phi(e^X)\cdot\Phi(e^Y). \tag{27}$$

So Φ is a Lie algebra homomorphism on the region U where every element is of the form e^X. Recall that a continuous homomorphism between (local) Lie groups is already smooth, so Φ is a homomorphism of Lie groups. The smooth map Φ really induces ϕ because the exponential mapping of a Lie group is natural. This concludes the proof of the local version of the First Lie Theorem.

3 Inducing a global map in the simply connected case

3.1 Proof sketch of the global First Lie Theorem

Problem: Not every element in G is necessarily of the form e^X. (We know the exponential maps is surjective in some special situations, e.g. if G is compact and connected, but not in general for every Lie group)
We need to extend Φ to an arbitrary $g \in G$ by some other means, and this can be done in a four-stage process [Stillwell, 2008, 9.6]:

1. Connect g to $\mathbb{1}$ by a path in G, show that there is a sequence of points
$$\mathbb{1} = g_1, g_2, \ldots, g_m = g \tag{28}$$
along the path such that
$$g_1, g_1^{-1}g_2, \ldots, g_{m-1}^{-1}g_m \tag{29}$$
all lie in U, so
$$\Phi(g_1), \Phi(g_1^{-1}g_2), \ldots, \Phi(g_{m-1}^{-1}g_m) \tag{30}$$
are defined. Observe that
$$g = g_1 \cdot g_1^{-1}g_2 \cdot \ldots \cdot g_{m-1}^{-1}g_m, \tag{31}$$
so we set
$$\Phi(g) \coloneqq \Phi(g_1) \cdot \Phi(g_1^{-1}g_2) \cdot \ldots \cdot \Phi(g_{m-1}^{-1}g_m). \tag{32}$$

2. Show that $\Phi(g)$ does not change if the sequence $g_1, g_2, \ldots, g_m$ is refined by inserting an extra point. Since any two sequences have a common refinement, obtained by inserting extra points, the value of $\Phi(g)$ is independent of the sequence of points on the path.

3. Show that $\Phi(g)$ only depends on the homotopy class of the path from $\mathbb{1}$ to g, by showing that $\Phi(g)$ does not change under a small deformation of the path. Since G is simply connected by assumption there is just one path between two points up to homotopy fixing the endpoints.

4. Check that Φ is a group homomorphism, and that it induces the Lie algebra homomorphism ϕ.

Combining the facts that there is always a local map and that in the simply connected case this can be extended uniquely, we have proved the global First Lie Theorem.

3.2 Consequences

Corollary 1. *Let G, H be simply connected Lie groups with isomorphic Lie algebras:*

$$\mathfrak{g} \cong_{(LieAlg)} \mathfrak{h},$$

then G and H are isomorphic as Lie groups:

$$G \cong_{(LieGrp)} H.$$

Corollary 2 ([Tits, 1983, III.4.2, Korollar 2])**.** *Let G be an analytical Lie group with Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(G)$ the group of Lie group automorphisms of G, $\mathrm{Aut}(\mathfrak{g})$ the group of Lie algebra automorphisms of $\mathfrak{g}$. Define the homomorphism*

$$\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g}) \tag{33}$$
$$\Phi \mapsto T_{\mathbb{1}}\Phi \tag{34}$$

- *G connected $\Rightarrow \mathrm{Aut}(G) \overset{d}{\hookrightarrow} \mathrm{Aut}\,\mathfrak{g}$ is injective*
- *G simply connected $\Rightarrow \mathrm{Aut}(G) \overset{\cong}{\to} \mathrm{Aut}\,\mathfrak{g}$ is an isomorphism*

4 Extra topics

4.1 Uniqueness and existence of an analytic structure

One can always automatically upgrade the smooth structure on a Lie group to an analytic structure, this is a consequence of the Campbell-Hausdorff-Baker-Dynkin formula. This analytic structure is unique in a sense we will make precise below. Thus the assumptions above that the Lie groups have an analytic atlas were unnecessary and we could also drop them.

We will now recall some definitions, give the important theorems and a constructions, but leave the proofs to [Duistermaat and Kolk, 2012, 1.6].

In this section we will write the product in logarithmic coordinates of $X, Y \in \mathfrak{g}$ as $\mu(X, Y)$, whenever this is defined for X and Y. For a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, let $\mathfrak{g}_e$ be the set of $X \in \mathfrak{g}$ such that $f(\mathrm{ad}\,X) = \frac{e^{\mathrm{ad}\,X} - \mathrm{id}}{\mathrm{ad}\,X}$ is invertible (for details consult [Duistermaat and Kolk, 2012, 1.5.3]). One can show that the set $\mathfrak{g}_e \times \mathfrak{g}_e$ is an open neighborhood of $(0, 0)$ in $\mathfrak{g} \times \mathfrak{g}$ and that μ is a real analytical mapping $\mathfrak{g}_e \times \mathfrak{g}_e \to \mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group G, with exponential mapping $\exp\colon \mathfrak{g} \to G$, then

$$\exp(X)\exp(Y) = \exp\mu(X, Y) \text{ for } (X, Y) \in \mathfrak{g}_e \times \mathfrak{g}_e.$$

Definition 3. *A real-analytic Lie group G is a group G that at the same time is a real-analytic manifold (i.e. the transition maps are locally given by a convergent power series) in such a way that the group operations multiplication $\cdot\colon G \times G \to G, (x, y) \mapsto xy$ and inversion ${}^{-1}\colon G \to G, x \mapsto x^{-1}$ are real-analytic mappings.*

To show the analyticity of a (smooth) Lie group G, take an open neighborhood of $0 \in \mathfrak{g} = \operatorname{Lie} G$, and V of $\mathbb{1} \in G$ such that

$$\exp \text{ is a diffeomorphism from } U \text{ to } V \tag{35}$$

and a neighborhood $0 \in U_0 \subset \mathfrak{g}$, such that for all $X, Y, Z \in U_0$:

$$(X, -Y) \in \mathfrak{g}_e \times \mathfrak{g}_e \tag{36}$$
$$(\mu(X, -Y), Z) \in \mathfrak{g}_e \times \mathfrak{g}_e \tag{37}$$
$$\mu(\mu(X, -Y), Z) \in U. \tag{38}$$

Recall that the differential of the exponential map at 0 is the identity,

$$T_0 \exp = \mathrm{id} \colon \mathfrak{g} \to \mathfrak{g},$$

so the differential is obviously invertible. If we now apply the inverse function theorem to exp, use that in logarithmic coordinates $\mu(0,0) = 0$, and that μ is continuous at $(0,0)$ we can conclude the existence of such neighborhoods U, V, U_0. To construct an analytic structure we need chart neighborhoods, and on those we have to define coordinate charts. For each $x \in G$ write

$$V_0^x = x(\exp U_0), \tag{39}$$

the left-translation of the open set U_0 by the group element x. On this set define a chart

$$\kappa^x \colon V_0^x \to \mathfrak{g} \cong \mathbb{R}^{\dim G} \tag{40}$$
$$y \mapsto \log(x^{-1}y) \tag{41}$$

Theorem 6 (Existence of analytic structure, [Duistermaat and Kolk, 2012, 1.6.3])**.** *The*

$$\kappa^x \colon V_0^x \to U_0 \textit{ for } x \in G$$

form a real-analytic atlas for G, making G into a real-analytic Lie group G_{an}, such that the identity in G is a $\mathscr{C}^\infty$ diffeomorphism between G and G_{an}.

The following proposition expresses the uniqueness of the real analytic structure of a Lie group G with the given Lie algebra $\mathfrak{g}$.

Proposition 1 (Uniqueness of analytic structure, [Duistermaat and Kolk, 2012, 1.6.4])**.** *If G is a real-analytic Lie group, then the exponential mapping* $\exp \colon G \to \mathfrak{g}$ *is real analytic. If G is provided with another structure of a real-analytic Lie group G' such that the identity $G \to G'$ is differentiable and $G' \to G$ is differentiable (at the neutral element $\mathbb{1} \in G, G'$), then the identity is a real-analytic diffeomorphism in both directions.*

Remark 2. *Warning: We need the assumption that the Lie algebra $\mathfrak{g}$ of G is specified in advance: For example if as sets we have $G \cong_{(Set)} G'$, but G' carries the discrete topology, the G' is a 0-dimensional Lie group (with the Lie algebra $\mathfrak{g}' = 0$). The identity $G' \to G$ is analytic, but in the other direction the identity $G \to G'$ is not even continuous if* $\dim G > 0$.

From now on we can assume all Lie groups to be real analytic, with the unique real analytical structure determined by its given real Lie algebra.

4.2 Classification of connected 2-dimensional Lie groups

The First Lie Theorem reduces the classification of simply connected analytical Lie groups to the classification of finite dimensional Lie algebras over $\mathbb{R}$. We know how to pass from the **simply-connected** real analytical groups to the **connected** real analytical groups: Every connected Lie group G is locally isomorphic to its universal cover $\widetilde{G}$. The quotients $\widetilde{G}/D$ of $\widetilde{G}$ by discrete normal subgroups D in the center of $\widetilde{G}$ are all the connected real analytical groups locally isomorphic to $\widetilde{G}$. For a proof of this see [Tits, 1983, II.4.6].

We now have all the tools necessary to classify the connected 2-dimensional real analytical Lie groups.

4.2.1 Enumerate the 2-dim. Lie algebras over the (arbitrary) field K

Let L be a 2-dimensional vector space over K with basis (x, y). The relations

$$[x, x] = [y, y] = 0 \tag{42}$$

$$[x, y] = -[y, x] = x \tag{43}$$

define a non-commutative Lie algebra which we will also call L.

Let $(A, [.,.])$ be a 2-dimensional Lie algebra over K, (e_1, e_2) a basis for A. Let A' be the subvectorspace spanned by the commutators $\{[a_1, a_2] | a_1, a_2 \in A\}$.

- Case 1: $A' = 0$, then A is commutative, thus isomorphic to K^2.
- Case 2: $A' \neq 0$, so A is not commutative. Thus $x := [e_1, e_2] \neq 0$ and $A' = K \cdot x$. Extend this to a basis of A, i.e. choose $y' \in A$ such that (x, y') is a basis of A. Then $[x, y'] = \alpha x$ with $\alpha \neq 0$. Let $y := \frac{1}{\alpha} y'$. (x, y) is a basis for A and $[x, y] = x$, so $A \cong L$ as Lie algebras.

This shows:

Proposition 2. *The commutative Lie algebra K^2 and the non-commutative algebra L are the only two-dimensional Lie algebras, up to isomorphism.*

4.2.2 Find the corresponding Lie groups

From now on $K = \mathbb{R}$. $(\mathbb{R}^2, +)$ is the simply connected Lie group belonging to the Lie algebra $\mathbb{R}^2$. Central discrete normal subgroups are up to base change $\{0\}, \mathbb{Z} \times 0$ and $\mathbb{Z} \times \mathbb{Z}$ with corresponding Lie groups $\mathbb{R}^2$ (the plane), $\mathbb{S}^1 \times \mathbb{R}$ (the cylinder) and $\mathbb{S}^1 \times \mathbb{S}^1$ (the torus).

Consider the affine transformations from $\mathbb{R}$ to itself with positive determinant, these form a non-commutative group under composition:

$$\text{Aff}_1(\mathbb{R}) = \{\mathbb{R} \to \mathbb{R}, x \mapsto a \cdot x + b | a, b \in \mathbb{R}, a > 0\}. \tag{44}$$

Topologically $\text{Aff}_1(\mathbb{R}) \approx \mathbb{R}_{\geq 0} \times \mathbb{R}$, so it is simply connected. $\text{Aff}_1(\mathbb{R})$ is isomorphic to the subgroup

$$\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in \mathbb{R}, a > 0\} < \text{Gl}_+(2, \mathbb{R}). \tag{45}$$

Proposition 3. *The Center of* $\mathrm{Aff}_1(\mathbb{R})$ *is trivial,* $\mathrm{Z}(\mathrm{Aff}_1(\mathbb{R})) = \{\mathbb{1}\}$.

Proof. For $c \in \mathbb{R}$ there is a $g \in \mathrm{Aff}_1(\mathbb{R})$ such that the fixed point set of g is exactly $\{c\}$, i.e. one can take a dilation with center c. For $z \in \mathrm{Z}(\mathrm{Aff}_1(\mathbb{R}))$ we have $z(c) = zg(c) = gz(c)$, so $z(c) = c$. But then z does not move any point, so $z = \mathbb{1}$. □

Corollary 3. *The Lie group* $\mathrm{Aff}_1(\mathbb{R})$ *is the only non-commutative connected 2-dimensional Lie group.*

Corollary 4. *We can list all the connected 2-dimensional real analytical Lie groups: Such a group is isomorphic to one of*

$$\mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}, \mathbb{S}^1 \times \mathbb{S}^1, \mathrm{Aff}_1(\mathbb{R}).$$

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